Propositional and Predicate Logic - VIII

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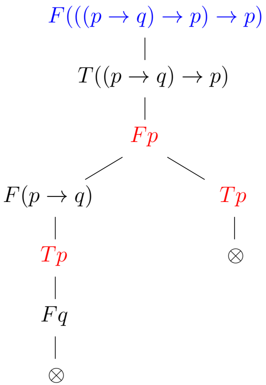
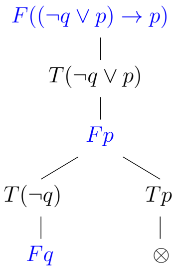
WS 2015/2016

Tableau method in propositional logic - a review

- A tableau is a binary tree that represents a search for a *counterexample*.
- Nodes are labeled by entries, i.e. formulas with a sign T / F that represents an assumption that the formula is true / false in some model.
- If this assumption is correct, then it is correct also for all the entries in some branch below that came from this entry.
- A branch is contradictory (it fails) if it contains $T\psi$, $F\psi$ for some ψ.
- A proof of formula φ is a contradictory tableau with root $F\varphi$, i.e. a tableau in which every branch is contradictory. If φ has a proof, it is valid.
- If a counterexample exists, there will be a branch in a finished tableau that provides us with this counterexample, but this branch can be infinite.
- We can construct a systematic tableau that is always finished.
- If φ is valid, the systematic tableau for φ is contradictory, i.e. it is a proof of φ; and in this case, it is also finite.

Tableau method in propositional logic - examples





$a)$ A tableau proof of the formula $((p \to q) \to p) \to p$.

$b)$ A finished tableau for $(\neg q \vee p) \to p$. The left branch provides us with a counterexample $v(p) = v(q) = 0$.

Tableau method in predicate logic - what is different

- Formulas in entries will always be sentences (closed formulas), i.e. formulas without free variables.
- We add new atomic tableaux for quantifiers.
- In these tableaux we substitute ground terms for quantified variables following certain rules.
- We extend the language by new (auxiliary) constant symbols (countably many) to represent *"witnesses"* of entries $T(\exists x)\varphi(x)$ and $F(\forall x)\varphi(x)$.
- In a finished noncontradictory branch containing an entry $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$ we have instances $T\varphi(x/t)$ resp. $F\varphi(x/t)$ for every ground term t (of the extended language).

Assumptions

1) *The formula φ that we want to prove (or refute) is a sentence.* If not, we can replace φ with its universal closure φ', since for every theory T,

$$T \models \varphi \quad \text{if and only if} \quad T \models \varphi'.$$

2) *We prove from a theory in a closed form, i.e. every axiom is a sentence.* By replacing every axiom ψ with its universal closure ψ' we obtain an equivalent theory since for every structure $\mathcal{A}$ (of the given language L),

$$\mathcal{A} \models \psi \quad \text{if and only if} \quad \mathcal{A} \models \psi'.$$

3) *The language L is countable.* Then every theory of L is countable. We denote by L_C the extension of L by new constant symbols $c_0, c_1, \ldots$ (countably many). Then there are countably many ground terms of L_C. Let t_i denote the i-th ground term (in some fixed enumeration).

4) *First, we assume that the language is without equality.*

Tableaux in predicate logic - examples

$$F((\exists x)\neg P(x) \rightarrow \neg(\forall x)P(x))$$
$$T(\exists x)\neg P(x)$$
$$F(\neg(\forall x)P(x))$$
$$T(\forall x)P(x)$$
$$T(\neg P(c)) \quad c \ \text{new}$$
$$FP(c)$$
$$T(\forall x)P(x)$$
$$TP(c)$$
$$\otimes$$

$$F(\neg(\forall x)P(x) \rightarrow (\exists x)\neg P(x))$$
$$T(\neg(\forall x)P(x))$$
$$F(\exists x)\neg P(x)$$
$$F(\forall x)P(x)$$
$$FP(d) \quad d \ \text{new}$$
$$F(\exists x)\neg P(x)$$
$$F(\neg P(d))$$
$$TP(d)$$
$$\otimes$$

Atomic tableaux - previous

An *atomic tableau* is one of the following trees (labeled by entries), where α is any atomic sentence and φ, ψ are any sentences, all of language L_C.

$T\alpha$	$F\alpha$	$\begin{array}{c} T(\varphi\wedge\psi) \\ \mid \\ T\varphi \\ \mid \\ T\psi \end{array}$	$\begin{array}{ccc} & F(\varphi\wedge\psi) & \\ / & & \backslash \\ F\varphi & & F\psi \end{array}$	$\begin{array}{ccc} & T(\varphi\vee\psi) & \\ / & & \backslash \\ T\varphi & & T\psi \end{array}$	$\begin{array}{c} F(\varphi\vee\psi) \\ \mid \\ F\varphi \\ \mid \\ F\psi \end{array}$
$\begin{array}{c} T(\neg\varphi) \\ \mid \\ F\varphi \end{array}$	$\begin{array}{c} F(\neg\varphi) \\ \mid \\ T\varphi \end{array}$	$\begin{array}{ccc} & T(\varphi\to\psi) & \\ / & & \backslash \\ F\varphi & & T\psi \end{array}$	$\begin{array}{c} F(\varphi\to\psi) \\ \mid \\ T\varphi \\ \mid \\ F\psi \end{array}$	$\begin{array}{ccc} & T(\varphi\leftrightarrow\psi) & \\ / & & \backslash \\ T\varphi & & F\varphi \\ \mid & & \mid \\ T\psi & & F\psi \end{array}$	$\begin{array}{ccc} & F(\varphi\leftrightarrow\psi) & \\ / & & \backslash \\ T\varphi & & F\varphi \\ \mid & & \mid \\ F\psi & & T\psi \end{array}$

Atomic tableaux - new

Atomic tableaux are also the following trees (labeled by entries), where φ is any formula of the language L_C with a free variable x, t is any ground term of L_C and c is a new constant symbol from $L_C \setminus L$.

$\sharp$ $T(\forall x)\varphi(x)$ $\vert$ $T\varphi(x/t)$ for any ground term t of L_C	$*$ $F(\forall x)\varphi(x)$ $\vert$ $F\varphi(x/c)$ for a *new* constant c	$*$ $T(\exists x)\varphi(x)$ $\vert$ $T\varphi(x/c)$ for a *new* constant c	$\sharp$ $F(\exists x)\varphi(x)$ $\vert$ $F\varphi(x/t)$ for any ground term t of L_C

Remark *The constant symbol c represents a "witness" of the entry $T(\exists x)\varphi(x)$ or $F(\forall x)\varphi(x)$. Since we need that no prior demands are put on c, we specify (in the definition of a tableau) which constant symbols c may be used.*

Tableau

A *finite tableau* from a theory T is a binary tree labeled with entries described

- (i) every atomic tableau is a finite tableau from T, whereas in case $(*)$ we may use any constant symbol $c \in L_C \setminus L$,
- (ii) if P is an entry on a branch V in a finite tableau from T, then by adjoining the atomic tableau for P at the end of branch V we obtain (again) a finite tableau from T, whereas in case $(*)$ we may use only a constant symbol $c \in L_C \setminus L$ that does not appear on V,
- (iii) if V is a branch in a finite tableau from T and $\varphi \in T$, then by adjoining $T\varphi$ at the end of branch V we obtain (again) a finite tableau from T.
- (iv) every finite tableau from T is formed by finitely many steps (i), (ii), (iii).

A *tableau* from T is a sequence $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ of finite tableaux from T such that τ_{n+1} is formed from τ_n by (ii) or (iii), formally $\tau = \cup\tau_n$.

Construction of tableaux

$$
\begin{array}{c}
F((\exists x)\neg P(x) \to \neg(\forall x)P(x)) \\
| \\
T(\exists x)\neg P(x) \\
| \\
F(\neg(\forall x)P(x)) \\
| \\
\hline
F(\neg(\forall x)P(x)) \\
| \\
T(\forall x)P(x) \\
\hline
T(\exists x)\neg P(x) \\
| \\
T(\neg P(c)) \quad c \ \text{new} \\
\hline
T(\neg P(c)) \\
| \\
FP(c) \\
\hline
T(\forall x)P(x) \\
| \qquad \text{choose} \\
TP(c) \quad t = c \\
| \\
\otimes
\end{array}
\qquad\qquad
\begin{array}{c}
F(\neg(\forall x)P(x) \to (\exists x)\neg P(x)) \\
| \\
T(\neg(\forall x)P(x)) \\
| \\
F(\exists x)\neg P(x) \\
\hline
T(\neg(\forall x)P(x)) \\
| \\
F(\forall x)P(x) \\
\hline
F(\forall x)P(x) \\
| \\
FP(d) \quad d \ \text{new} \\
\hline
F(\exists x)\neg P(x) \\
| \qquad \text{choose} \\
F(\neg P(d)) \quad t = d \\
\hline
F(\neg P(d)) \\
| \\
TP(d) \\
| \\
\otimes
\end{array}
$$

Convention

$$\begin{array}{c}
F((\exists x)\neg P(x) \to \neg(\forall x)P(x)) \\
| \\
T(\exists x)\neg P(x) \\
| \\
F(\neg(\forall x)P(x)) \\
| \\
T(\forall x)P(x) \\
| \\
T(\neg P(c)) \ c \ \text{ new} \\
| \\
FP(c) \\
| \\
\boxed{\begin{array}{c} T(\forall x)P(x) \\ | \\ TP(c) \end{array}} \\
| \\
\otimes
\end{array}
\qquad\qquad
\begin{array}{c}
F(\neg(\forall x)P(x) \to (\exists x)\neg P(x)) \\
| \\
T(\neg(\forall x)P(x)) \\
| \\
F(\exists x)\neg P(x) \\
| \\
F(\forall x)P(x) \\
| \\
FP(d) \quad d \ \text{ new} \\
| \\
\boxed{\begin{array}{c} F(\exists x)\neg P(x) \\ | \\ F(\neg P(d)) \end{array}} \\
| \\
TP(d) \\
| \\
\otimes
\end{array}$$

We will not write the entry that is expanded again on the branch, except in cases when the entry is in the form of $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$.

Tableau proof

- A branch V in a tableau τ is *contradictory* if it contains entries $T\varphi$ and $F\varphi$ for some sentence φ, otherwise V is *noncontradictory*.
- A tableau τ is *contradictory* if every branch in τ is contradictory.
- A *tableau proof* (*proof by tableau*) of a sentence φ from a theory T is a contradictory tableau from T with $F\varphi$ in the root.
- A sentence φ is *(tableau) provable* from T, denoted by $T \vdash \varphi$, if it has a tableau proof from T.
- A *refutation* of a sentence φ by *tableau* from a theory T is a contradictory tableau from T with the root entry $T\varphi$.
- A sentence φ is *(tableau) refutable* from T if it has a refutation by tableau from T, i.e. $T \vdash \neg\varphi$.

Examples

- $F((\forall x)(P(x) \to Q(x)) \to ((\forall x)P(x) \to (\forall x)Q(x)))$
  - $T(\forall x)(P(x) \to Q(x))$
  - $F((\forall x)P(x) \to (\forall x)Q(x))$
  - $T(\forall x)P(x)$
  - $F(\forall x)Q(x)$
  - $FQ(c)$ $\quad c$ new
  - $T(\forall x)P(x)$
  - $TP(c)$
  - $T(\forall x)(P(x) \to Q(x))$
  - $T(P(c) \to Q(c))$
    - $FP(c)$
      - $\otimes$
    - $TQ(c)$
      - $\otimes$

- $F((\forall x)(\varphi(x) \wedge \psi(x)) \leftrightarrow ((\forall x)\varphi(x) \wedge (\forall x)\psi(x)))$
  - $T((\forall x)(\varphi(x) \wedge \psi(x)))$
    - $F((\forall x)\varphi(x) \wedge (\forall x)\psi(x))$
      - $F(\forall x)\varphi(x)$
        - $F\varphi(c)$ $\quad c$ new
        - $T((\forall x)(\varphi(x) \wedge \psi(x)))$
        - $T(\varphi(c) \wedge \psi(c))$
        - $T\varphi(c)$
        - $T\psi(c)$
        - $\otimes$
      - $F(\forall x)\psi(x)$
        - $F\psi(d)$ $\quad d$ new
        - $T((\forall x)(\varphi(x) \wedge \psi(x)))$
        - $T(\varphi(d) \wedge \psi(d))$
        - $T\varphi(d)$
        - $T\psi(d)$
        - $\otimes$
  - $F((\forall x)(\varphi(x) \wedge \psi(x)))$
    - $T((\forall x)\varphi(x) \wedge (\forall x)\psi(x))$
    - $T(\forall x)\varphi(x)$
    - $T(\forall x)\psi(x)$
    - $F(\varphi(e) \wedge \psi(e))$ $\quad e$ new
      - $F\varphi(e)$
        - $T(\forall x)\varphi(x)$
        - $T\varphi(e)$
        - $\otimes$
      - $F\psi(e)$
        - $T(\forall x)\psi(x)$
        - $T\psi(e)$
        - $\otimes$

Finished tableau

A finished noncontradictory branch should provide us with a counterexample.

An occurrence of an entry P in a node v of a tableau τ is *i-th* if v has exactly $i-1$ predecessors labeled by P; and is *reduced* on a branch V through v if

$a)$ P is neither in form of $T(\forall x)\varphi(x)$ nor $F(\exists x)\varphi(x)$ and P occurs on V as a root of an atomic tableau, i.e. it was already expanded on V, or

$b)$ P is in form of $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$, P has an $(i+1)$-th occurrence on V, and V contains an entry $T\varphi(x/t_i)$ resp. $F\varphi(x/t_i)$ where t_i is the i-th ground term (of the language L_C).

Let V be a branch in a tableau τ from a theory T. We say that

- V is *finished* if it is contradictory, or every occurrence of an entry on V is reduced on V and, moreover, V contains $T\varphi$ for every $\varphi \in T$,
- τ is *finished* if every branch in τ is finished.

Systematic tableau - construction

Let R be an entry and $T = \{\varphi_0, \varphi_1, \dots\}$ be a (possibly infinite) theory.

(1) We take the atomic tableau for R as τ_0. In case $(*)$ we choose any $c \in L_C \setminus L$, in case $(\sharp)$ we take t_1 for t. Till possible, proceed as follows.

(2) Let v be the leftmost node in the smallest level as possible in tableau τ_n containing an occurrence of an entry P that is not reduced on some noncontradictory branch through v. (If v does not exist, we take $\tau'_n = \tau_n$.)

(3a) If P is neither $T(\forall x)\varphi(x)$ nor $F(\exists x)\varphi(x)$, let τ'_n be the tableau obtained from τ_n by adjoining the atomic tableau for P to every noncontradictory branch through v. In case $(*)$ we choose c_i for the smallest possible i.

(3b) If P is $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$ and it has i-th occurrence in v, let τ'_n be the tableau obtained from τ_n by adjoining atomic tableau for P to every noncontradictory branch through v, where we take the term t_i for t.

(4) Let τ_{n+1} be the tableau obtained from τ'_n by adjoining $T\varphi_n$ to every noncontradictory branch that does not contain $T\varphi_n$ yet. (If φ_n does not exist, we take $\tau_{n+1} = \tau'_n$.)

The *systematic tableau* for R from T is the result $\tau = \cup\tau_n$ of this construction.

Systematic tableau - an example

$$T((\exists y)(\neg R(y,y) \vee P(y,y)) \wedge (\forall x)R(x,x))$$

$$T(\exists y)(\neg R(y,y) \vee P(y,y))$$

$$T(\forall x)R(x,x)$$

$$T(\neg R(c_0,c_0) \vee P(c_0,c_0)) \qquad c_0 \ \text{new}$$

$$T(\forall x)R(x,x)$$

$$TR(c_0,c_0) \qquad (\text{assuming that } t_1 = c_0)$$

$$\begin{array}{cc}
T(\neg R(c_0,c_0)) & TP(c_0,c_0) \\
T(\forall x)R(x,x) & T(\forall x)R(x,x) \\
TR(t_2,t_2) & TR(t_2,t_2) \\
FR(c_0,c_0) & T(\forall x)R(x,x) \\
\otimes & TR(t_3,t_3) \\
 & \vdots
\end{array}$$

Systematic tableau - being finished

Proposition *Every systematic tableau is finished.*

Proof Let $\tau = \cup\tau_n$ be a systematic tableau from $T = \{\varphi_0, \varphi_1, \dots\}$ with root R and let P be an entry in a node v of the tableau τ.

- There are only finitely many entries in τ in levels up to the level of v.
- If the occurrence of P in v was unreduced on some noncontradictory branch in τ, it would be found in some step (2) and reduced by $(3a)$, $(3b)$.
- By step (4) every $\varphi_n \in T$ will be (no later than) in τ_{n+1} on every noncontradictory branch.
- Hence the systematic tableau τ has all branches finished. $\square$

Proposition *If a systematic tableau τ is a proof (from a theory T), it is finite.*

Proof Suppose that τ is infinite. Then by König's lemma, τ contains an infinite branch. This branch is noncontradictory since in the construction only noncontradictory branches are prolonged. But this contradicts the assumption that τ is a contradictory tableau. $\square$

Equality

Axioms of equality for a language L with equality are

(i) $x = x$

(ii) $x_1 = y_1 \ \wedge \ \cdots \ \wedge \ x_n = y_n \ \ \rightarrow \ \ f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$
for each n-ary function symbol f of the language L.

(iii) $x_1 = y_1 \ \wedge \ \cdots \ \wedge \ x_n = y_n \ \ \rightarrow \ \ (R(x_1, \ldots, x_n) \rightarrow R(y_1, \ldots, y_n))$
for each n-ary relation symbol R of the language L including $=$.

A *tableau proof* from a theory T in a language L *with equality* is a tableau proof from T^* where T^* denotes the extension of T by adding axioms of equality for L *(resp. their universal closures).*

Remark *In context of logic programming the equality often has other meaning than in mathematics (identity). For example in Prolog,* $t_1 = t_2$ *means that* t_1 *and* t_2 *are unifiable.*

Congruence and quotient structure

Let $\sim$ be an equivalence on A, $f : A^n \to A$, and $R \subseteq A^n$ for $n \in \mathbb{N}$. Then $\sim$ is

- a *congruence for the function* f if for every $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$
$$x_1 \sim y_1 \ \wedge \ \cdots \ \wedge \ x_n \sim y_n \quad \Rightarrow \quad f(x_1, \ldots, x_n) \sim f(y_1, \ldots, y_n),$$
- a *congruence for the relation* R if for every $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$
$$x_1 \sim y_1 \ \wedge \ \cdots \ \wedge \ x_n \sim y_n \quad \Rightarrow \quad (R(x_1, \ldots, x_n) \Leftrightarrow R(y_1, \ldots, y_n)).$$

Let an equivalence $\sim$ on A be a congruence for every function and relation in a structure $\mathcal{A} = \langle A, \mathcal{F}^A, \mathcal{R}^A \rangle$ of language $L = \langle \mathcal{F}, \mathcal{R} \rangle$. Then the *quotient* (*structure*) of $\mathcal{A}$ by $\sim$ is the structure $\mathcal{A}/\!\sim\, = \langle A/\!\sim, \mathcal{F}^{A/\sim}, \mathcal{R}^{A/\sim} \rangle$ where
$$f^{A/\sim}([x_1]_\sim, \ldots, [x_n]_\sim) = [f^A(x_1, \ldots, x_n)]_\sim$$
$$R^{A/\sim}([x_1]_\sim, \ldots, [x_n]_\sim) \Leftrightarrow R^A(x_1, \ldots, x_n)$$
for each $f \in \mathcal{F}$, $R \in \mathcal{R}$, and $x_1, \ldots, x_n \in A$, i.e. the functions and relations are defined from $\mathcal{A}$ using representatives.

Example: $\underline{\mathbb{Z}}_p$ *is the quotient of* $\underline{\mathbb{Z}} = \langle \mathbb{Z}, +, -, 0 \rangle$ *by the congruence modulo* p.

Role of axioms of equality

Let $\mathcal{A}$ be a structure of a language L in which the equality is interpreted as a relation $=^A$ satisfying the axioms of equality for L, i.e. not necessarily the identity relation.

1) From axioms (i) and (iii) it follows that the relation $=^A$ is an equivalence.
2) Axioms (ii) and (iii) express that the relation $=^A$ is a congruence for every function and relation in $\mathcal{A}$.
3) If $\mathcal{A} \models T^*$ then also $(\mathcal{A}/{=^A}) \models T^*$ where $\mathcal{A}/{=^A}$ is the quotient of $\mathcal{A}$ by $=^A$. Moreover, the equality is interpreted in $\mathcal{A}/{=^A}$ as the identity relation.

On the other hand, in every model in which the equality is interpreted as the identity relation, all axioms of equality evidently hold.

Soundness

We say that a model $\mathcal{A}$ *agrees* with an entry P, if P is $T\varphi$ and $\mathcal{A} \models \varphi$ or if P is $F\varphi$ and $\mathcal{A} \models \neg\varphi$, i.e. $\mathcal{A} \not\models \varphi$. Moreover, $\mathcal{A}$ *agrees* with a branch V if $\mathcal{A}$ agrees with every entry on V.

Lemma *Let $\mathcal{A}$ be a model of a theory T of a language L that agrees with the root entry R in a tableau $\tau = \cup\tau_n$ from T. Then $\mathcal{A}$ can be expanded to the language L_C so that it agrees with some branch V in τ.*

Remark *It suffices to expand $\mathcal{A}$ only by constants c^A such that $c \in L_C \setminus L$ occurs on V, other constants may be defined arbitrarily.*

Proof By induction on n we find a branch V_n in τ_n and an expansion $\mathcal{A}_n$ of $\mathcal{A}$ by constants c^A for all $c \in L_C \setminus L$ on V_n s.t. $\mathcal{A}_n$ agrees with V_n and $V_{n-1} \subseteq V_n$.

Assume we have a branch V_n in τ_n and an expansion $\mathcal{A}_n$ that agrees with V_n.

- If τ_{n+1} is formed from τ_n without extending the branch V_n, we take $V_{n+1} = V_n$ and $\mathcal{A}_{n+1} = \mathcal{A}_n$.
- If τ_{n+1} is formed from τ_n by appending $T\varphi$ to V_n for some $\varphi \in T$, let V_{n+1} be this branch and $\mathcal{A}_{n+1} = \mathcal{A}_n$. Since $\mathcal{A} \models \varphi$, $\mathcal{A}_{n+1}$ agrees with V_{n+1}.

Soundness - proof (cont.)

- Otherwise τ_{n+1} is formed from τ_n by appending an atomic tableau to V_n for some entry P on V_n. By induction we know that $\mathcal{A}_n$ agrees with P.

(i) If P is formed by a logical connective, we take $\mathcal{A}_{n+1} = \mathcal{A}_n$ and verify that V_n can always be extended to a branch V_{n+1} agreeing with $\mathcal{A}_{n+1}$.

(ii) If P is in form $T(\forall x)\varphi(x)$, let V_{n+1} be the (unique) extension of V_n to a branch in τ_{n+1}, i.e. by the entry $T\varphi(x/t)$. Let $\mathcal{A}_{n+1}$ be any expansion of by new constants from t. Since $\mathcal{A}_n \models (\forall x)\varphi(x)$, we have $\mathcal{A}_{n+1} \models \varphi(x/t)$. Analogously for P in form $F(\exists x)\varphi(x)$.

(iii) If P is in form $T(\exists x)\varphi(x)$, let V_{n+1} be the (unique) extension of V_n to a branch in τ_{n+1}, i.e. by the entry $T\varphi(x/c)$. Since $\mathcal{A}_n \models (\exists x)\varphi(x)$, there is some $a \in A$ with $\mathcal{A}_n \models \varphi(x)[e(x/a)]$ for every assignment e. Let $\mathcal{A}_{n+1}$ be the expansion of $\mathcal{A}_n$ by a new constant $c^A = a$. Then $\mathcal{A}_{n+1} \models \varphi(x/c)$. Analogously for P in form $F(\forall x)\varphi(x)$.

The base step for $n = 0$ follows from similar analysis of atomic tableaux for the root entry R applying the assumption that $\mathcal{A}$ agrees with R. $\square$

Theorem on soundness

We will show that the tableau method in predicate logic is sound.

Theorem *For every theory T and sentence φ, if φ is tableau provable from T, then φ is valid in T, i.e. $T \vdash \varphi \;\Rightarrow\; T \models \varphi$.*

Proof

- Let φ be tableau provable from a theory T, i.e. there is a contradictory tableau τ from T with the root entry $F\varphi$.
- Suppose for a contradiction that φ is not valid in T, i.e. there exists a model $\mathcal{A}$ of the theory T in which φ is not true (a counterexample).
- Since $\mathcal{A}$ agrees with the root entry $F\varphi$, by the previous lemma, $\mathcal{A}$ can be expanded to the language L_C so that it agrees with some branch in τ.
- But this is impossible, since every branch of τ is contradictory, i.e. it contains a pair of entries $T\psi$, $F\psi$ for some sentence ψ. $\square$