Propositional and Predicate Logic - IV

Petr Gregor

KTIML MFF UK

ZS 2015/2016

Introductory examples

$$
\begin{array}{c}
F(((p \to q) \to p) \to p) \\
| \\
T((p \to q) \to p) \\
| \\
Fp \\
| \\
T((p \to q) \to p) \\
\diagup \qquad \diagdown \\
\begin{array}{cc}
F(p \to q) & Tp \\
| & | \\
F(p \to q) & \otimes \\
| & \\
Tp & \\
| & \\
Fq & \\
| & \\
\otimes &
\end{array}
\end{array}
\qquad\qquad
\begin{array}{c}
F((\neg q \vee p) \to p) \\
| \\
T(\neg q \vee p) \\
| \\
Fp \\
| \\
T(\neg q \vee p) \\
\diagup \qquad \diagdown \\
\begin{array}{cc}
T(\neg q) & Tp \\
| & | \\
T(\neg q) & \otimes \\
| & \\
Fq &
\end{array}
\end{array}
$$

Atomic tableaux

An *atomic tableau* is one of the following trees (labeled by entries), where p is any propositional letter and φ, ψ are any propositions.

Tp	Fp	$T(\varphi \wedge \psi)$ $\vert$ $T\varphi$ $\vert$ $T\psi$	$F(\varphi \wedge \psi)$ $/ \quad \backslash$ $F\varphi \qquad F\psi$	$T(\varphi \vee \psi)$ $/ \quad \backslash$ $T\varphi \qquad T\psi$	$F(\varphi \vee \psi)$ $\vert$ $F\varphi$ $\vert$ $F\psi$
$T(\neg\varphi)$ $\vert$ $F\varphi$	$F(\neg\varphi)$ $\vert$ $T\varphi$	$T(\varphi \rightarrow \psi)$ $/ \quad \backslash$ $F\varphi \qquad T\psi$	$F(\varphi \rightarrow \psi)$ $\vert$ $T\varphi$ $\vert$ $F\psi$	$T(\varphi \leftrightarrow \psi)$ $/ \quad \backslash$ $T\varphi \qquad F\varphi$ $\vert \qquad\quad \vert$ $T\psi \qquad F\psi$	$F(\varphi \leftrightarrow \psi)$ $/ \quad \backslash$ $T\varphi \qquad F\varphi$ $\vert \qquad\quad \vert$ $F\psi \qquad T\psi$

Tableaux

A *finite tableau* is a binary tree labeled with entries described (inductively) by

- (i) every atomic tableau is a finite tableau,
- (ii) if P is an entry on a branch V in a finite tableau τ and τ' is obtained from τ by adjoining the atomic tableaux for P at the end of branch V, then τ' is also a finite tableau,
- (iii) every finite tableau is formed by a finite number of steps (i), (ii).

A *tableau* is a sequence $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ (finite or infinite) of finite tableaux such that τ_{n+1} is formed from τ_n by an application of (ii), formally $\tau = \cup\tau_n$.

Remark *It is not specified how to choose the entry P and the branch V for expansion. This will be specified in systematic tableaux.*

Tableau proofs

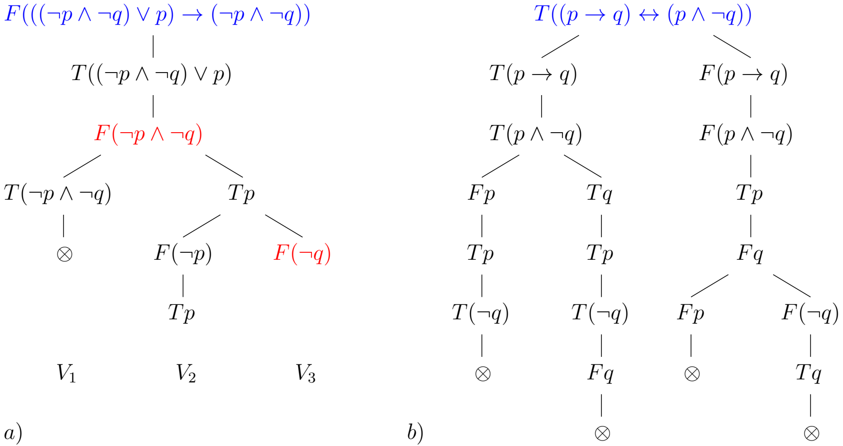
Let P be an entry on a branch V in a tableau τ. We say that

- the entry P is *reduced* on V if it occurs on V as a root of an atomic tableau, i.e. it was already expanded on V during the construction of τ,
- the branch V is *contradictory* if it contains entries $T\varphi$ and $F\varphi$ for some proposition φ, otherwise V is *noncontradictory*. The branch V is *finished* if it is contradictory or every entry on V is already reduced on V,
- the tableau τ is *finished* if every branch in τ is finished, and τ is *contradictory* if every branch in τ is contradictory.

A *tableau proof* (*proof by tableau*) of φ is a contradictory tableau with the root entry $F\varphi$. φ is *(tableau) provable*, denoted by $\vdash \varphi$, if it has a tableau proof.

Similarly, a *refutation* of φ by *tableau* is a contradictory tableau with the root entry $T\varphi$. φ is *(tableau) refutable* if it has a refutation by tableau, i.e. $\vdash \neg\varphi$.

Examples



a) $F(\neg p \wedge \neg q)$ not reduced on V_1, V_1 contradictory, V_2 finished, V_3 unfinished,

b) a (tableau) refutation of φ: $(p \to q) \leftrightarrow (p \wedge \neg q)$, i.e. $\vdash \neg\varphi$.

Tableau from a theory

How to add axioms of a given theory into a proof?

A *finite tableau from a theory* T is generalized tableau with an additional rule

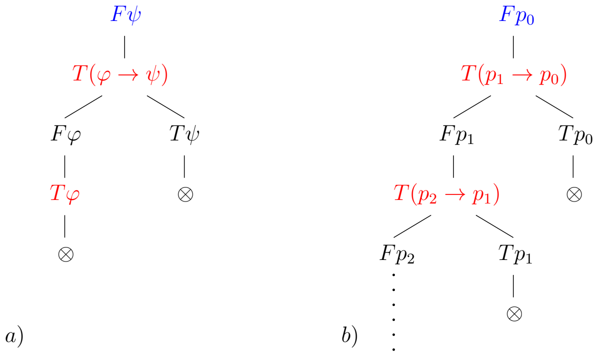
(ii)' if V is a branch of a finite tableau (from T) and $\varphi \in T$, then by adjoining $T\varphi$ at the end of V we obtain (again) a finite tableau from T.

We generalize other definitions by appending *"from T"*.

- a *tableau from T* is a sequence $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ of finite tableaux from T such that τ_{n+1} is formed from τ_n applying (ii) or (ii)', formally $\tau = \cup\tau_n$,
- a *tableau proof* of φ *from T* is a contradictory tableaux from T with $F\varphi$ in the root. $T \vdash \varphi$ denotes that φ is *(tableau) provable from T*.
- a *refutation* of φ by a *tableau from T* is a contradictory tableau from T with the root entry $T\varphi$.

Unlike in previous definitions, a branch V of a tableau from T is *finished*, if it is contradictory, or every entry on V is already reduced on V and, moreover, V contains $T\varphi$ for every $\varphi \in T$.

Examples of tableaux from theories



a) A tableau proof of ψ from $T = \{\varphi, \varphi \to \psi\}$, so $T \vdash \psi$.

b) A finished tableau with the root Fp_0 from $T = \{p_{n+1} \to p_n \mid n \in \mathbb{N}\}$.
All branches are finished, the leftmost branch is noncontradictory and infinite. It provides us with the (only one) model of T in which p_0 is false.

Systematic tableaux

We describe a systematic construction that leads to a finished tableau.

Let R be an entry and $T = \{\varphi_0, \varphi_1, \dots\}$ be a (possibly infinite) theory.

(1) We take the atomic tableau for R as τ_0. Till possible, proceed as follows.

(2) Let P be the leftmost entry in the smallest level as possible of the tableau τ_n s.t. P is not reduced on some noncontradictory branch through P.

(3) Let τ_n' be the tableau obtained from τ_n by adjoining the atomic tableau for P to every noncontradictory branch through P. (If P does not exists, we take $\tau_n' = \tau_n$.)

(4) Let τ_{n+1} be the tableau obtained from τ_n' by adjoining $T\varphi_n$ to every noncontradictory branch that does not contain $T\varphi_n$ yet. (If φ_n does not exists, we take $\tau_{n+1} = \tau_n'$.)

The *systematic tableau* from T for the entry R is the result of the above construction, i.e. $\tau = \cup\tau_n$.

Systematic tableau - being finished

Proposition *Every systematic tableau is finished.*

Proof Let $\tau = \cup \tau_n$ be a systematic tableau from $T = \{\varphi_0, \varphi_1, \ldots\}$ with root R.

- If a branch is noncontradictory in τ, its prefix in every τ_n is noncontradictory as well.
- If an entry P in unreduced on some branch in τ, it is unreduced on its prefix in every τ_n as well (assuming P occurs on this prefix).
- There are only finitely many entries in τ in levels up to the level of P.
- Thus, if P was unreduced on some noncontradictory branch in τ, it would be considered in some step (2) and reduced by step (3).
- By step (4) every $\varphi_n \in T$ will be (no later than) in τ_{n+1} on every noncontradictory branch.
- Hence the systematic tableau τ has all branches finished. $\square$

Finiteness of proofs

Proposition *For every contradictory tableau $\tau = \cup\tau_n$ there is some n such that τ_n is a contradictory finite tableau.*

- *Proof* Let S be the set of nodes in τ that have no pair of contradictory entries $T\varphi$, $F\varphi$ amongst their predecessors.
- If S was infinite, then by König's lemma, the subtree of τ induced by S would contain an infinite brach, and thus τ would not be contradictory.
- Since S is finite, for some m all nodes of S belong to levels up to m.
- Thus every node in level $m + 1$ has a pair of contradictory entries amongst its predecessors.
- Let n be such that τ_n agrees with τ at least up to the level $m + 1$.
- Then every branch in τ_n is contradictory. □

Corollary *If a systematic tableau (from a theory) is a proof, it is finite.*

Proof In its construction, only noncontradictory branches are extended. □

Soundness

We say the an entry P *agrees* with an assignment v, if P is $T\varphi$ and $\overline{v}(\varphi) = 1$, or if P is $F\varphi$ and $\overline{v}(\varphi) = 0$. A branch V *agrees* with v, if every entry on V agrees with v.

Lemma *Let v be a model of a theory T that agrees with the root entry of a tableau $\tau = \cup\tau_n$ from T. Then τ contains a branch that agrees with v.*

Proof By induction we find a sequence $V_0, V_1, \ldots$ so that for every n, V_n is a branch in τ_n agreeing with v and V_n is contained in V_{n+1}.

- By considering all atomic tableaux we verify that base of induction holds.
- If τ_{n+1} is obtained from τ_n without extending V_n, we put $V_{n+1} = V_n$.
- If τ_{n+1} is obtained from τ_n by adjoining $T\varphi$ to V_n for some $\varphi \in T$, then let V_{n+1} be this branch. Since v is a model of φ, V_{n+1} agrees with v.
- Otherwise τ_{n+1} is obtained from τ_n by adjoining the atomic tableau for some entry P on V_n to the end of V_n. Since P agrees with v and atomic tableaux are verified, V_n can be extended to V_{n+1} as required. $\square$

Theorem on soundness

We will show that the tableau method in propositional logic is sound.

Theorem *For every theory T and proposition φ, if φ is tableau provable from T, then φ is valid in T, i.e. $T \vdash \varphi \;\Rightarrow\; T \models \varphi$.*

Proof

- Let φ be tableau provable from a theory T, i.e. there is a contradictory tableau τ from T with the root entry $F\varphi$.
- Suppose for a contradiction that φ is not valid in T, i.e. there exists a model v of the theory T if which φ is false (a counterexample).
- Since the root entry $F\varphi$ agrees with v, by the previous lemma, there is a branch in the tableau τ that agrees with v.
- But this is impossible, since every branch of τ is contradictory, i.e. it contains a pair of entries $T\psi$, $F\psi$ for some ψ. $\square$

Completeness

A noncontradictory branch in a finished tableau gives us a counterexample.

Lemma *Let V be a noncontradictory branch of a finished tableau τ.*

Then V agrees with the following assignment v.

$$v(p) = \begin{cases} 1 & \text{if } Tp \text{ occurs on } V \\ 0 & \text{otherwise} \end{cases}$$

Proof By induction on the structure of formulas in entries occurring on V.

- For an entry Tp on V, where p is a letter, we have $\overline{v}(p) = 1$ by definition.
- For an entry Fp on V, Tp in not on V since V is noncontradictory, thus $\overline{v}(p) = 0$ by definition of v.
- For an entry $T(\varphi \wedge \psi)$ on V, we have $T\varphi$ and $T\psi$ on V since τ is finished. By induction, we have $\overline{v}(\varphi) = \overline{v}(\psi) = 1$, and thus $\overline{v}(\varphi \wedge \psi) = 1$.
- For an entry $F(\varphi \wedge \psi)$ on V, we have $F\varphi$ or $F\psi$ on V since τ is finished. By induction, we have $\overline{v}(\varphi) = 0$ or $\overline{v}(\psi) = 0$, and thus $\overline{v}(\varphi \wedge \psi) = 0$.
- For other entries similarly as in previous two cases. $\square$

Theorem on completeness

We will show that the tableau method in propositional logic is complete.

Theorem *For every theory T and proposition φ, if φ is valid in T, then φ is tableau provable from T, i.e. $T \models \varphi \;\Rightarrow\; T \vdash \varphi$.*

Proof Let φ be valid in T. We will show that an arbitrary finished tableau (e.g. *systematic*) τ from theory T with the root entry $F\varphi$ is contradictory.

- If not, let V be some noncontradictory branch in τ.
- By the previous lemma, there exists an assignment v such that V agrees with v, in particular in the root entry $F\varphi$, i.e. $\overline{v}(\varphi) = 0$.
- Since V is finished, it contains $T\psi$ for every $\psi \in T$.
- Thus v is a model of theory T (since V agrees with v).
- But this contradicts the assumption that φ is valid in T.

Hence the tableau τ is a proof of φ from T. □

Properties of theories

We introduce syntactic variants of previous semantically defined notions.

Let T be a theory over $\mathbb{P}$. If φ is provable from T, we say that φ is a *theorem* of T. The set of theorems of T is denoted by

$$\mathrm{Thm}^{\mathbb{P}}(T) = \{\varphi \in \mathrm{VF}_{\mathbb{P}} \mid T \vdash \varphi\}.$$

We say that a theory T is

- *inconsistent* if $T \vdash \bot$, otherwise T is *consistent*,
- *complete* if it is consistent and every proposition is provable or refutable from T, i.e. $T \vdash \varphi$ or $T \vdash \neg\varphi$ for every $\varphi \in \mathrm{VF}_{\mathbb{P}}$,
- *extension* of a theory T' over $\mathbb{P}'$ if $\mathbb{P}' \subseteq \mathbb{P}$ and $\mathrm{Thm}^{\mathbb{P}'}(T') \subseteq \mathrm{Thm}^{\mathbb{P}}(T)$; we say that an extension T of a theory T' is *simple* if $\mathbb{P} = \mathbb{P}'$; and *conservative* if $\mathrm{Thm}^{\mathbb{P}'}(T') = \mathrm{Thm}^{\mathbb{P}}(T) \cap \mathrm{VF}_{\mathbb{P}'}$,
- *equivalent* with a theory T' if T is an extension of T' and vice-versa.

Corollaries

From the soundness and completeness of the tableau method it follows that these syntactic definitions agree with their semantic variants.

Corollary *For every theory T and propositions φ, ψ over $\mathbb{P}$,*

- $T \vdash \varphi$ *if and only if* $T \models \varphi$,
- $\mathrm{Thm}^{\mathbb{P}}(T) = \theta^{\mathbb{P}}(T)$,
- T *is inconsistent if and only if* T *is unsatisfiable, i.e. it has no model,*
- T *is complete if and only if* T *is semantically complete, i.e. it has a single model,*
- $T, \varphi \vdash \psi$ *if and only if* $T \vdash \varphi \rightarrow \psi$ *(Deduction theorem).*

Remark Deduction theorem can be proved directly by transformations of tableaux.

Theorem on compactness

Theorem *A theory T has a model iff every finite subset of T has a model.*

Proof 1 The implication from left to right is obvious. If T has no model, then it is inconsistent, i.e. $\perp$ is provable by a systematic tableau τ from T. Since τ is finite, $\perp$ is provable from some finite $T' \subseteq T$, i.e. T' has no model. $\square$

Remark *This proof is based on finiteness of proofs, soundness and completeness. We present an alternative proof (applying König's lemma).*

Proof 2 Let $T = \{\varphi_i \mid i \in \mathbb{N}\}$. Consider a tree S on (certain) finite binary strings σ ordered by being a prefix. We put $\sigma \in S$ if and only if there exists an assignment v with prefix σ such that $v \models \varphi_i$ for every $i \leq \mathrm{lth}(\sigma)$.

Observation *S has an infinite branch if and only if T has a model.*

Since $\{\varphi_i \mid i \in n\} \subseteq T$ has a model for every $n \in \mathbb{N}$, every level in S is nonempty. Thus S is infinite and moreover binary, hence by König's lemma, S contains an infinite branch. $\square$

Application of compactness

A graf (V, E) is *k-colorable* if there exists $c\colon V \to k$ such that $c(u) \neq c(v)$ for every edge $\{u, v\} \in E$.

Theorem *A countably infinite graph $G = (V, E)$ is k-colorable if and only if every finite subgraph of G is k-colorable.*

Proof The implication $\Rightarrow$ is obvious. Assume that every finite subgraph of G is k-colorable. Consider $\mathbb{P} = \{p_{u,i} \mid u \in V, i \in k\}$ and a theory T with axioms

$$
\begin{array}{ll}
p_{u,0} \vee \cdots \vee p_{u,k-1} & \text{for every } u \in V, \\
\neg(p_{u,i} \wedge p_{u,j}) & \text{for every } u \in V, i < j < k, \\
\neg(p_{u,i} \wedge p_{v,i}) & \text{for every } \{u, v\} \in E, i < k.
\end{array}
$$

Then G is k-colorable if and only if T has a model. By compactness, it suffices to show that every finite $T' \subseteq T$ has a model. Let G' be the subgraph of G induced by vertices u such that $p_{u,i}$ appears in T' for some i. Since G' is k-colorable by the assumption, the theory T' has a model. $\square$