Exam test

February 9, 2016

1. Consider the following two propositions over a set of propositional letters $\mathbb{P} = \{p, q, r, s, t\}$:

$$\begin{aligned} \varphi: &\quad (p \vee r) \wedge (\neg p \vee q) \wedge (\neg q \vee \neg r), \\ \psi: &\quad (\neg s \vee \neg r) \wedge (\neg t \vee s) \wedge \neg p. \end{aligned}$$

   (a) Decide whether $\varphi \wedge \psi$ is satisfiable by using an implication graph. If yes, find some assignment satisfying $\varphi \wedge \psi$. *(2b)*

   (b) Find by tableau method the set $M^{\mathbb{P}'}(\varphi)$ of all models of φ over $\mathbb{P}' = \{p, q, r\}$. *(2b)*

   (c) How many nonequivalent simple extensions the theory $\{\varphi\}$ over $\mathbb{P}'$ has? How many of them are complete? *(2b)*

   (d) Is the theory $\{\varphi, \psi\}$ over $\mathbb{P}$ a conservative extension of a theory $\{\varphi\}$ over $\mathbb{P}'$? Give an explanation. *(2b)*

2. In what follows by a *son/daughter* of a person x we mean a man/woman that has x as a parent, and by an *aunt from father's side* of a person x we mean a daughter of the father of the father of x or a daughter of the mother of the father of x. Consider the following statements:

   (i) *Both the father and the mother of each person are his(her) parents, the father is a man and the mother is a woman.*

   (ii) *If a woman has a son, then she also has a daughter.*

   Prove by resolution that it follows:

   (iii) *Everyone has an aunt from father's side.*

   More specifically:

   (a) Express the above statements as <u>sentences</u> φ_1, φ_2, φ_3 of language $L = \langle P, M, W, f, m\rangle$ without equality, where P is binary relation symbol and $P(x, y)$ means that "x is a parent of y", M, W are unary relation symbols and $M(x)$, $W(x)$ means that "x is a man",resp. "x is a woman", and f, m are unary function symbols and $f(x)$, $m(x)$ represent "the father of x", resp. "the mother of x". *(2b)*

   (b) Applying skolemisation of sentences from (a) or their negations find an open theory T (possibly in an extended language) that is unsatisfiable if and only if $\{\varphi_1, \varphi_2\} \models \varphi_3$. *(2b)*

   (c) By transformation of axioms of T into CNF find a theory T' equivalent to T such that all axioms of T' are clauses. Write T' in a clausal form. *(2b)*

   (d) Prove by resolution that T' is unsatisfiable. Present the resolution refutation by a resolution tree. Write down the unification used in each step. *(4b)*

   (e) Find a conjunction of ground instances of axioms of T' that is unsatisfiable. *Hint: use unifications from (d). (2b)*

3. Let T be a theory of language $L = \langle f, g, a\rangle$ with equality where f, g, a are (respectively) binary, unary and nullary function symbols, with the following axioms

$$\begin{aligned} f(x, f(y, z)) &= f(f(x, y), z), \\ f(a, x) = x &\quad \wedge \quad f(x, a) = x, \\ f(x, g(x)) = a &\quad \wedge \quad f(g(x), x) = a. \end{aligned}$$

(a) Is $(\forall x)(\forall y) f(x, y) = f(y, x)$ valid / contradictory / independent in T? Give an explanation. *(2b)*

(b) Consider the structure $\underline{\mathbb{Z}_4} = \langle \{0, 1, 2, 3\}, +, -, 0\rangle$ of language L where $+, -$ are standard addition and (unary) minus modulo 4. Is the theory $\mathrm{Th}(\underline{\mathbb{Z}_4})$ a simple extension of the theory T? Give an explanation. *(2b)*

(c) Find all substructures of $\underline{\mathbb{Z}_4}$. Are they models of T? Give an explanation. *(2b)*

(d) Let $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, -, \cdot, 0, 1\rangle$ be the structure of rational numbers with standard operations. Is there a reduct of $\underline{\mathbb{Q}}$ that is a model of T? Give an explanation. *(2b)*