Propositional and Predicate Logic - IX

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The canonical model

From a noncontradictory branch V of a finished tableau we build a model that agrees with V. We build it on available (syntactical) objects - ground terms.

Let V be a noncontradictory branch of a finished tableau from a theory T of a language $L = \langle \mathcal{F}, \mathcal{R} \rangle$. The *canonical model* from V is the L_C-structure $\mathcal{A} = \langle A, \mathcal{F}^A, \mathcal{R}^A \rangle$ where

(1) A is the set of all ground terms of the language L_C,

(2) $f^A(t_{i_1}, \ldots, t_{i_n}) = f(t_{i_1}, \ldots, t_{i_n})$
for every n-ary function symbol $f \in \mathcal{F} \cup (L_C \setminus L)$ a $t_{i_1}, \ldots, t_{i_n} \in A$.

(3) $R^A(t_{i_1}, \ldots, t_{i_n}) \Leftrightarrow TR(t_{i_1}, \ldots, t_{i_n})$ is an entry on V
for every n-ary relation symbol $R \in \mathcal{R}$ or equality and $t_{i_1}, \ldots, t_{i_n} \in A$.

Remark *The expression $f(t_{i_1}, \ldots, t_{i_n})$ on the right side of (2) is a ground term of L_C, i.e. an element of A. Informally, to stress that it is a syntactical object*

$$f^A(t_{i_1}, \ldots, t_{i_n}) = \text{“}f(t_{i_1}, \ldots, t_{i_n})\text{”}$$

The canonical model - an example

Let $T = \{(\forall x)R(f(x))\}$ be a theory of a language $L = \langle R, f, d\rangle$. The systematic tableau for $F\neg R(d)$ from T contains a single branch V, which is noncontradictory.

The canonical model $\mathcal{A} = \langle A, R^A, f^A, d^A, c_i^A\rangle_{i\in\mathbb{N}}$ from V is for language L_C and

$$\begin{aligned}
A &= \{d, f(d), f(f(d)), \ldots, c_0, f(c_0), f(f(c_0)), \ldots, c_1, f(c_1), f(f(c_1)), \ldots\},\\
d^A &= d, \quad c_i^A = c_i \text{ for } i \in \mathbb{N},\\
f^A(d) &= \text{“}f(d)\text{”},\ f^A(f(d)) = \text{“}f(f(d))\text{”},\ f^A(f(f(d))) = \text{“}f(f(f(d)))\text{”},\ \ldots\\
R^A &= \{d, f(d), f(f(d)), \ldots, f(c_0), f(f(c_0)), \ldots, f(c_1), f(f(c_1)), \ldots\}.
\end{aligned}$$

The reduct of $\mathcal{A}$ to the language L is $\mathcal{A}' = \langle A, R^A, f^A, d^A\rangle$.

The canonical model with equality

If L is with equality, T^* is the extension of T by the axioms of equality for L.

If we require that the equality is interpreted as the identity, we have to take the quotient of the canonical model $\mathcal{A}$ by the congruence $=^A$.

By (3), for the relation $=^A$ in $\mathcal{A}$ from V it holds that for every $t_{i_1}, t_{i_2} \in A$,

$$t_{i_1} =^A t_{i_2} \quad \Leftrightarrow \quad T(t_{i_1} = t_{i_2}) \text{ is an entry on } V.$$

Since V is finished and contains the axioms of equality, the relation $=^A$ is a congruence for all functions and relations in $\mathcal{A}$.

The *canonical model with equality* from V is the quotient $\mathcal{A}/{=^A}$.

Observation *For every formula φ,*

$$\mathcal{A} \models \varphi \quad \Leftrightarrow \quad (\mathcal{A}/{=^A}) \models \varphi,$$

where $=$ is interpreted in $\mathcal{A}$ by the relation $=^A$, while in $\mathcal{A}/{=^A}$ by the identity.

Remark *$\mathcal{A}$ is a countably infinite model, but $\mathcal{A}/{=^A}$ can be finite.*

The canonical model with equality - an example

Let $T = \{(\forall x)R(f(x)),\ (\forall x)(x = f(f(x)))\}$ be of $L = \langle R, f, d\rangle$ with equality. The systematic tableau for $F\neg R(d)$ from T^* contains a noncontradictory V.

In the canonical model $\mathcal{A} = \langle A, R^A, =^A, f^A, d^A, c_i^A\rangle_{i\in\mathbb{N}}$ from V we have that

$$s =^A t \quad \Leftrightarrow \quad t = f(\cdots(f(s)\cdots) \text{ or } s = f(\cdots(f(t)\cdots),$$

where f is applied $2i$-times for some $i \in \mathbb{N}$.

The canonical model with equality from V is
$\mathcal{B} = (\mathcal{A}/=^A) = \langle A/=^A, R^B, f^B, d^B, c_i^B\rangle_{i\in\mathbb{N}}$ where

$$(A/=^A) = \{[d]_{=^A}, [f(d)]_{=^A}, [c_0]_{=^A}, [f(c_0)]_{=^A}, [c_1]_{=^A}, [f(c_1)]_{=^A}, \ldots\},$$
$$d^B = [d]_{=^A}, \quad c_i^B = [c_i]_{=^A} \text{ for } i \in \mathbb{N},$$
$$f^B([d]_{=^A}) = [f(d)]_{=^A},\ f^B([f(d)]_{=^A}) = [f(f(d))]_{=^A} = [d]_{=^A},\ \ldots$$
$$R^B = (A/=^A).$$

The reduct of $\mathcal{B}$ to the language L is $\mathcal{B}' = \langle A/=^A, R^B, f^B, d^B\rangle$.

Completeness

Lemma *The canonical model $\mathcal{A}$ from a noncontr. finished V agrees with V.*

Proof By induction on the structure of a sentence in an entry on V.

- For atomic φ, if $T\varphi$ is on V, then $\mathcal{A} \models \varphi$ by (3). If $F\varphi$ is on V, then $T\varphi$ is not on V since V is noncontradictory, so $\mathcal{A} \models \neg\varphi$ by (3).
- If $T(\varphi \wedge \psi)$ is on V, then $T\varphi$ and $T\psi$ are on V since V is finished. By induction, $\mathcal{A} \models \varphi$ and $\mathcal{A} \models \psi$, and thus $\mathcal{A} \models \varphi \wedge \psi$.
- If $F(\varphi \wedge \psi)$ is on V, then $F\varphi$ or $F\psi$ is on V since V is finished. By induction, $\mathcal{A} \models \neg\varphi$ or $\mathcal{A} \models \neg\psi$, and thus $\mathcal{A} \models \neg(\varphi \wedge \psi)$.
- For other connectives similarly as in previous two cases.
- If $T(\forall x)\varphi(x)$ is on V, then $T\varphi(x/t)$ is on V for every $t \in A$ since V is finished. By induction, $\mathcal{A} \models \varphi(x/t)$ for every $t \in A$, and thus $\mathcal{A} \models (\forall x)\varphi(x)$. Similarly for $F(\exists x)\varphi(x)$ on V.
- If $T(\exists x)\varphi(x)$ is on V, then $T\varphi(x/c)$ is on V for some $c \in A$ since V is finished. By induction, $\mathcal{A} \models \varphi(x/c)$, and thus $\mathcal{A} \models (\exists x)\varphi(x)$. Similarly for $F(\forall x)\varphi(x)$ on V. $\square$

Theorem on completeness

We will show that the tableau method in predicate logic is complete.

Theorem *For every theory T and sentence φ, if φ is valid in T, then φ is tableau provable from T, i.e. $T \models \varphi \;\Rightarrow\; T \vdash \varphi$.*

Proof Let φ be valid in T. We will show that an arbitrary finished tableau (e.g. systematic) τ from a theory T with the root entry $F\varphi$ is contradictory.

- If not, then there is some noncontradictory branch V in τ.
- By the previous lemma, there is a structure $\mathcal{A}$ for L_C that agrees with V, in particular with the root entry $F\varphi$, i.e. $\mathcal{A} \models \neg\varphi$.
- Let $\mathcal{A}'$ be the reduct of $\mathcal{A}$ to the language L. Then $\mathcal{A}' \models \neg\varphi$.
- Since V is finished, it contains $T\psi$ for every $\psi \in T$.
- Thus $\mathcal{A}'$ is a model of T (as $\mathcal{A}'$ agrees with $T\psi$ for every $\psi \in T$).
- But this contradicts the assumption that φ is valid in T.

Therefore the tableau τ is a proof of φ from T. $\square$

Properties of theories

We introduce syntactic variants of previous semantical definitions.

Let T be a theory of a language L. If a sentence φ is provable from T, we say that φ is a *theorem* of T. The set of theorems of T is denoted by

$$\mathrm{Thm}^L(T) = \{\varphi \in \mathrm{Fm}_L \mid T \vdash \varphi\}.$$

We say that a theory T is

- *inconsistent* if $T \vdash \bot$, otherwise T is *consistent*,
- *complete* if it is consistent and every sentence is provable or refutable from T, i.e. $T \vdash \varphi$ or $T \vdash \neg\varphi$.
- an *extension* of a theory T' of L' if $L' \subseteq L$ and $\mathrm{Thm}^{L'}(T') \subseteq \mathrm{Thm}^L(T)$, we say that an extension T of a theory T' is *simple* if $L = L'$; and *conservative* if $\mathrm{Thm}^{L'}(T') = \mathrm{Thm}^L(T) \cap \mathrm{Fm}_{\mathrm{L}'}$,
- *equivalent* with a theory T' if T is an extension of T' and vice-versa.

Corollaries

From the soundness and completeness of the tableau method it follows that these syntactic definitions agree with their semantic variants.

Corollary *For every theory T and sentences φ, ψ of a language L,*

- $T \vdash \varphi$ *if and only if* $T \models \varphi$,
- $\mathrm{Thm}^L(T) = \theta^L(T)$,
- T *is inconsistent if and only if* T *is unsatisfiable, i.e. it has no model,*
- T *is complete if and only if* T *is semantically complete, i.e. it has a single model, up to elementarily equivalence,*
- $T, \varphi \vdash \psi$ *if and only if* $T \vdash \varphi \to \psi$ *(Deduction theorem).*

Remark Deduction theorem can be proved directly by transformations of tableaux.

Existence of a countable model and compactness

Theorem *Every consistent theory T of a countable language L without equality has a countably infinite model.*

Proof Let τ be the systematic tableau from T with $F\bot$ in the root. Since τ is finished and contains a noncontradictory branch V as $\bot$ is not provable from T, there exists a canonical model $\mathcal{A}$ from V. Since $\mathcal{A}$ agrees with V, its reduct to the language L is a desired countably infinite model of T. $\square$

Remark *This is a weak version of so called Löwenheim-Skolem theorem. In a countable language with equality the canonical model with equality is countable (i.e. finite or countably infinite).*

Theorem *A theory T has a model iff every finite subset of T has a model.*

Proof The implication from left to right is obvious. If T has no model, then it is inconsistent, i.e. $\bot$ is provable by a systematic tableau τ from T. Since τ is finite, $\bot$ is provable from some finite $T' \subseteq T$, i.e. T' has no model. $\square$

Non-standard model of natural numbers

Let $\underline{\mathbb{N}} = \langle \mathbb{N}, S, +, \cdot, 0, \leq \rangle$ be the standard model of natural numbers.

Let $\mathrm{Th}(\underline{\mathbb{N}})$ denote the set of all sentences that are valid in $\underline{\mathbb{N}}$. For $n \in \mathbb{N}$ let $\underline{n}$ denote the term $S(S(\cdots(S(0))\cdots))$, so called the *n-th numeral*, where S is applied n-times.

Consider the following theory T where c is a new constant symbol.

$$T = \mathrm{Th}(\underline{\mathbb{N}}) \cup \{\underline{n} < c \mid n \in \mathbb{N}\}$$

Observation *Every finite subset of T has a model.*

Thus by the compactness theorem, T has a model $\mathcal{A}$. It is a non-standard model of natural numbers. Every sentence from $\mathrm{Th}(\underline{\mathbb{N}})$ is valid in $\mathcal{A}$ but it contains an element c^A that is greater then every $n \in \mathbb{N}$ (i.e. the value of the term $\underline{n}$ in $\mathcal{A}$).

Extensions of theories

We show that introducing new definitions has only an "auxiliary character".

Proposition *Let T be a theory of L and T' be a theory of L' where $L \subseteq L'$.*

- (i) *T' is an extension of T if and only if the reduct $\mathcal{A}$ of every model $\mathcal{A}'$ of T' to the language L is a model of T,*
- (ii) *T' is a conservative extension of T if T' is an extension of T and every model $\mathcal{A}$ of T can be expanded to the language L' on a model $\mathcal{A}'$ of T'.*

Proof

- $(i)a)$ If T' is an extension of T and φ is any axiom of T, then $T' \models \varphi$. Thus $\mathcal{A}' \models \varphi$ and also $\mathcal{A} \models \varphi$, which implies that $\mathcal{A}$ is a model of T.
- $(i)b)$ If $\mathcal{A}$ is a model of T and $T \models \varphi$ where φ is of L, then $\mathcal{A} \models \varphi$ and also $\mathcal{A}' \models \varphi$. This implies that $T' \models \varphi$ and thus T' is an extension of T.
- (ii) If $T' \models \varphi$ where φ is of L and $\mathcal{A}$ is a model of T, then in its expansion $\mathcal{A}'$ that models T' we have $\mathcal{A}' \models \varphi$. Thus also $\mathcal{A} \models \varphi$, and hence $T \models \varphi$. Therefore T' is conservative. $\square$

Extensions by definition of a relation symbol

Let T be a theory of L, $\psi(x_1, \dots, x_n)$ be a formula of L in free variables $x_1, \dots, x_n$ and L' denote the language L with a new n-ary relation symbol R.

The *extension* of T *by definition of R* with the formula ψ is the theory T' of L' obtained from T by adding the axiom

$$R(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n)$$

Observation *Every model of T can be uniquely expanded to a model of T'.*

Corollary *T' is a conservative extension of T.*

Proposition *For every formula φ' of L' there is φ of L s.t. $T' \models \varphi' \leftrightarrow \varphi$.*

Proof Replace each subformula $R(t_1, \dots, t_n)$ in φ with $\psi'(x_1/t_1, \dots, x_n/t_n)$, where ψ' is a suitable variant of ψ allowing all substitutions. $\square$

For example, the symbol $\leq$ can be defined in arithmetics by the axiom

$$x \leq y \leftrightarrow (\exists z)(x + z = y)$$

Extensions by definition of a function symbol

Let T be a theory of a language L and $\psi(x_1, \ldots, x_n, y)$ be a formula of L in free variables $x_1, \ldots, x_n, y$ such that

$$T \models (\exists y)\psi(x_1, \ldots, x_n, y) \qquad \text{(existence)}$$

$$T \models \psi(x_1, \ldots, x_n, y) \wedge \psi(x_1, \ldots, x_n, z) \rightarrow y = z \qquad \text{(uniqueness)}$$

Let L' denote the language L with a new n-ary function symbol f.

The *extension* of T *by definition of* f with the formula ψ is the theory T' of L' obtained from T by adding the axiom

$$f(x_1, \ldots, x_n) = y \leftrightarrow \psi(x_1, \ldots, x_n, y)$$

Remark *In particular, if* ψ *is* $t(x_1, \ldots, x_n) = y$ *where* t *is a term and* $x_1, \ldots, x_n$ *are the variables in* t*, both the conditions of existence and uniqueness hold.*

For example binary $-$ *can be defined using* $+$ *and unary* $-$ *by the axiom*

$$x - y = z \leftrightarrow x + (-y) = z$$

Extensions by definition of a function symbol (cont.)

Observation *Every model of T can be uniquely expanded to a model of T'.*

Corollary *T' is a conservative extension of T.*

Proposition *For every formula φ' of L' there is φ of L s.t. $T' \models \varphi' \leftrightarrow \varphi$.*

Proof It suffices to consider φ' with a single occurrence of f. If φ' has more, we may proceed inductively. Let φ^* denote the formula obtained from φ' by replacing the term $f(t_1, \ldots, t_n)$ with a new variable z. Let φ be the formula

$$(\exists z)(\varphi^* \ \wedge\ \psi'(x_1/t_1, \ldots, x_n/t_n, y/z)),$$

where ψ' is a suitable variant of ψ allowing all substitutions.

Let $\mathcal{A}$ be a model of T', e be an assignment, and $a = f^A(t_1, \ldots, t_n)[e]$. By the two conditions, $\mathcal{A} \models \psi'(x_1/t_1, \ldots, x_n/t_n, y/z)[e]$ if and only if $e(z) = a$. Thus

$$\mathcal{A} \models \varphi[e] \quad \Leftrightarrow \quad \mathcal{A} \models \varphi^*[e(z/a)] \quad \Leftrightarrow \quad \mathcal{A} \models \varphi'[e]$$

for every assignment e, i.e. $\mathcal{A} \models \varphi' \leftrightarrow \varphi$ and so $T' \models \varphi' \leftrightarrow \varphi$. $\square$

Extensions by definitions

A theory T' of L' is called an *extension* of a theory T of L *by definitions* if it is obtained from T by successive definitions of relation and function symbols.

Corollary *Let T' be an extension of a theory T by definitions. Then*

- *every model of T can be uniquely expanded to a model of T',*
- *T' is a conservative extension of T,*
- *for every formula φ' of L' there is a formula φ of L such that $T' \models \varphi' \leftrightarrow \varphi$.*

For example, in $T = \{(\exists y)(x + y = 0), (x + y = 0) \wedge (x + z = 0) \rightarrow y = z\}$ of $L = \langle +, 0, \leq \rangle$ with equality we can define $<$ and unary $-$ by the axioms

$$\begin{aligned} -x = y \quad &\leftrightarrow \quad x + y = 0 \\ x < y \quad &\leftrightarrow \quad x \leq y \ \wedge \ \neg(x = y) \end{aligned}$$

Then the formula $-x < y$ is equivalent in this extension to a formula

$$(\exists z)((z \leq y \ \wedge \ \neg(z = y)) \ \wedge \ x + z = 0).$$