

Example of test problems

December 20, 2022

1. Let $T = \{(\exists x)P(x, x), (\forall x)(\exists y)R(x, y), (\forall u)(\forall v)(\exists x)(\forall y)(R(x, y) \rightarrow \neg P(u, v))\}$ be a theory of the language $L = \langle P, R \rangle$ without equality, where P, R are binary relation symbols.
 - (a) Using Skolemization, find a theory T' (over a suitably extended language) equisatisfiable with T and axiomatized only by universal sentences.
 - (b) Use the tableau method to prove that T' is unsatisfiable.
 - (c) Let T'' be a theory consisting of matrices of all the axioms of T' . Find an unsatisfiable conjunction of ground instances of axioms of T'' . *Hint: use the tableau from (b).*
 - (d) Is the sentence $(\forall x)P(x, x)$ true / contradictory / independent in T ? Justify your answers.
2. Let $T = \{(\forall x)(P(x) \rightarrow (\exists y)R(x, y)), (\exists y)((\exists x)R(y, x) \rightarrow \neg(\forall z)P(z)), (\forall x)P(x)\}$ be a theory in language $L = \langle P, R \rangle$ without equality where P, R is unary resp. binary relation symbol.
 - (a) Applying skolemization find a theory T' (in a some extended language) such that T' is equisatisfiable with T and all axioms of T' are universal sentences.
 - (b) Prove by tableau method that T' is unsatisfiable.
 - (c) Let T'' denote the set of open matrices of axioms of T' , so T'' is an open theory equivalent to T' . Find a conjunction of ground instances of axioms of T'' that is unsatisfiable. *Hint: use the tableau from (b).*
 - (d) Find some complete extension of T or explain why no such extension exists.
3. We know that:
 - (i) Aristotle is Greek and Caesar is Roman and Dido is Carthaginian.
 - (ii) No Greek is Roman.
 - (iii) No Carthaginian is Greek.
 - (iv) Only Carthaginians were born in Carthage.

Using resolution, we want to prove that:

- (v) There exists someone who was not born in Carthage and who is not Roman.

In particular:

- (a) Express the statements as sentences $\varphi_1, \dots, \varphi_5$ in the language $L = \langle G, R, C, B, a, c, d \rangle$ without equality, where G, R, C, B are unary relation symbols and $G(x), R(x), C(x), B(x)$ mean that “ x is Greek / Roman / Carthaginian” and “ x was born in Carthage”, respectively, and a, c, d are constant symbols denoting Aristotle, Caesar, and Dido.
- (b) Using Skolemization, find an open theory T (possibly in an extended language) which is unsatisfiable, if and only if $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \models \varphi_5$. Convert T to CNF and write it in set representation.
- (c) Prove by resolution that T is not satisfiable. Draw the resolution refutation in the form of a resolution tree. Write the unification used in every step.
- (d) Find a conjunction of ground instances of axioms of T which is unsatisfiable.
- (e) Do the statements (i) to (iv) imply that “Caesar was not born in Carthage”? Justify your answer.

4. Let T be the following theory in the language $L = \langle R, A \rangle$ without equality, where R is a binary relation symbol and A is a unary relation symbol:

$$T = \{(\exists x)(\forall y)(R(x, y) \rightarrow R(y, x)), \\ (\forall x)((\exists y)(R(x, y) \wedge R(y, x)) \rightarrow \neg A(x)), \\ (\exists y)R(x, y), \\ \neg(\exists x)\neg A(x)\}$$

- (a) Using Skolemization, find an open theory T' (over a suitable extension of L) equisatisfiable with T .
- (b) Convert T' to an equivalent theory S in CNF. Write S in set representation.
- (c) Find a resolution refutation of the theory S . Draw the resolution tree and in each step, write the unification used.
- (d) Find an unsatisfiable conjunction of ground instances of axioms of S *Hint: Use the unifications from (c)*.
- (e) Does the theory T have a complete simple extension? If yes, give an example. If not, explain why.
5. We know that:

- (i) Parents are older than their children.
- (ii) “Being a parent” is an asymmetric relation.
- (iii) “Being older” is a transitive relation.
- (iv) Tom is a father of Mary, Mary is not older than Bob, Bob is a son of Jane.

Use the resolution method to show that:

- (v) Tom is older than Bob or Mary is not a mother of Jane.

In particular:

- (a) Express the statements (i) to (v) by open formulæ φ_1 to φ_5 of the language $L = \langle P, O, t, m, b, j \rangle$ without equality, where P, O are binary relation symbols, $P(x, y)$, $O(x, y)$ denotes that “ x is a parent of y ”, “ x is older than y ” (respectively), and t, m, b, j are constant symbols denoting Tom, Mary, Bob, and Jane (respectively).
- (b) By transforming to CNF, find an open theory S in set representation which is unsatisfiable, if and only if φ_5 is valid in the theory $T = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$.
- (c) Prove by resolution that S is not satisfiable. Depict the resolution refutation of S by a resolution tree. At every step, write down the unification used.
- (d) Find an unsatisfiable conjunction of ground instances of axioms of S .
- (e) Is the theory T complete? Justify your answer.