Propositional and Predicate Logic - VIII

Petr Gregor

KTIML MFF UK

WS 2022/23

Theorem on constants

Theorem *Let φ be a formula in a language L with free variables $x_1, \dots, x_n$ and let T be a theory in L. Let L' be the extension of L with new constant symbols $c_1, \dots, c_n$ and let T' denote the theory T in L'. Then*

$$T \models \varphi \quad \textit{if and only if} \quad T' \models \varphi(x_1/c_1, \dots, x_n/c_n).$$

Proof (⇒) If $\mathcal{A}'$ is a model of T', let $\mathcal{A}$ be the reduct of $\mathcal{A}'$ to L. Since $\mathcal{A} \models \varphi[e]$ for every assignment e, we have in particular

$$\mathcal{A} \models \varphi[e(x_1/c_1^{A'}, \dots, x_n/c_n^{A'})], \quad \text{i.e.} \ \mathcal{A}' \models \varphi(x_1/c_1, \dots, x_n/c_n).$$

(⇐) If $\mathcal{A}$ is a model of T and e an assignment, let $\mathcal{A}'$ be the expansion of A into L' by setting $c_i^{A'} = e(x_i)$ for every i. Since $\mathcal{A}' \models \varphi(x_1/c_1, \dots, x_n/c_n)[e']$ for every assignment e', we have

$$\mathcal{A}' \models \varphi[e(x_1/c_1^{A'}, \dots, x_n/c_n^{A'})], \quad \text{i.e.} \ \mathcal{A} \models \varphi[e]. \quad \square$$

Extensions of theories

We show that introducing new definitions has only an "auxiliary character".

Proposition *Let T be a theory of L and T' be a theory of L' where $L \subseteq L'$.*

(i) *T' is an extension of T if and only if the reduct $\mathcal{A}$ of every model $\mathcal{A}'$ of T' to the language L is a model of T,*

(ii) *T' is a conservative extension of T if T' is an extension of T and every model $\mathcal{A}$ of T can be expanded to the language L' on a model $\mathcal{A}'$ of T'.*

Proof

$(i)a)$ If T' is an extension of T and φ is any axiom of T, then $T' \models \varphi$. Thus $\mathcal{A}' \models \varphi$ and also $\mathcal{A} \models \varphi$, which implies that $\mathcal{A}$ is a model of T.

$(i)b)$ If $\mathcal{A}$ is a model of T and $T \models \varphi$ where φ is of L, then $\mathcal{A} \models \varphi$ and also $\mathcal{A}' \models \varphi$. This implies that $T' \models \varphi$ and thus T' is an extension of T.

(ii) If $T' \models \varphi$ where φ is of L and $\mathcal{A}$ is a model of T, then in its expansion $\mathcal{A}'$ that models T' we have $\mathcal{A}' \models \varphi$. Thus also $\mathcal{A} \models \varphi$, and hence $T \models \varphi$. Therefore T' is conservative. ☐

Extensions by definition of a relation symbol

Let T be a theory of L, $\psi(x_1, \ldots, x_n)$ be a formula of L in free variables $x_1, \ldots, x_n$ and L' denote the language L with a new n-ary relation symbol R.

The *extension* of T *by definition of R* with the formula ψ is the theory T' of L' obtained from T by adding the axiom

$$R(x_1, \ldots, x_n) \;\leftrightarrow\; \psi(x_1, \ldots, x_n)$$

Observation *Every model of T can be uniquely expanded to a model of T'.*

Corollary *T' is a conservative extension of T.*

Proposition *For every formula φ' of L' there is φ of L s.t. $T' \models \varphi' \leftrightarrow \varphi$.*

Proof Replace each subformula $R(t_1, \ldots, t_n)$ in φ with $\psi'(x_1/t_1, \ldots, x_n/t_n)$, where ψ' is a suitable variant of ψ allowing all substitutions. $\square$

For example, the symbol $\leq$ can be defined in arithmetics by the axiom

$$x \leq y \;\leftrightarrow\; (\exists z)(x + z = y)$$

Extensions by definition of a function symbol

Let T be a theory of a language L and $\psi(x_1, \ldots, x_n, y)$ be a formula of L in free variables $x_1, \ldots, x_n, y$ such that

$$T \models (\exists y)\psi(x_1, \ldots, x_n, y) \qquad \text{(existence)}$$

$$T \models \psi(x_1, \ldots, x_n, y) \ \wedge \ \psi(x_1, \ldots, x_n, z) \ \rightarrow \ y = z \qquad \text{(uniqueness)}$$

Let L' denote the language L with a new n-ary function symbol f.

The *extension* of T *by definition of f* with the formula ψ is the theory T' of L' obtained from T by adding the axiom

$$f(x_1, \ldots, x_n) = y \ \leftrightarrow \ \psi(x_1, \ldots, x_n, y)$$

Remark *In particular, if* ψ *is* $t(x_1, \ldots, x_n) = y$ *where* t *is a term and* $x_1, \ldots, x_n$ *are the variables in* t*, both the conditions of existence and uniqueness hold.*

For example binary $-$ *can be defined using* $+$ *and unary* $-$ *by the axiom*

$$x - y = z \ \leftrightarrow \ x + (-y) = z$$

Extensions by definition of a function symbol (cont.)

Observation *Every model of T can be uniquely expanded to a model of T'.*

Corollary *T' is a conservative extension of T.*

Proposition *For every formula φ' of L' there is φ of L s.t. $T' \models \varphi' \leftrightarrow \varphi$.*

Proof It suffices to consider φ' with a single occurrence of f. If φ' has more, we may proceed inductively. Let φ^* denote the formula obtained from φ' by replacing the term $f(t_1, \ldots, t_n)$ with a new variable z. Let φ be the formula

$$(\exists z)(\varphi^* \;\wedge\; \psi'(x_1/t_1, \ldots, x_n/t_n, y/z)),$$

where ψ' is a suitable variant of ψ allowing all substitutions.

Let $\mathcal{A}$ be a model of T', e be an assignment, and $a = f^A(t_1, \ldots, t_n)[e]$. By the two conditions, $\mathcal{A} \models \psi'(x_1/t_1, \ldots, x_n/t_n, y/z)[e]$ if and only if $e(z) = a$. Thus

$$\mathcal{A} \models \varphi[e] \quad \Leftrightarrow \quad \mathcal{A} \models \varphi^*[e(z/a)] \quad \Leftrightarrow \quad \mathcal{A} \models \varphi'[e]$$

for every assignment e, i.e. $\mathcal{A} \models \varphi' \leftrightarrow \varphi$ and so $T' \models \varphi' \leftrightarrow \varphi$. □

Extensions by definitions

A theory T' of L' is called an *extension* of a theory T of L *by definitions* if it is obtained from T by successive definitions of relation and function symbols.

Corollary *Let T' be an extension of a theory T by definitions. Then*

- *every model of T can be uniquely expanded to a model of T',*
- *T' is a conservative extension of T,*
- *for every formula φ' of L' there is a formula φ of L such that $T' \models \varphi' \leftrightarrow \varphi$.*

For example, in $T = \{(\exists y)(x + y = 0), (x + y = 0) \wedge (x + z = 0) \rightarrow y = z\}$ of $L = \langle +, 0, \leq \rangle$ with equality we can define $<$ and unary $-$ by the axioms

$$\begin{aligned} -x = y \quad &\leftrightarrow \quad x + y = 0 \\ x < y \quad &\leftrightarrow \quad x \leq y \ \wedge \ \neg(x = y) \end{aligned}$$

Then the formula $-x < y$ is equivalent in this extension to a formula

$$(\exists z)((z \leq y \ \wedge \ \neg(z = y)) \ \wedge \ x + z = 0).$$

Definable sets

We interested in which sets can be defined within a given structure.

- *A set defined by a formula* $\varphi(x_1, \ldots, x_n)$ *in structure* $\mathcal{A}$ is the set
$$\varphi^{\mathcal{A}}(x_1, \ldots, x_n) = \{(a_1, \ldots, a_n) \in A^n \mid \mathcal{A} \models \varphi[e(x_1/a_1, \ldots, x_n/a_n)]\}.$$
  Shortly, $\varphi^{\mathcal{A}}(\overline{x}) = \{\overline{a} \in A^{|\overline{x}|} \mid \mathcal{A} \models \varphi[e(\overline{x}/\overline{a})]\}$, where $|\overline{x}| = n$.
- *A set defined by a formula* $\varphi(\overline{x}, \overline{y})$ *with parameters* $\overline{b} \in A^{|\overline{y}|}$ in $\mathcal{A}$ is
$$\varphi^{\mathcal{A}, \overline{b}}(\overline{x}, \overline{y}) = \{\overline{a} \in A^{|\overline{x}|} \mid \mathcal{A} \models \varphi[e(\overline{x}/\overline{a}, \overline{y}/\overline{b})]\}.$$
  Example: $E(x, y)^{\mathcal{G}, b}$ *is the set of neighbors of a vertex* b *in a graph* $\mathcal{G}$.
- For a structure $\mathcal{A}$, a set $B \subseteq A$, and $n \in \mathbb{N}$ let $\mathrm{Df}^n(\mathcal{A}, B)$ denote the class of definable sets $D \subseteq A^n$ in the structure $\mathcal{A}$ with parameters from B.

Observation $\mathrm{Df}^n(\mathcal{A}, B)$ *is closed under complements, union, intersection and it contains* $\emptyset$, A^n. *Thus it forms a subalgebra of the set algebra* $\underline{\mathcal{P}}(A^n)$.

Example - database queries

Movie	*name*	*director*	*actor*
	Lidé z Maringotek	M. Frič	J. Tříska
	Po strništi bos	J. Svěrák	Z. Svěrák
	Po strništi bos	J. Svěrák	J. Tříska
	...	...	...

Program	*cinema*	*name*	*time*
	Světozor	Po strništi bos	13:15
	Mat	Po strništi bos	16:15
	Mat	Lidé z Maringotek	18:30
	...	...	...

Where and when can I see a movie with J. Tříska?

select *Program.cinema, Program.time* **from** *Movie, Program*
where *Movie.name* = *Program.name* **and** *actor* = 'J. Tříska';

Equivalently, it is the set $\varphi^{\mathcal{D}}(x, y)$ defined by the formula $\varphi(x, y)$

$$(\exists n)(\exists d)(P(x, n, y) \wedge M(n, d, \text{'J. Tříska'}))$$

in the structure $\mathcal{D} = \langle D, Movie, Program, c^D \rangle_{c \in D}$ of $L = \langle M, P, c \rangle_{c \in D}$, where $D = \{$'Po strništi bos', 'J. Tříska', 'Mat', '13:15', $\ldots\}$ and $c^D = c$ for any $c \in D$.

Boolean algebras

The theory of *Boolean algebras* has the language $L = \langle -, \wedge, \vee, 0, 1\rangle$ with equality and the following axioms.

$x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (asociativity of ∧)

$x \vee (y \vee z) = (x \vee y) \vee z$ (asociativity of ∨)

$x \wedge y = y \wedge x$ (commutativity of ∧)

$x \vee y = y \vee x$ (commutativity of ∨)

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (distributivity of ∧ over ∨)

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ (distributivity of ∨ over ∧)

$x \wedge (x \vee y) = x, \quad x \vee (x \wedge y) = x$ (absorption)

$x \vee (-x) = 1, \quad x \wedge (-x) = 0$ (complementation)

$0 \neq 1$ (non-triviality)

The smallest model is $\underline{2} = \langle \{0, 1\}, -_1, \wedge_1, \vee_1, 0, 1\rangle$. Finite Boolean algebras are (up to isomorphism) $\langle \{0, 1\}^n, -_n, \wedge_n, \vee_n, 0_n, 1_n\rangle$ for $n \in \mathbb{N}^+$, where the operations *(on binary n-tuples)* are the coordinate-wise operations of $\underline{2}$.

Relations of propositional and predicate logic

- Propositional formulas over connectives $\neg$, $\wedge$, $\vee$ (eventually with $\top$, $\bot$) can be viewed as Boolean terms. Then the truth value of φ in a given assignment is the value of the term in the Boolean algebra $\underline{2}$.
- Lindenbaum-Tarski algebra over $\mathbb{P}$ is Boolean algebra (also for $\mathbb{P}$ infinite).
- If we represent atomic subformulas in an open formula φ (without equality) with propositional letters, we obtain a proposition that is valid if and only if φ is valid.
- Propositional logic can be introduced as a fragment of predicate logic using nullary relation symbols (*syntax*) and nullary relations (*semantics*) since $A^0 = \{\emptyset\} = 1$, so $R^A \subseteq A^0$ is either $R^A = \emptyset = 0$ or $R^A = \{\emptyset\} = 1$.

Tableau method in propositional logic - a review

- A tableau is a binary tree that represents a search for a *counterexample*.
- Nodes are labeled by entries, i.e. formulas with a sign T / F that represents an assumption that the formula is true / false in some model.
- If this assumption is correct, then it is correct also for all the entries in some branch below that came from this entry.
- A branch is contradictory (it fails) if it contains $T\psi$, $F\psi$ for some ψ.
- A proof of formula φ is a contradictory tableau with root $F\varphi$, i.e. a tableau in which every branch is contradictory. If φ has a proof, it is valid.
- If a counterexample exists, there will be a branch in a finished tableau that provides us with this counterexample, but this branch can be infinite.
- We can construct a systematic tableau that is always finished.
- If φ is valid, the systematic tableau for φ is contradictory, i.e. it is a proof of φ; and in this case, it is also finite.

Tableau method in predicate logic - what is different

- Formulas in entries will always be sentences (closed formulas), i.e. formulas without free variables.
- We add new atomic tableaux for quantifiers.
- In these tableaux we substitute ground terms for quantified variables following certain rules.
- We extend the language by new (auxiliary) constant symbols (countably many) to represent *"witnesses"* of entries $T(\exists x)\varphi(x)$ and $F(\forall x)\varphi(x)$.
- In a finished noncontradictory branch containing an entry $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$ we have instances $T\varphi(x/t)$ resp. $F\varphi(x/t)$ for every ground term t (of the extended language).

Assumptions

1) *The formula φ that we want to prove (or refute) is a sentence.* If not, we can replace φ with its universal closure φ', since for every theory T,

$$T \models \varphi \quad \text{if and only if} \quad T \models \varphi'.$$

2) *We prove from a theory in a closed form, i.e. every axiom is a sentence.* By replacing every axiom ψ with its universal closure ψ' we obtain an equivalent theory since for every structure $\mathcal{A}$ (of the given language L),

$$\mathcal{A} \models \psi \quad \text{if and only if} \quad \mathcal{A} \models \psi'.$$

3) *The language L is countable.* Then every theory of L is countable. We denote by L_C the extension of L by new constant symbols $c_0, c_1, \dots$ (countably many). Then there are countably many ground terms of L_C. Let t_i denote the i-th ground term (in some fixed enumeration).

4) *First, we assume that the language is without equality.*

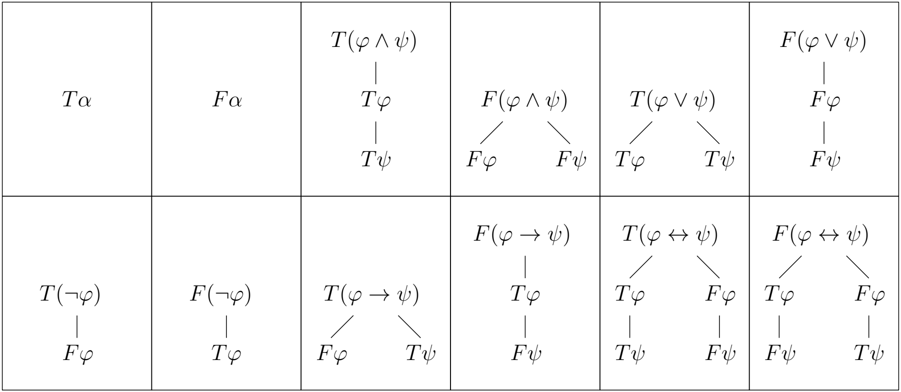
Tableaux in predicate logic - examples

$$\begin{array}{c}
F((\exists x)\neg P(x) \rightarrow \neg(\forall x)P(x)) \\
| \\
T(\exists x)\neg P(x) \\
| \\
F(\neg(\forall x)P(x)) \\
| \\
T(\forall x)P(x) \\
| \\
T(\neg P(c)) \quad c \ \text{new} \\
| \\
FP(c) \\
| \\
T(\forall x)P(x) \\
| \\
TP(c) \\
| \\
\otimes
\end{array}$$

$$\begin{array}{c}
F(\neg(\forall x)P(x) \rightarrow (\exists x)\neg P(x)) \\
| \\
T(\neg(\forall x)P(x)) \\
| \\
F(\exists x)\neg P(x) \\
| \\
F(\forall x)P(x) \\
| \\
FP(d) \quad d \ \text{new} \\
| \\
F(\exists x)\neg P(x) \\
| \\
F(\neg P(d)) \\
| \\
TP(d) \\
| \\
\otimes
\end{array}$$

Atomic tableaux - previous

An *atomic tableau* is one of the following trees (labeled by entries), where α is any atomic sentence and φ, ψ are any sentences, all of language L_C.



Atomic tableaux - new

Atomic tableaux are also the following trees (labeled by entries), where φ is any formula of the language L_C with a free variable x, t is any ground term of L_C and c is a new constant symbol from $L_C \setminus L$.

$\sharp$	$*$	$*$	$\sharp$
$T(\forall x)\varphi(x)$ $\vert$ $T\varphi(x/t)$	$F(\forall x)\varphi(x)$ $\vert$ $F\varphi(x/c)$	$T(\exists x)\varphi(x)$ $\vert$ $T\varphi(x/c)$	$F(\exists x)\varphi(x)$ $\vert$ $F\varphi(x/t)$
for any ground term t of L_C	for a *new* constant c	for a *new* constant c	for any ground term t of L_C

Remark *The constant symbol c represents a “witness” of the entry $T(\exists x)\varphi(x)$ or $F(\forall x)\varphi(x)$. Since we need that no prior demands are put on c, we specify (in the definition of a tableau) which constant symbols c may be used.*

Tableau

A *finite tableau* from a theory T is a binary tree labeled with entries described

(i) every atomic tableau is a finite tableau from T, whereas in case $(*)$ we may use any constant symbol $c \in L_C \setminus L$,

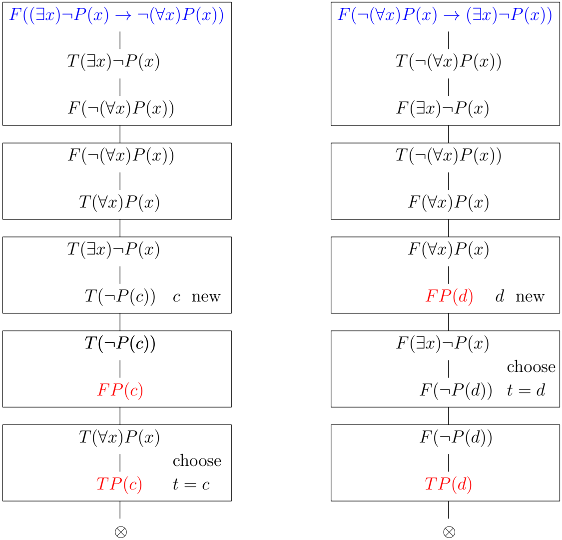
(ii) if P is an entry on a branch V in a finite tableau from T, then by adjoining the atomic tableau for P at the end of branch V we obtain (again) a finite tableau from T, whereas in case $(*)$ we may use only a constant symbol $c \in L_C \setminus L$ that does not appear on V,

(iii) if V is a branch in a finite tableau from T and $\varphi \in T$, then by adjoining $T\varphi$ at the end of branch V we obtain (again) a finite tableau from T.

(iv) every finite tableau from T is formed by finitely many steps (i), (ii), (iii).

A *tableau* from T is a sequence $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ of finite tableaux from T such that τ_{n+1} is formed from τ_n by (ii) or (iii), formally $\tau = \cup\tau_n$.

Construction of tableaux



$F((\exists x)\neg P(x) \rightarrow \neg(\forall x)P(x))$

$T(\exists x)\neg P(x)$

$F(\neg(\forall x)P(x))$

$F(\neg(\forall x)P(x))$

$T(\forall x)P(x)$

$T(\exists x)\neg P(x)$

$T(\neg P(c))$ $\quad c$ new

$T(\neg P(c))$

$FP(c)$

$T(\forall x)P(x)$

$TP(c)$ $\quad$ choose $t = c$

$\otimes$

$F(\neg(\forall x)P(x) \rightarrow (\exists x)\neg P(x))$

$T(\neg(\forall x)P(x))$

$F(\exists x)\neg P(x)$

$T(\neg(\forall x)P(x))$

$F(\forall x)P(x)$

$F(\forall x)P(x)$

$FP(d)$ $\quad d$ new

$F(\exists x)\neg P(x)$

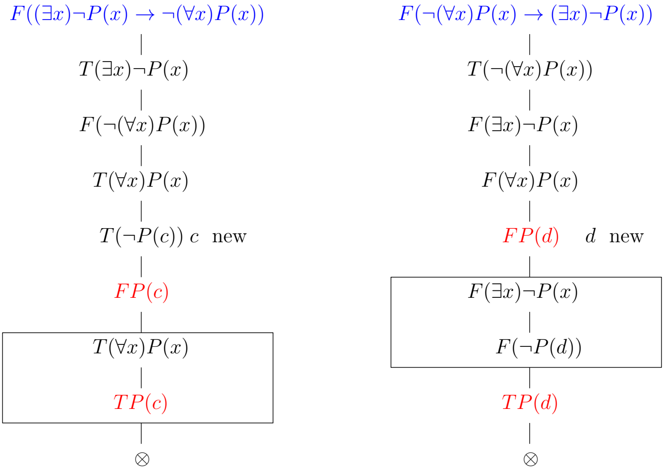
$F(\neg P(d))$ $\quad$ choose $t = d$

$F(\neg P(d))$

$TP(d)$

$\otimes$

Convention



$F((\exists x)\neg P(x) \to \neg(\forall x)P(x))$

$T(\exists x)\neg P(x)$

$F(\neg(\forall x)P(x))$

$T(\forall x)P(x)$

$T(\neg P(c))$ c new

$FP(c)$

$T(\forall x)P(x)$

$TP(c)$

$\otimes$

$F(\neg(\forall x)P(x) \to (\exists x)\neg P(x))$

$T(\neg(\forall x)P(x))$

$F(\exists x)\neg P(x)$

$F(\forall x)P(x)$

$FP(d)$ d new

$F(\exists x)\neg P(x)$

$F(\neg P(d))$

$TP(d)$

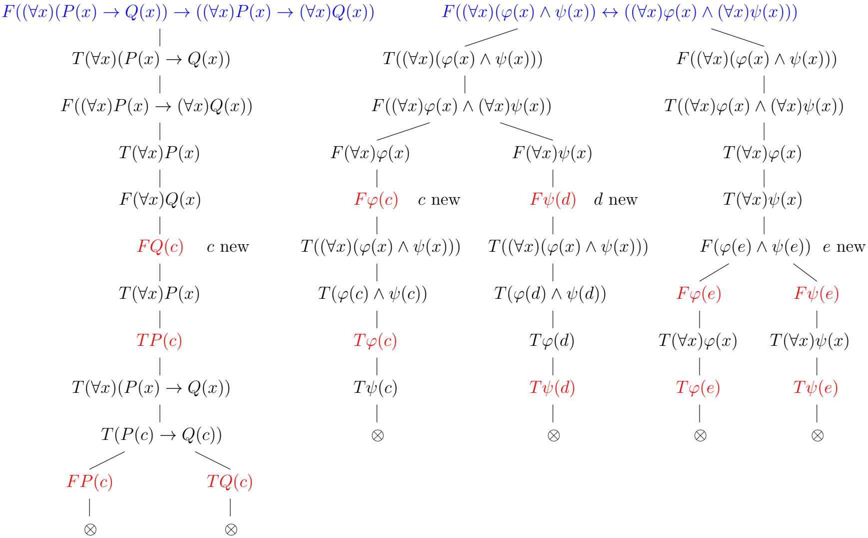
$\otimes$

We will not write the entry that is expanded again on the branch, except in cases when the entry is in the form of $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$.

Tableau proof

- A branch V in a tableau τ is *contradictory* if it contains entries $T\varphi$ and $F\varphi$ for some sentence φ, otherwise V is *noncontradictory*.
- A tableau τ is *contradictory* if every branch in τ is contradictory.
- A *tableau proof* (*proof by tableau*) of a sentence φ from a theory T is a contradictory tableau from T with $F\varphi$ in the root.
- A sentence φ is *(tableau) provable* from T, denoted by $T \vdash \varphi$, if it has a tableau proof from T.
- A *refutation* of a sentence φ by *tableau* from a theory T is a contradictory tableau from T with the root entry $T\varphi$.
- A sentence φ is *(tableau) refutable* from T if it has a refutation by tableau from T, i.e. $T \vdash \neg\varphi$.

Examples



$F((\forall x)(P(x) \rightarrow Q(x)) \rightarrow ((\forall x)P(x) \rightarrow (\forall x)Q(x))$

$T(\forall x)(P(x) \rightarrow Q(x))$

$F((\forall x)P(x) \rightarrow (\forall x)Q(x))$

$T(\forall x)P(x)$

$F(\forall x)Q(x)$

$FQ(c)$ c new

$T(\forall x)P(x)$

$TP(c)$

$T(\forall x)(P(x) \rightarrow Q(x))$

$T(P(c) \rightarrow Q(c))$

$FP(c)$ $\quad$ $TQ(c)$

$\otimes$ $\quad$ $\otimes$

$F((\forall x)(\varphi(x) \wedge \psi(x)) \leftrightarrow ((\forall x)\varphi(x) \wedge (\forall x)\psi(x)))$

$T((\forall x)(\varphi(x) \wedge \psi(x)))$ $\quad$ $F((\forall x)(\varphi(x) \wedge \psi(x)))$

$F((\forall x)\varphi(x) \wedge (\forall x)\psi(x))$ $\quad$ $T((\forall x)\varphi(x) \wedge (\forall x)\psi(x))$

$F(\forall x)\varphi(x)$ $\quad$ $F(\forall x)\psi(x)$ $\quad$ $T(\forall x)\varphi(x)$

$F\varphi(c)$ c new $\quad$ $F\psi(d)$ d new $\quad$ $T(\forall x)\psi(x)$

$T((\forall x)(\varphi(x) \wedge \psi(x)))$ $\quad$ $T((\forall x)(\varphi(x) \wedge \psi(x)))$ $\quad$ $F(\varphi(e) \wedge \psi(e))$ e new

$T(\varphi(c) \wedge \psi(c))$ $\quad$ $T(\varphi(d) \wedge \psi(d))$ $\quad$ $F\varphi(e)$ $\quad$ $F\psi(e)$

$T\varphi(c)$ $\quad$ $T\varphi(d)$ $\quad$ $T(\forall x)\varphi(x)$ $\quad$ $T(\forall x)\psi(x)$

$T\psi(c)$ $\quad$ $T\psi(d)$ $\quad$ $T\varphi(e)$ $\quad$ $T\psi(e)$

$\otimes$ $\quad$ $\otimes$ $\quad$ $\otimes$ $\quad$ $\otimes$