

Propositional and Predicate Logic - IX

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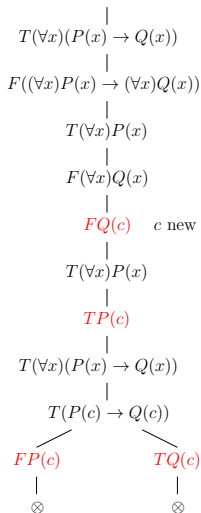
WS 2022/2023

Tableau proof

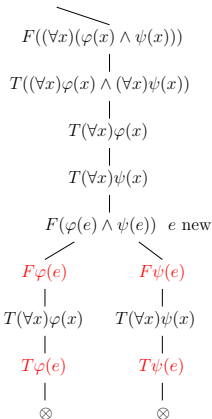
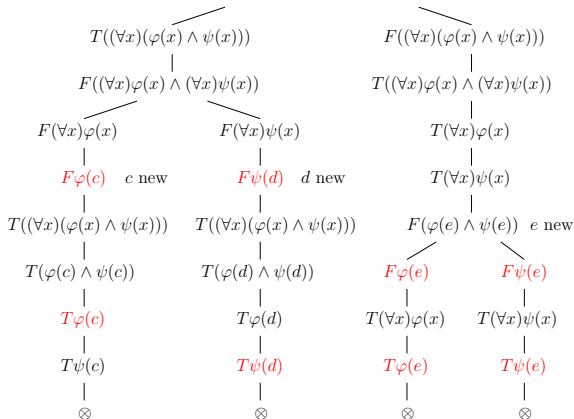
- A branch V in a tableau τ is *contradictory* if it contains entries $T\varphi$ and $F\varphi$ for some sentence φ , otherwise V is *noncontradictory*.
- A tableau τ is *contradictory* if every branch in τ is contradictory.
- A *tableau proof* (*proof by tableau*) of a sentence φ from a theory T is a *contradictory tableau* from T with $F\varphi$ in the root.
- A sentence φ is *(tableau) provable* from T , denoted by $T \vdash \varphi$, if it has a tableau proof from T .
- A *refutation* of a sentence φ by *tableau* from a theory T is a *contradictory tableau* from T with the root entry $T\varphi$.
- A sentence φ is *(tableau) refutable* from T if it has a refutation by tableau from T , i.e. $T \vdash \neg\varphi$.

Examples

$$F((\forall x)(P(x) \rightarrow Q(x)) \rightarrow ((\forall x)P(x) \rightarrow (\forall x)Q(x)))$$



$$F((\forall x)(\varphi(x) \wedge \psi(x)) \leftrightarrow ((\forall x)\varphi(x) \wedge (\forall x)\psi(x)))$$



Finished tableau

A finished noncontradictory branch should provide us with a *counterexample*.

An occurrence of an entry P in a node ν of a tableau τ is *i -th* if ν has exactly $i - 1$ predecessors labeled by P ; and is *reduced* on a branch V through ν if

- a) P is neither in form of $T(\forall x)\varphi(x)$ nor $F(\exists x)\varphi(x)$ and P occurs on V as a root of an atomic tableau, i.e. it was already expanded on V , or
- b) P is in form of $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$, P has an $(i + 1)$ -th occurrence on V , and V contains an entry $T\varphi(x/t_i)$ resp. $F\varphi(x/t_i)$ where t_i is the i -th ground term (of the language L_C).

Let V be a branch in a tableau τ from a theory T . We say that

- V is *finished* if it is contradictory, or every occurrence of an entry on V is reduced on V and, moreover, V contains $T\varphi$ for every $\varphi \in T$,
- τ is *finished* if every branch in τ is finished.

Systematic tableau - construction

Let R be an entry and $T = \{\varphi_0, \varphi_1, \dots\}$ be a (possibly infinite) theory.

- (1) We take the atomic tableau for R as τ_0 . In case (*) we choose any $c \in L_C \setminus L$, in case (#) we take t_1 for t . Till possible, proceed as follows.
- (2) Let v be the **leftmost** node in the **smallest** level as possible in tableau τ_n containing an occurrence of an entry P that is not reduced on some noncontradictory branch **through** v . (If v does not exist, we take $\tau'_n = \tau_n$.)
- (3a) If P is neither $T(\forall x)\varphi(x)$ nor $F(\exists x)\varphi(x)$, let τ'_n be the tableau obtained from τ_n by adjoining the atomic tableau for P to every noncontradictory branch through v . In case (*) we choose c_i for the smallest possible i .
- (3b) If P is $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$ and it has i -th occurrence in v , let τ'_n be the tableau obtained from τ_n by adjoining atomic tableau for P to every noncontradictory branch through v , where we take the term t_i for t .
- (4) Let τ_{n+1} be the tableau obtained from τ'_n by adjoining $T\varphi_n$ to every noncontradictory branch that does not contain $T\varphi_n$ yet. (If φ_n does not exist, we take $\tau_{n+1} = \tau'_n$.)

The **systematic tableau** for R from T is the result $\tau = \bigcup \tau_n$ of this construction.

Systematic tableau - an example

$$T((\exists y)(\neg R(y, y) \vee P(y, y)) \wedge (\forall x)R(x, x))$$

$$T(\exists y)(\neg R(y, y) \vee P(y, y))$$

$$T(\forall x)R(x, x)$$

$$T(\neg R(c_0, c_0) \vee P(c_0, c_0)) \quad c_0 \text{ new}$$

$$T(\forall x)R(x, x)$$

$$TR(c_0, c_0) \quad (\text{assuming that } t_1 = c_0)$$

$$T(\neg R(c_0, c_0))$$

$$TP(c_0, c_0)$$

$$T(\forall x)R(x, x)$$

$$T(\forall x)R(x, x)$$

$$TR(t_2, t_2)$$

$$TR(t_2, t_2)$$

$$FR(c_0, c_0)$$

$$T(\forall x)R(x, x)$$

$$\otimes$$

$$TR(t_3, t_3)$$

$$\vdots$$

Systematic tableau - being finished

Proposition Every systematic tableau is *finished*.

Proof Let $\tau = \cup \tau_n$ be a systematic tableau from $T = \{\varphi_0, \varphi_1, \dots\}$ with root R and let P be an entry in a node ν of the tableau τ .

- There are only finitely many entries in τ in levels up to the level of ν .
- If the occurrence of P in ν was unreduced on some noncontradictory branch in τ , it would be found in some step (2) and reduced by (3a), (3b).
- By step (4) every $\varphi_n \in T$ will be (no later than) in τ_{n+1} on every noncontradictory branch.
- Hence the systematic tableau τ has all branches finished. \square

Proposition If a systematic tableau τ is a proof (from a theory T), it is finite.

Proof Suppose that τ is infinite. Then by König's lemma, τ contains an infinite branch. This branch is noncontradictory since in the construction only noncontradictory branches are prolonged. But this contradicts the assumption that τ is a contradictory tableau. \square

Equality

Axioms of equality for a language L with equality are

(i) $x = x$

(ii) $x_1 = y_1 \wedge \cdots \wedge x_n = y_n \rightarrow f(x_1, \dots, x_n) = f(y_1, \dots, y_n)$

for each n -ary function symbol f of the language L .

(iii) $x_1 = y_1 \wedge \cdots \wedge x_n = y_n \rightarrow (R(x_1, \dots, x_n) \rightarrow R(y_1, \dots, y_n))$

for each n -ary relation symbol R of the language L including $=$.

A *tableau proof* from a theory T in a language L *with equality* is a tableau proof from T^* where T^* denotes the extension of T by adding axioms of equality for L (resp. their universal closures).

Remark In context of logic programming the equality often has other meaning than in mathematics (identity). For example in Prolog, $t_1 = t_2$ means that t_1 and t_2 are unifiable.

Congruence and quotient structure

Let \sim be an equivalence on A , $f : A^n \rightarrow A$, and $R \subseteq A^n$ for $n \in \mathbb{N}$. Then \sim is

- a *congruence for the function* f if for every $x_1, \dots, x_n, y_1, \dots, y_n \in A$

$$x_1 \sim y_1 \wedge \dots \wedge x_n \sim y_n \Rightarrow f(x_1, \dots, x_n) \sim f(y_1, \dots, y_n),$$
- a *congruence for the relation* R if for every $x_1, \dots, x_n, y_1, \dots, y_n \in A$

$$x_1 \sim y_1 \wedge \dots \wedge x_n \sim y_n \Rightarrow (R(x_1, \dots, x_n) \Leftrightarrow R(y_1, \dots, y_n)).$$

Let an equivalence \sim on A be a congruence for every function and relation in a structure $\mathcal{A} = \langle A, \mathcal{F}^A, \mathcal{R}^A \rangle$ of language $L = \langle \mathcal{F}, \mathcal{R} \rangle$. Then the *quotient (structure)* of \mathcal{A} by \sim is the structure $\mathcal{A}/\sim = \langle A/\sim, \mathcal{F}^{A/\sim}, \mathcal{R}^{A/\sim} \rangle$ where

$$f^{A/\sim}([x_1]_{\sim}, \dots, [x_n]_{\sim}) = [f^A(x_1, \dots, x_n)]_{\sim}$$

$$R^{A/\sim}([x_1]_{\sim}, \dots, [x_n]_{\sim}) \Leftrightarrow R^A(x_1, \dots, x_n)$$

for each $f \in \mathcal{F}$, $R \in \mathcal{R}$, and $x_1, \dots, x_n \in A$, i.e. the functions and relations are defined from \mathcal{A} using *representatives*.

Example: $\underline{\mathbb{Z}}_p$ is the quotient of $\underline{\mathbb{Z}} = \langle \mathbb{Z}, +, -, 0 \rangle$ by the congruence modulo p .

Role of axioms of equality

Let \mathcal{A} be a structure of a language L in which the equality is interpreted as a relation $=^A$ satisfying the axioms of equality for L , i.e. not necessarily the identity relation.

- 1) From axioms (i) and (iii) it follows that the relation $=^A$ is an **equivalence**.
- 2) Axioms (ii) and (iii) express that the relation $=^A$ is a **congruence** for every function and relation in \mathcal{A} .
- 3) If $\mathcal{A} \models T^*$ then also $(\mathcal{A}/=^A) \models T^*$ where $\mathcal{A}/=^A$ is the **quotient** of \mathcal{A} by $=^A$. Moreover, the equality is interpreted in $\mathcal{A}/=^A$ as the identity relation.

On the other hand, in every model in which the equality is interpreted as the identity relation, all axioms of equality evidently hold.

Soundness

We say that a model \mathcal{A} *agrees* with an entry P , if P is $T\varphi$ and $\mathcal{A} \models \varphi$ or if P is $F\varphi$ and $\mathcal{A} \models \neg\varphi$, i.e. $\mathcal{A} \not\models \varphi$. Moreover, \mathcal{A} *agrees* with a branch V if \mathcal{A} agrees with every entry on V .

Lemma *Let \mathcal{A} be a model of a theory T of a language L that agrees with the root entry R in a tableau $\tau = \cup \tau_n$ from T . Then \mathcal{A} can be *expanded* to the language L_C so that it agrees with *some* branch V in τ .*

Remark *It suffices to expand \mathcal{A} only by constants c^A such that $c \in L_C \setminus L$ occurs on V , other constants may be defined arbitrarily.*

Proof By induction on n we find a branch V_n in τ_n and an expansion \mathcal{A}_n of \mathcal{A} by constants c^A for all $c \in L_C \setminus L$ on V_n s.t. \mathcal{A}_n agrees with V_n and $V_{n-1} \subseteq V_n$.

Assume we have a branch V_n in τ_n and an expansion \mathcal{A}_n that agrees with V_n .

- If τ_{n+1} is formed from τ_n without extending the branch V_n , we take $V_{n+1} = V_n$ and $\mathcal{A}_{n+1} = \mathcal{A}_n$.
- If τ_{n+1} is formed from τ_n by appending $T\varphi$ to V_n for some $\varphi \in T$, let V_{n+1} be this branch and $\mathcal{A}_{n+1} = \mathcal{A}_n$. Since $\mathcal{A} \models \varphi$, \mathcal{A}_{n+1} agrees with V_{n+1} .

Soundness - proof (cont.)

- Otherwise τ_{n+1} is formed from τ_n by appending an atomic tableau to V_n for some entry P on V_n . By induction we know that \mathcal{A}_n agrees with P .
- (i) If P is formed by a **logical connective**, we take $\mathcal{A}_{n+1} = \mathcal{A}_n$ and verify that V_n can always be extended to a branch V_{n+1} agreeing with \mathcal{A}_{n+1} .
- (ii) If P is in form $T(\forall x)\varphi(x)$, let V_{n+1} be the (unique) extension of V_n to a branch in τ_{n+1} , i.e. by the entry $T\varphi(x/t)$. Let \mathcal{A}_{n+1} be **any** expansion by new constants from t . Since $\mathcal{A}_n \models (\forall x)\varphi(x)$, we have $\mathcal{A}_{n+1} \models \varphi(x/t)$. Analogously for P in form $F(\exists x)\varphi(x)$.
- (iii) If P is in form $T(\exists x)\varphi(x)$, let V_{n+1} be the (unique) extension of V_n to a branch in τ_{n+1} , i.e. by the entry $T\varphi(x/c)$. Since $\mathcal{A}_n \models (\exists x)\varphi(x)$, there is some $a \in A$ with $\mathcal{A}_n \models \varphi(x)[e(x/a)]$ for every assignment e . Let \mathcal{A}_{n+1} be the expansion of \mathcal{A}_n by a new constant $c^A = a$. Then $\mathcal{A}_{n+1} \models \varphi(x/c)$. Analogously for P in form $F(\forall x)\varphi(x)$.

The base step for $n = 0$ follows from similar analysis of atomic tableaux for the root entry R applying the assumption that \mathcal{A} agrees with R . \square

Theorem on soundness

We will show that the tableau method in predicate logic is *sound*.

Theorem For every theory T and sentence φ , if φ is tableau provable from T , then φ is valid in T , i.e. $T \vdash \varphi \Rightarrow T \models \varphi$.

Proof

- Let φ be tableau provable from a theory T , i.e. there is a contradictory tableau τ from T with the root entry $F\varphi$.
- Suppose for a contradiction that φ is not valid in T , i.e. there exists a model \mathcal{A} of the theory T in which φ is not true (a *counterexample*).
- Since \mathcal{A} agrees with the root entry $F\varphi$, by the previous lemma, \mathcal{A} can be expanded to the language L_C so that it agrees with some branch in τ .
- But this is impossible, since every branch of τ is contradictory, i.e. it contains a pair of entries $T\psi, F\psi$ for some sentence ψ . \square

The canonical model

From a noncontradictory branch V of a finished tableau we build a model that agrees with V . We build it on available (syntactical) objects - **ground terms**.

Let V be a noncontradictory branch of a finished tableau from a theory T of a language $L = \langle \mathcal{F}, \mathcal{R} \rangle$. The **canonical model** from V is the L_C -structure $\mathcal{A} = \langle A, \mathcal{F}^A, \mathcal{R}^A \rangle$ where

- (1) A is the set of all ground terms of the language L_C ,
- (2) $f^A(t_{i_1}, \dots, t_{i_n}) = f(t_{i_1}, \dots, t_{i_n})$
for every n -ary function symbol $f \in \mathcal{F} \cup (L_C \setminus L)$ and $t_{i_1}, \dots, t_{i_n} \in A$.
- (3) $R^A(t_{i_1}, \dots, t_{i_n}) \Leftrightarrow TR(t_{i_1}, \dots, t_{i_n})$ is an entry on V
for every n -ary relation symbol $R \in \mathcal{R}$ or **equality** and $t_{i_1}, \dots, t_{i_n} \in A$.

Remark The expression $f(t_{i_1}, \dots, t_{i_n})$ on the right side of (2) is a ground term of L_C , i.e. an element of A . Informally, to indicate that it is a syntactical object

$$f^A(t_{i_1}, \dots, t_{i_n}) = "f(t_{i_1}, \dots, t_{i_n})"$$

The canonical model - an example

Let $T = \{(\forall x)R(f(x))\}$ be a theory of a language $L = \langle R, f, d \rangle$. The systematic tableau for $F\neg R(d)$ from T contains a single branch V , which is noncontradictory.

The canonical model $\mathcal{A} = \langle A, R^A, f^A, d^A, c_i^A \rangle_{i \in \mathbb{N}}$ from V is for language L_C and

$$A = \{d, f(d), f(f(d)), \dots, c_0, f(c_0), f(f(c_0)), \dots, c_1, f(c_1), f(f(c_1)), \dots\},$$

$$d^A = d, \quad c_i^A = c_i \text{ for } i \in \mathbb{N},$$

$$f^A(d) = "f(d)", \quad f^A(f(d)) = "f(f(d))", \quad f^A(f(f(d))) = "f(f(f(d)))", \quad \dots$$

$$R^A = \{d, f(d), f(f(d)), \dots, f(c_0), f(f(c_0)), \dots, f(c_1), f(f(c_1)), \dots\}.$$

The reduct of \mathcal{A} to the language L is $\mathcal{A}' = \langle A, R^A, f^A, d^A \rangle$.

The canonical model with equality

If L is with equality, T^* is the extension of T by the axioms of equality for L .

If we require that the equality is interpreted as the identity, we have to take the quotient of the canonical model \mathcal{A} by the congruence $=^A$.

By (3), for the relation $=^A$ in \mathcal{A} from V it holds that for every $s, t \in A$,

$$s =^A t \Leftrightarrow T(s = t) \text{ is an entry on } V.$$

Since V is finished and contains the axioms of equality, the relation $=^A$ is a **congruence** for all functions and relations in \mathcal{A} .

The **canonical model with equality** from V is the quotient $\mathcal{A}/=^A$.

Observation For every formula φ ,

$$\mathcal{A} \models \varphi \Leftrightarrow (\mathcal{A}/=^A) \models \varphi,$$

where $=$ is interpreted in \mathcal{A} by the relation $=^A$, while in $\mathcal{A}/=^A$ by the identity.

Remark \mathcal{A} is a countably infinite model, but $\mathcal{A}/=^A$ can be finite.

The canonical model with equality - an example

Let $T = \{(\forall x)R(f(x)), (\forall x)(x = f(f(x)))\}$ be of $L = \langle R, f, d \rangle$ with equality.

The systematic tableau for $F\neg R(d)$ from T^* contains a noncontradictory V .

In the canonical model $\mathcal{A} = \langle A, R^A, =^A, f^A, d^A, c_i^A \rangle_{i \in \mathbb{N}}$ from V we have that

$$s =^A t \iff t = f(\dots(f(s)\dots)) \text{ or } s = f(\dots(f(t)\dots)),$$

where f is applied $2i$ -times for some $i \in \mathbb{N}$.

The canonical model with equality from V is

$\mathcal{B} = (\mathcal{A}/=^A) = \langle A/=^A, R^B, f^B, d^B, c_i^B \rangle_{i \in \mathbb{N}}$ where

$$(A/=^A) = \{[d]_{=^A}, [f(d)]_{=^A}, [c_0]_{=^A}, [f(c_0)]_{=^A}, [c_1]_{=^A}, [f(c_1)]_{=^A}, \dots\},$$

$$d^B = [d]_{=^A}, \quad c_i^B = [c_i]_{=^A} \text{ for } i \in \mathbb{N},$$

$$f^B([d]_{=^A}) = [f(d)]_{=^A}, \quad f^B([f(d)]_{=^A}) = [f(f(d))]_{=^A} = [d]_{=^A}, \quad \dots$$

$$R^B = (A/=^A).$$

The reduct of \mathcal{B} to the language L is $\mathcal{B}' = \langle A/=^A, R^B, f^B, d^B \rangle$.

Completeness

Lemma *The canonical model \mathcal{A} from a noncontr. finished V agrees with V .*

Proof By induction on the structure of a sentence in an entry on V .

- For **atomic** φ , if $T\varphi$ is on V , then $\mathcal{A} \models \varphi$ by (3). If $F\varphi$ is on V , then $T\varphi$ is not on V since V is noncontradictory, so $\mathcal{A} \models \neg\varphi$ by (3).
- If $T(\varphi \wedge \psi)$ is on V , then $T\varphi$ and $T\psi$ are on V since V is finished. By induction, $\mathcal{A} \models \varphi$ and $\mathcal{A} \models \psi$, and thus $\mathcal{A} \models \varphi \wedge \psi$.
- If $F(\varphi \wedge \psi)$ is on V , then $F\varphi$ or $F\psi$ is on V since V is finished. By induction, $\mathcal{A} \models \neg\varphi$ or $\mathcal{A} \models \neg\psi$, and thus $\mathcal{A} \models \neg(\varphi \wedge \psi)$.
- For other connectives similarly as in previous two cases.
- If $T(\forall x)\varphi(x)$ is on V , then $T\varphi(x/t)$ is on V for every $t \in A$ since V is finished. By induction, $\mathcal{A} \models \varphi(x/t)$ for every $t \in A$, and thus $\mathcal{A} \models (\forall x)\varphi(x)$. Similarly for $F(\exists x)\varphi(x)$ on V .
- If $T(\exists x)\varphi(x)$ is on V , then $T\varphi(x/c)$ is on V for some $c \in A$ since V is finished. By induction, $\mathcal{A} \models \varphi(x/c)$, and thus $\mathcal{A} \models (\exists x)\varphi(x)$. Similarly for $F(\forall x)\varphi(x)$ on V . \square

Theorem on completeness

We will show that the tableau method in predicate logic is **complete**.

Theorem For every theory T and sentence φ , if φ is valid in T , then φ is tableau provable from T , i.e. $T \models \varphi \Rightarrow T \vdash \varphi$.

Proof Let φ be valid in T . We will show that an arbitrary **finished** tableau (e.g. **systematic**) τ from a theory T with the root entry $F\varphi$ is **contradictory**.

- If not, then there is some noncontradictory branch V in τ .
- By the previous lemma, there is a structure \mathcal{A} for L_C that agrees with V , in particular with the root entry $F\varphi$, i.e. $\mathcal{A} \models \neg\varphi$.
- Let \mathcal{A}' be the reduct of \mathcal{A} to the language L . Then $\mathcal{A}' \models \neg\varphi$.
- Since V is finished, it contains $T\psi$ for every $\psi \in T$.
- Thus \mathcal{A}' is a model of T (as \mathcal{A}' agrees with $T\psi$ for every $\psi \in T$).
- But this contradicts the assumption that φ is valid in T .

Therefore the tableau τ is a proof of φ from T . \square

Properties of theories

We introduce syntactic variants of previous semantical definitions.

Let T be a theory of a language L . If a sentence φ is provable from T , we say that φ is a *theorem* of T . The set of theorems of T is denoted by

$$\text{Thm}^L(T) = \{\varphi \in \text{Fm}_L \mid T \vdash \varphi\}.$$

We say that a theory T is

- *inconsistent* if $T \vdash \perp$, otherwise T is *consistent*,
- *complete* if it is consistent and every sentence is provable or refutable from T , i.e. $T \vdash \varphi$ or $T \vdash \neg\varphi$.
- an *extension* of a theory T' of L' if $L' \subseteq L$ and $\text{Thm}^{L'}(T') \subseteq \text{Thm}^L(T)$, we say that an extension T of a theory T' is *simple* if $L = L'$; and *conservative* if $\text{Thm}^{L'}(T') = \text{Thm}^L(T) \cap \text{Fm}_{L'}$,
- *equivalent* with a theory T' if T is an extension of T' and vice-versa.

Corollaries

From the soundness and completeness of the tableau method it follows that these syntactic definitions agree with their semantic variants.

Corollary For every theory T and sentences φ, ψ of a language L ,

- $T \vdash \varphi$ if and only if $T \models \varphi$,
- $\text{Thm}^L(T) = \theta^L(T)$,
- T is inconsistent if and only if T is unsatisfiable, i.e. it has no model,
- T is complete if and only if T is semantically complete, i.e. it has a single model, up to elementary equivalence,
- $T, \varphi \vdash \psi$ if and only if $T \vdash \varphi \rightarrow \psi$ (*Deduction theorem*).

Remark Deduction theorem can be proved directly by transformations of tableaux.

Existence of a countable model and compactness

Theorem *Every consistent theory T of a countable language L without equality has a **countably infinite** model.*

Proof Let τ be the systematic tableau from T with $F\perp$ in the root. Since τ is finished and contains a noncontradictory branch V as \perp is not provable from T , there exists a **canonical model** \mathcal{A} from V . Since \mathcal{A} agrees with V , its reduct to the language L is a desired countably infinite model of T . \square

Remark *This is a weak version of so called **Löwenheim-Skolem theorem**. In a countable language with **equality** the canonical model with equality is **countable** (i.e. finite or countably infinite).*

Theorem *A theory T has a model iff every **finite** subset of T has a model.*

Proof The implication from left to right is obvious. If T has no model, then it is inconsistent, i.e. \perp is provable by a systematic tableau τ from T . Since τ is finite, \perp is provable from some finite $T' \subseteq T$, i.e. T' has no model. \square

Non-standard model of natural numbers

Let $\underline{\mathbb{N}} = \langle \mathbb{N}, S, +, \cdot, 0, \leq \rangle$ be the standard model of natural numbers.

Let $\text{Th}(\underline{\mathbb{N}})$ denote the set of all **sentences** that are valid in $\underline{\mathbb{N}}$. For $n \in \mathbb{N}$ let \underline{n} denote the term $S(S(\dots(S(0))\dots))$, so called the *n -th numeral*, where S is applied n -times.

Consider the following theory T where c is a new constant symbol.

$$T = \text{Th}(\underline{\mathbb{N}}) \cup \{ \underline{n} < c \mid n \in \mathbb{N} \}$$

Observation Every finite subset of T has a model.

Thus by the compactness theorem, T has a model \mathcal{A} . It is a *non-standard model of natural numbers*. Every sentence from $\text{Th}(\underline{\mathbb{N}})$ is valid in \mathcal{A} but it contains an element $c^{\mathcal{A}}$ that is greater than every $n \in \mathbb{N}$ (i.e. the value of the term \underline{n} in \mathcal{A}).