Predicate and Propositional Logic - Seminar 10

Dec 15, 2014

1. Let $L(x, y)$ represent that *"there is a flight from x to y"* and let $S(x, y)$ represent that *"there is a connection from x to y"*. Assume that

   (a) From Prague you can flight to Bratislava, London and New York, and from New York to Paris,

   (b) $(\forall x)(\forall y)(L(x, y) \to L(y, x))$,

   (c) $(\forall x)(\forall y)(L(x, y) \to S(x, y))$,

   (d) $(\forall x)(\forall y)(\forall z)(S(x, y) \wedge L(y, z) \to S(x, z))$.

   Prove by tableau method that there is a connection from Bratislava to Paris.

2. Let T^* be a theory with axioms of equality. Prove by tableau method that

   (a) $T^* \models x = y \;\to\; y = x$ (symmetry of $=$)

   (b) $T^* \models (x = y \;\wedge\; y = z) \;\to\; x = z$ (transitivity of $=$)

   Hint: To show (a) apply the axiom of equality (*iii*) for $x_1 = x$, $x_2 = x$, $y_1 = y$ a $y_2 = x$, to show (b) apply (*iii*) for $x_1 = x$, $x_2 = y$, $y_1 = x$ a $y_2 = z$.

3. Prove the theorem on constants syntactically by transformations of tableaux.

   Theorem 1. *Let φ be a formula of a language L with free variables $x_1, \ldots, x_n$ and let T be a theory in L. Let L' denote the extension of L with new constant symbols $c_1, \ldots, c_n$ and let T' denote the theory T in L'. Then*

   $$T \vdash (\forall x_1) \ldots (\forall x_n)\varphi \quad \textit{if and only if} \quad T' \vdash \varphi(x_1/c_1, \ldots, x_n/c_n).$$

4. Let L be a language with equality containing a binary relation symbol $\leq$ and let T be a theory of L such that T has an infinite model and the axioms of linear ordering are valid in T. Applying the compactness theorem show that T has a model $\mathcal{A}$ with an *infinite decreasing chain*; that is, there are elements c_i for every $i \in \mathbb{N}$ in A such that

   $$\cdots < c_{n+1} < c_n < \cdots < c_0.$$

   (This show that the notion of *well-ordering* is not definable in a first-order language.)

5. Convert the following formulas into the prenex normal form.

   (a) $(\forall y)((\exists x)P(x, y) \to Q(y, z)) \wedge (\exists y)((\forall x)R(x, y) \vee Q(x, y))$

   (b) $(\exists x)R(x, y) \leftrightarrow (\forall y)P(x, y)$

   (c) $\neg((\forall x)(\exists y)P(x, y) \to (\exists x)(\exists y)R(x, y)) \wedge (\forall x)\neg(\exists y)Q(x, y)$

6. Find Skolem variants of the formulas in PNF from the previous problem.

7. Verify that (thus, a Skolem variant does not have to be equivalent to the original formula)

   (a) $\models (\forall x)P(x, f(x)) \to (\forall x)(\exists y)P(x, y)$

   (b) $\not\models (\forall x)(\exists y)P(x, y) \to (\forall x)P(x, f(x))$

8. Let T' be the extension of $T = \{(\exists y)(x + y = 0), (x + y = 0) \wedge (x + z = 0) \to y = z\}$ in $L = \langle +, 0, \leq\rangle$ with equality by definitions of $<$ and unary $-$ with axioms

   $$\begin{aligned} -x = y \quad &\leftrightarrow \quad x + y = 0 \\ x < y \quad &\leftrightarrow \quad x \leq y \;\wedge\; \neg(x = y) \end{aligned}$$

   Find formulas of L that are equivalent in T' to the following formulas.

(a) $x + (-x) = 0$
(b) $x + (-y) < x$
(c) $-(x + y) < -x$

9. The theory T of *fields* in $L = \langle +, -, \cdot, 0, 1\rangle$ contains one axiom φ that is not open:
$$x \neq 0 \ \rightarrow \ (\exists y)(x \cdot y = 1).$$
We know that $T \models 0 \cdot y = 0$ and $T \models \ (x \neq 0 \ \wedge \ x \cdot y = 1 \ \wedge \ x \cdot z = 1) \ \rightarrow \ y = z$.
   (a) Find a Skolem variant φ_S of φ with a new function symbol f.
   (b) Let T' be the theory obtained from T by replacing φ with φ_S. Is $T' \models \varphi$?
   (c) Can every model of T be *uniquely* expanded to a model of T'?
10. Let T denote the (previous) theory of fields. Let ψ be the formula $x \cdot y = 1 \vee (x = 0 \wedge y = 0)$.
   (a) Do the conditions of existence and uniqueness hold in T for $\psi(x, y)$ and the variable y?
   (b) Find an extension T^* of T by definition of a function symbol f with the formula ψ.
   (c) Is T^* equivalent to the theory T' from the previous problem?
   (d) Find a formula of the original language L that is equivalent in T^* to the formula
$$f(x \cdot y) = f(x) \cdot f(y)$$
11. Find Herbrand universe and an example of a Herbrand structure for the following languages.
   (a) $L = \langle P, Q, f, a, b\rangle$ where P, Q are unary resp. binary relation symbols, f is a unary function symbol, a, b are constant symbols.
   (b) $L = \langle P, f, g, a\rangle$ where P is a binary relation s., f, g unary function s., a constant symbol.
12. Find Herbrand models for the following theories or find unsatisfiable conjunctions of ground instances of their axioms. Assume that the language has constant symbols a, b.
   (a) $T = \{\neg P(x) \vee Q(f(x), y), \neg Q(x, b), P(a)\}$
   (b) $T = \{\neg P(x) \vee Q(f(x), y), Q(x, b), P(a)\}$
   (c) $T = \{P(x, f(x)), \neg P(x, g(x))\}$
   (d) $T = \{P(x, f(x)), \neg P(x, g(x)), P(g(x), f(y)) \rightarrow P(x, y)\}$
13. Transform the following formulas to equisatisfiable formulas in clausal form.
   (a) $(\forall y)(\exists x)P(x, y)$
   (b) $\neg(\forall y)(\exists x)P(x, y)$
   (c) $\neg(\exists x)((P(x) \rightarrow P(a)) \wedge (P(x) \rightarrow P(b)))$
   (d) $(\exists x)(\forall y)(\exists z)(P(x, z) \wedge P(z, y) \rightarrow R(x, y))$
14. We know that
   (a) If a brick is on (another) brick, then it is not on the ground.
   (b) Every brick is on (another) brick or on the ground.
   (c) No brick is on a brick that is on (another) brick.

   Express these facts in a first-order language and prove by resolution (via grounding) that if a brick is on another brick, the lower brick is on the ground.
15. We know that
   (a) Every barber shaves all who do not shave themselves.
   (b) No barber shaves someone who shaves himself.

   Express these facts in a first-order language and prove by resolution (via grounding) that no barber exists.