Predicate and Propositional Logic - Seminar 9

Dec 8, 2014

1. Let $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, -, \cdot, 0, 1 \rangle$ be the structure of rational numbers with standard operations (thus forming a *field*).
   (a) Is there a reduct of $\underline{\mathbb{Q}}$ that is a model of theory T of groups (from the previous seminar)?
   (b) Can we expand the reduct $\langle \mathbb{Q}, \cdot, 1 \rangle$ to a model of T?
   (c) Does $\underline{\mathbb{Q}}$ contain a substructure that is not elementarily equivalent to $\underline{\mathbb{Q}}$?
   (d) Let $Th(\underline{\mathbb{Q}})$ denote the set of all sentences that are valid in $\underline{\mathbb{Q}}$. Is $Th(\underline{\mathbb{Q}})$ a complete theory?

2. Let $T = \{x = c_1 \vee x = c_2 \vee x = c_3\}$ be a theory of $L = \langle c_1, c_2, c_3 \rangle$ with equality.
   (a) Is T (semantically) consistent?
   (b) Are all models of T elementarily equivalent? That is, is T (semantically) complete?
   (c) Find all simple complete extensions of T.
   (d) Is a theory $T' = T \cup \{x = c_1 \vee x = c_4\}$ of the language $L = \langle c_1, c_2, c_3, c_4 \rangle$ an extension of T? Is T' a simple extension of T? Is T' a conservative extension of T?

3. Assume that
   (a) all guilty persons are liars,
   (b) at least one of the accused is also a witness,
   (c) no witness lies.

   Prove by tableau method that not all accused are guilty.

4. Let $L(x, y)$ represent that *"there is a flight from x to y"* and let $S(x, y)$ represent that *"there is a connection from x to y"*. Assume that
   (a) From Prague you can flight to Bratislava, London and New York, and from New York to Paris,
   (b) $(\forall x)(\forall y)(L(x, y) \rightarrow L(y, x))$,
   (c) $(\forall x)(\forall y)(L(x, y) \rightarrow S(x, y))$,
   (d) $(\forall x)(\forall y)(\forall z)(S(x, y) \wedge L(y, z) \rightarrow S(x, z))$.

   Prove by tableau method that there is a connection from Bratislava to Paris.

5. Let φ, ψ be sentences or formulas in a free variable x, denoted by $\varphi(x)$, $\psi(x)$. Find tableau proofs of the following formulas.
   (a) $(\exists x)(\varphi(x) \vee \psi(x)) \leftrightarrow (\exists x)\varphi(x) \vee (\exists x)\psi(x)$,
   (b) $(\forall x)(\varphi(x) \wedge \psi(x)) \leftrightarrow (\forall x)\varphi(x) \wedge (\forall x)\psi(x)$,
   (c) $(\varphi \vee (\forall x)\psi(x)) \rightarrow (\forall x)(\varphi \vee \psi(x))$ where x is not free in φ,
   (d) $(\varphi \wedge (\exists x)\psi(x)) \rightarrow (\exists x)(\varphi \wedge \psi(x))$ where x is not free in φ.
   (e) $(\exists x)(\varphi \rightarrow \psi(x)) \rightarrow (\varphi \rightarrow (\exists x)\psi(x))$ where x is not free in φ,
   (f) $(\exists x)(\varphi \wedge \psi(x)) \rightarrow (\varphi \wedge (\exists x)\psi(x))$ where x is not free in φ,
   (g) $(\exists x)(\varphi(x) \rightarrow \psi) \rightarrow ((\forall x)\varphi(x) \rightarrow \psi)$ where x is not free in ψ,
   (h) $((\exists x)\varphi(x) \rightarrow \psi) \rightarrow (\forall x)(\varphi(x) \rightarrow \psi)$ where x is not free in ψ.

6. Let T^* be a theory with axioms of equality. Prove by tableau method that

   (a) $T^* \models x = y \ \rightarrow \ y = x$ (symmetry of $=$)

   (b) $T^* \models (x = y \ \wedge \ y = z) \ \rightarrow \ x = z$ (transitivity of $=$)

   Hint: To show (a) apply the axiom of equality (*iii*) for $x_1 = x$, $x_2 = x$, $y_1 = y$ a $y_2 = x$, to show (b) apply (*iii*) for $x_1 = x$, $x_2 = y$, $y_1 = x$ a $y_2 = z$.

7. Prove the theorem on constants syntactically by transformations of tableaux.

   Věta 1. *Let φ be a formula of a language L with free variables $x_1, \ldots, x_n$ and let T be a theory in L. Let L' denote the extension of L with new constant symbols $c_1, \ldots, c_n$ and let T' denote the theory T in L'. Then*

   $$T \vdash (\forall x_1) \ldots (\forall x_n)\varphi \quad \textit{if and only if} \quad T' \vdash \varphi(x_1/c_1, \ldots, x_n/c_n).$$

8. Prove the deduction theorem syntactically by transformations of tableaux.

   Věta 2. *For every theory T (in a closed form) and sentences φ, ψ,*

   $$T \vdash \varphi \rightarrow \psi \quad \textit{if and only if} \quad T, \varphi \vdash \psi.$$

9. Let L be a language with equality containing a binary relation symbol $\leq$ and let T be a theory of L such that T has an infinite model and the axioms of linear ordering are valid in T. Applying the compactness theorem show that T has a model $\mathcal{A}$ with an *infinite decreasing chain*; that is, there are elements c_i for every $i \in \mathbb{N}$ in A such that

   $$\cdots < c_{n+1} < c_n < \cdots < c_0.$$

   (This show that the notion of *well-ordering* is not definable in a first-order language.)