## Predicate and Propositional Logic - Tutorial 3

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1. Consider an infinite theory $T = \{p_i \to (p_{i+1} \vee q_{i+1}), q_i \to (p_{i+1} \vee q_{i+1}) \mid i \in \mathbb{N}\}$ over $\mathrm{var}(T)$.
   - (a) Which propositions in the form $p_i \to p_j$ are logical consequences of $T$?
   - (b) Which propositions in the form $p_i \to (p_j \vee q_j)$ are logical consequences of $T$?
   - (c) Determine all models of the theory $T$.
2. Prove or disprove (or find the correct relation) that for every theory $T$ and propositions $\varphi$, $\psi$ over $\mathbb{P}$ it holds
   - (a) $T \models \varphi$, if and only if $T \not\models \neg\varphi$
   - (b) $T \models \varphi$ and $T \models \psi$, if and only if $T \models \varphi \wedge \psi$
   - (c) $T \models \varphi$ or $T \models \psi$, if and only if $T \models \varphi \vee \psi$
   - (d) $T \models \varphi \to \psi$ and $T \models \psi \to \chi$, if and only if $T \models \varphi \to \chi$
3. Prove or disprove (or find the correct relation). For every theories $T$ and $S$ over $\mathbb{P}$
   - (a) $S \subseteq T \Rightarrow \theta^{\mathbb{P}}(T) \subseteq \theta^{\mathbb{P}}(S)$
   - (b) $\theta^{\mathbb{P}}(S \cup T) = \theta^{\mathbb{P}}(S) \cup \theta^{\mathbb{P}}(T)$
   - (c) $\theta^{\mathbb{P}}(S \cap T) = \theta^{\mathbb{P}}(S) \cap \theta^{\mathbb{P}}(T)$
4. Let $|\mathbb{P}| = n$ and $\varphi \in \mathrm{VF}_{\mathbb{P}}$ with $|M(\varphi)| = m$.
   - (a) What is the number of nonequivalent propositions $\psi$ such that $\varphi \models \psi$ or $\psi \models \varphi$?
   - (b) What is the number of nonequivalent theories over $\mathbb{P}$ in which $\varphi$ is valid? What is the number of nonequivalent *complete* theories over $\mathbb{P}$ in which $\varphi$ is valid?
   - (c) What is the number of nonequivalent theories $T$ over $\mathbb{P}$ such that $T \cup \{\varphi\}$ is satisfiable?
   - (d) Let, moreover, $\{\varphi, \psi\}$ be an unsatisfiable theory with $|M(\psi)| = p$. What is the number of nonequivalent propositions $\chi$ such that $\varphi \vee \psi \models \chi$? What is the number of nonequivalent theories in which $\varphi \vee \psi$ is valid?
5. Let $T = \{q \to (\neg p \to r),\ \neg r \to (\neg p \wedge q),\ (s \to r) \to p\}$ be a theory over the language $\mathbb{P} = \{p, q, r, s\}$.
   - (a) Axiomatize the theory $T$ by a proposition in CNF.
   - (b) Find all models of the theory $T$.
   - (c) Is the theory $T$ an extension of the theory $S = \{q \leftrightarrow \neg r\}$ over the language $\{q, r\}$? Is $T$ a conservative extension of $S$? Justify.
   - (d) Determine the number of mutually inequivalent propositions in the language $\mathbb{P}$ that are *contradictory* in both the theories $T$ and $S$. Justify.
6. Let $T = \{p \vee q \to r, \neg(p \to \neg s)\}$ be a theory over the propositional language $\mathbb{P} = \{p, q, r, s\}$.
   - (a) Is the proposition $q \to p$ valid in the theory $T$? Is it contradictory? Is it independent? Justify.
   - (b) Find all models of the theory $T$.
   - (c) Find a theory $S$ over the language $\mathbb{P}' = \{p, q, r\}$ such that $T$ is a conservative extension of $S$. Axiomatize $S$ by a proposition in CNF. Justify why $S$ has the desired property.
   - (d) Determine the number of mutually inequivalent propositions $\varphi$ over $\mathbb{P}$ such that $\varphi$ is valid in $T$ and independent in $S$.

7. Let $T = \{p,\ \neg q \to \neg r,\ \neg q \to \neg s,\ r \to p,\ \neg s \to \neg p\}$ be a theory over the language $\mathbb{P} = \{p, q, r, s\}$.

   (a) Using the implication graph, show that $T$ is satisfiable.

   (b) Find all models of the theory $T$ and axiomatize $M^{\mathbb{P}}(T)$ by a proposition in CNF.

   (c) Determine, and justify, the number of mutually

      ($i$) inequivalent propositions over $\mathbb{P}$ that are independent in $T$,

      ($ii$) $T$-inequivalent propositions over $\mathbb{P}$ which are independent in $T$.

8. Let $T = \{(r \to p) \to \neg q,\ \neg q \to p, \neg(r \wedge q), r \to \neg s\}$ be a theory in the language $\mathbb{P} = \{p, q, r, s\}$.

   (a) Axiomatize $M^{\mathbb{P}}(T)$ by a proposition in CNF.

   (b) Is the theory $T$ a conservative extension of some theory over the language $\{p, q, r\}$? Justify.

   (c) Determine the number of mutually inequivalent noncontradictory extensions of the theory $T$ over the language $\{p, q, r, s, t\}$. Justify.

9. Let $T = \{p \to \neg q \wedge r, q \vee r, (q \wedge s) \leftrightarrow r\}$ be a theory over the language $\mathbb{P} = \{p, q, r, s\}$.

   (a) Is the proposition $q \to p$ valid in the theory $T$? Is it contradictory? Is it independent? Justify your answers.

   (b) Axiomatize $M(T)$ by a proposition in CNF.

   (c) Determine the number of mutually inequivalent theories $S$ over $\mathbb{P}' = \{r, s\}$ such that $T$ is a conservative extension of $S$. How many of them are complete?