Exam from logic - the test

February 4th, 2015

1. Let $C_n = \langle V, E\rangle$ be an undirected cycle of length $n \geq 2$; that is, the graph with the vertex set $V = \{1, 2, \ldots, n\}$ and the edge set $E = \{\{i, i+1\} \mid 1 \leq i < n\} \cup \{\{n, 1\}\}$. We say that a set $A \subseteq E$ is a *perfect matching* if every vertex is incident with exactly one edge of A.

   We want to show (by resolution in propositional logic) that the cycle C_n has no perfect matching for an odd n (in particular for $n = 3$). Let $\mathbb{P}_n = \{p_{\{i,j\}} \mid \{i, j\} \in E\}$ be a set of propositional letters, where $p_{\{i,j\}}$ represents that *"the edge $\{i, j\}$ is in the set A"*.

   (a) Write a proposition φ_i, where $1 \leq i \leq n$, over $\mathbb{P}_n$ expressing that *"the vertex i is incident with exactly one edge of A"*. With use of propositions φ_i, write a theory T_n expressing that *"the set A is a perfect matching."* *(2 pts)*

   (b) Now let us fix $n = 3$. Transform the theory T_3 into the clausal form. *(2 pts)*

   (c) Show by resolution that T_3 is unsatisfiable. *(4 pts)*

   (d) Is $T_3 \vdash_{LI} \square$? Give an explanation. *(2 pts)*

2. We consider the following two statements:

   (i) There is a student such that if he passes the exam from logic then all students pass the exam from logic.

   (ii) There is a student such that if he passes the exam from logic then all students pass the exam from algebra.

   (a) Write the statements (i), (ii) formally as sentences φ, ψ (respectively) in the language $L = \langle P, R\rangle$ without equality where P, R are unary relation symbols and $P(x)$, $R(x)$ represent that *"the student x passes the exam from logic"*, resp. *"the student x passes the exam from algebra"*. *(2 pts)*

   (b) Determine by the tableaux method which of the formulas φ, ψ are logically valid. Provide the associated finished tableaux. *(4 pts)*

   (c) Choose any noncontradictory branch V in one of the tableaux from (b) and construct a canonical model $\mathcal{A}$ from the branch V. *(2 pts)*

   (d) Is the theory $\{\psi\}$ of the language L a conservative extension of the theory $\{\varphi\}$ of the language $L' = \langle P\rangle$? Give an explanation. *(2 pts)*

3. Let $T = \{(\forall x)(\exists y) f(y) = x,\ f(x) = f(y) \rightarrow x = y\}$ be a theory of the language $L = \langle f\rangle$ with equality, where f is a unary function symbol.

   (a) Find an extension T' of the theory T by definition of a new unary function symbol g such that $T' \models f(g(x)) = x$. *(2 pts)*

   (b) Write a formula $\varphi(x, y)$ in the language L such that $T' \models \varphi(x, y) \leftrightarrow g(g(x)) = y$. *(2 pts)*

   (c) Is the sentence $(\forall x)(\exists y) g(y) = x$ valid / contradictory / independent in T'? Give an explanation. *(2 pts)*

   (d) Let $\mathcal{A} = \langle \mathbb{Z}, f^A\rangle$, where $f^A(m) = m + 2$ for every $m \in \mathbb{Z}$. How many mutually nonisomorphic substructures the structure $\mathcal{A}$ has? *(2 pts)*