Exam test

Jan 27th, 2016

1. Three suspects of a crime mutually evaluate their allegations:

   (i) *The first one says that the second one lies.*

   (ii) *The second one says that the third one lies.*

   (iii) *The third one says that both the first one and second one lie.*

   Assume that each of them either always tells the truth or always lies. Let propositional letters p_1, p_2, p_3 represent (respectively) that *the first, the second, the third one tells the truth* and let $\mathbb{P} = \{p_1, p_2, p_3\}$.

   (a) Write propositions (in the form of equivalences) φ_1, φ_2, φ_3 over $\mathbb{P}$ representing our knowledge from each statement (i), (ii), (iii). *(2b)*

   (b) Find all models of a theory $T = \{\varphi_1, \varphi_2, \varphi_3\}$ over $\mathbb{P}$ and write an elementary conjunction ψ that is equivalent to T. *Hint: T is complete. (2b)*

   (c) Transform $T \cup \{\neg\psi\}$ into a set representation (i.e. clausal form). *(2b)*

   (d) Prove by resolution that $T \models \psi$. Present the resolution refutation by a resolution tree. *(4b)*

2. Let $T = \{(\forall x)(P(x) \to R(x,x)), (\exists x)(\forall y)(\neg P(y) \to R(x,y)), (\forall x)(\exists y)(\neg R(x,y) \wedge \neg R(y,y))\}$ be a theory in language $L = \langle P, R\rangle$ without equality where P, R is unary resp. binary relation symbol.

   (a) Applying skolemization find a theory T' (in a some extended language) such that T' is equisatisfiable with T and all axioms of T' are universal sentences. *(2b)*

   (b) Prove by tableau method that T' is unsatisfiable. *(4b)*

   (c) Let T'' denote the set of open matrices of axioms of T', so T'' is an open theory equivalent to T'. Find a conjunction of ground instances of axioms of T'' that is unsatisfiable. *Hint: use the tableau from (b). (2b)*

   (d) Is T complete? Give an explanation. *(2b)*

3. Let $T = \{x \leq x, x \leq y \wedge y \leq x \to x = y, x \leq y \wedge y \leq z \to x \leq z, x \leq y \vee y \leq x\}$ be a theory *(of linear orderings)* in language $L = \langle \leq \rangle$ with equality. Let $\varphi(x,y)$ denote the formula $x < y \wedge \neg(\exists z)(x < z \wedge z < y)$ where $x < y$ is a shortcut for "$x \leq y \wedge \neg(x = y)$".

   (a) Is $(\forall x)(\exists y)\varphi$ provable / refutable / independent in T? Give an explanation. *(2b)*

   (b) Let $T' = T \cup \{(\forall x)(\exists y)\varphi\}$. Do the conditions of existence and uniqueness hold in T' for the definition of $f(x) = y$ by formula $\varphi(x,y)$? Give an explanation. *(2b)*

   (c) Is the theory $T' \cup \{f(x) = y \leftrightarrow \varphi(x,y)\}$ a conservative extension of T'? Give an explanation. *(2b)*

   (d) Is the theory T' ω-categorical? Give an explanation. *(2b)*