Predicate and Propositional Logic - Tutorial 2

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1. Consider a theory $T = \{\neg q \to (\neg p \vee q),\ \neg p \to q,\ r \to q\}$. Which of the following propositions are valid, contradictory, independent, satisfiable, equivalent in T?
   (a) p, q, r, s
   (b) $p \vee q$, $p \vee r$, $p \vee s$, $q \vee s$
   (c) $p \wedge q$, $q \wedge s$, $p \to q$, $s \to q$

2. Prove or disprove that the following sets of connectives are adequate.
   (a) $\{\downarrow\}$ where $\downarrow$ is Peirce arrow (NOR)
   (b) $\{\uparrow\}$ where $\uparrow$ is Sheffer stroke (NAND)
   (c) $\{\vee, \to, \leftrightarrow\}$, $\{\vee, \wedge, \to\}$

3. Transform the following propositions into DNF and CNF a) by using truth tables (determining the models), b) by using transformation rules.
   (a) $(\neg p \vee q) \to (\neg q \wedge r)$
   (b) $(\neg p \to (\neg q \to r)) \to p$
   (c) $((p \to \neg q) \to \neg r) \to \neg p$

4. Applying the implication graph determine whether the following proposition in 2-CNF is satisfiable or not; and if yes, find a satisfying assignment.

$$(p_0 \vee p_2) \wedge (p_0 \vee \neg p_3) \wedge (p_1 \vee \neg p_3) \wedge (p_1 \vee \neg p_4) \wedge (p_2 \vee \neg p_4) \wedge (p_0 \vee \neg p_5) \wedge \\ (p_1 \vee \neg p_5) \wedge (p_2 \vee \neg p_5) \wedge (\neg p_1 \vee \neg p_6) \wedge (p_4 \vee p_6) \wedge (p_5 \vee p_6) \wedge p_1$$

5. Applying unit propagation determine whether the following Horn formula is satisfiable; and if yes, find a satisfying assignment.

$$(\neg p_1 \vee \neg p_3 \vee p_2) \wedge (\neg p_1 \vee p_2) \wedge p_1 \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge \\ (\neg p_2 \vee \neg p_4 \vee p_1) \wedge (p_4 \vee \neg p_3 \vee \neg p_2) \wedge (\neg p_4 \vee p_5)$$

6. Find both DNF and CNF representations of the Boolean function $\mathrm{maj}\colon \{0,1\}^3 \to \{0,1\}$ defined as the majority of the three (truth) values.

7. Let $\mathrm{maj}_n\colon (\{0,1\}^n)^3 \to \{0,1\}^n$ be the coordinate-wise majority function; that is, for example

$$\mathrm{maj}_4((0,1,0,1),(1,1,0,0),(1,1,0,0)) = (1,1,0,0)$$

   We say that a set $K \subseteq \{0,1\}^n$ is a *median* set if it is closed under maj_n.
   (a) Show that for every 2-CNF proposition φ it holds that $M(\varphi)$ is a median set.
   (b)* Show that for every median set $K \subseteq \{0,1\}^n$ there exists a 2-CNF proposition φ over n variables such that $M(\varphi) = K$.