Predicate and Propositional Logic - Seminar 8

Nov 22, 2022

1. Determine whether the following holds for every formula φ.
   (a) $\varphi \models (\forall x)\varphi$
   (b) $\models \varphi \to (\forall x)\varphi$
   (c) $\varphi \models (\exists x)\varphi$
   (d) $\models \varphi \to (\exists x)\varphi$

2. The theory of groups T is of language $L = \langle +, -, 0 \rangle$ with equality where $+$ is a binary function symbol, $-$ is a unary function symbol, 0 is a constant symbol, and has it axioms

$$x + (y + z) = (x + y) + z$$
$$0 + x = x = x + 0$$
$$x + (-x) = 0 = (-x) + x$$

   Are the following formulas valid / contradictory / independent in T?
   (a) $x + y = y + x$
   (b) $x + y = x \ \to \ y = 0$
   (c) $x + y = 0 \ \to \ y = -x$
   (d) $-(x + y) = (-y) + (-x)$

3. Consider a structure $\underline{\mathbb{Z}_4} = \langle \{0, 1, 2, 3\}, +, -, 0 \rangle$ where $+$ is the binary addition modulo 4 and $-$ is the unary function for the *inverse* element of $+$ with respect to the *neutral* element 0.
   (a) Is $\underline{\mathbb{Z}_4}$ a model of the theory T from the previous example (i.e. is it a *group*)?
   (b) Determine all substructures $\underline{\mathbb{Z}_4}\langle a \rangle$ generated by some $a \in \mathbb{Z}_4$.
   (c) Does $\underline{\mathbb{Z}_4}$ contain also other substructures?
   (d) Is every substructure of $\underline{\mathbb{Z}_4}$ a model of T?
   (e) Is every substructure of $\underline{\mathbb{Z}_4}$ elementarily equivalent to $\underline{\mathbb{Z}_4}$?
   (f) Is every substructure of a *commutative* group (i.e. a group that satisfies also 2(a)) again a commutative group?

4. Let $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, -, \cdot, 0, 1 \rangle$ be the structure of rational numbers with standard operations (thus forming a *field*).
   (a) Is there a reduct of $\underline{\mathbb{Q}}$ that is a model of T from the previous exercises?
   (b) Can we expand the reduct $\langle \mathbb{Q}, \cdot, 1 \rangle$ to a model of T?
   (c) Does $\underline{\mathbb{Q}}$ contain a substructure that is not elementary equivalent to $\underline{\mathbb{Q}}$?
   (d) Let $Th(\underline{\mathbb{Q}})$ denote the set of all sentences that are valid in $\underline{\mathbb{Q}}$. Is $Th(\underline{\mathbb{Q}})$ a complete theory?

5. Let $T = \{x = c_1 \lor x = c_2 \lor x = c_3\}$ be a theory of $L = \langle c_1, c_2, c_3 \rangle$ with equality.
   (a) Is T (semantically) consistent?
   (b) Are all models of T elementarily equivalent? That is, is T (semantically) complete?
   (c) Find all simple complete extensions of T.
   (d) Is a theory $T' = T \cup \{x = c_1 \lor x = c_4\}$ of the language $L = \langle c_1, c_2, c_3, c_4 \rangle$ an extension of T? Is T' a simple extension of T? Is T' a conservative extension of T?

6. Let T' be the extension of $T = \{(\exists y)(x + y = 0), (x + y = 0) \wedge (x + z = 0) \to y = z\}$ in $L = \langle +, 0, \leq \rangle$ with equality by definitions of $<$ and unary $-$ with axioms

$$\begin{aligned} -x = y &\quad\leftrightarrow\quad x + y = 0 \\ x < y &\quad\leftrightarrow\quad x \leq y \ \wedge\ \neg(x = y) \end{aligned}$$

Find formulas of L that are equivalent in T' to the following formulas.

(a) $x + (-x) = 0$

(b) $x + (-y) < x$

(c) $-(x + y) < -x$

7. Consider the following database as a relational structure $\mathcal{D} = \langle D, Movies, Program, c^D \rangle_{c \in D}$ of language $L = \langle F, P, c \rangle_{c \in D}$ with equality where $D = \{$'Po strništi bos', 'J. Tříska', 'Mat', '13:15', ... $\}$ and $c^D = c$ for every $c \in D$. Write formulas that define in $\mathcal{D}$ tables of

(a) movies in which a director is acting,

(b) cinemas and times where and when one can see a movie in which a director is acting,

(c) directors that act in movies that are on program in the cinema Mat,

(d) actors or directors whose movie is not on a program in any cinema.

Movie	*name*	*director*	*actor*
	Lidé z Maringotek	M. Frič	J. Tříska
	Po strništi bos	J. Svěrák	Z. Svěrák
	Po strništi bos	J. Svěrák	J. Tříska
	...	...	...

Program	*cinema*	*name*	*time*
	Světozor	Po strništi bos	13:15
	Mat	Po strništi bos	16:15
	Mat	Lidé z Maringotek	18:30
	...	...	...

8. Let $L = \langle F \rangle$ be a language with equality where F is a binary function symbol. Write formulas that define (without parameters) the following sets in the following structures:

(a) the interval $(0, \infty)$ in $\mathcal{A} = \langle \mathbb{R}, \cdot \rangle$ where $\cdot$ is the standard multiplication of real numbers,

(b) the set $\{(x, 1/x) \mid x \neq 0\}$ in the same structure $\mathcal{A}$,

(c) the set of all at most one-element subsets of $\mathbb{N}$ in $\mathcal{B} = \langle \mathcal{P}(\mathbb{N}), \cup \rangle$.