

## Predicate and Propositional Logic - Tutorials 11, 12

Dec 13 and 20, 2022

- Convert the following formulas into the prenex normal form.

- $(\forall y)((\exists x)P(x, y) \rightarrow Q(y, z)) \wedge (\exists y)((\forall x)R(x, y) \vee Q(x, y))$
- $(\exists x)R(x, y) \leftrightarrow (\forall y)P(x, y)$
- $\neg((\forall x)(\exists y)P(x, y) \rightarrow (\exists x)(\exists y)R(x, y)) \wedge (\forall x)\neg(\exists y)Q(x, y)$

- Find Skolem variants of the formulas in PNF from the previous problem.

- Verify that (thus, a Skolem variant does not have to be equivalent to the original formula)

- $\models (\forall x)P(x, f(x)) \rightarrow (\forall x)(\exists y)P(x, y)$
- $\not\models (\forall x)(\exists y)P(x, y) \rightarrow (\forall x)P(x, f(x))$

- The theory  $T$  of *fields* in  $L = \langle +, -, \cdot, 0, 1 \rangle$  contains one axiom  $\varphi$  that is not open:

$$x \neq 0 \rightarrow (\exists y)(x \cdot y = 1).$$

We know that  $T \models 0 \cdot y = 0$  and  $T \models (x \neq 0 \wedge x \cdot y = 1 \wedge x \cdot z = 1) \rightarrow y = z$ .

- Find a Skolem variant  $\varphi_S$  of  $\varphi$  with a new function symbol  $f$ .
  - Let  $T'$  be the theory obtained from  $T$  by replacing  $\varphi$  with  $\varphi_S$ . Is  $T' \models \varphi$ ?
  - Can every model of  $T$  be *uniquely* expanded to a model of  $T'$ ?
- Let  $T$  denote the (previous) theory of fields. Let  $\psi$  be the formula  $x \cdot y = 1 \vee (x = 0 \wedge y = 0)$ .
    - Do the conditions of existence and uniqueness hold in  $T$  for  $\psi(x, y)$  and the variable  $y$ ?
    - Find an extension  $T^*$  of  $T$  by definition of a function symbol  $f$  with the formula  $\psi$ .
    - Is  $T^*$  equivalent to the theory  $T'$  from the previous problem?
    - Find a formula of the original language  $L$  that is equivalent in  $T^*$  to the formula

$$f(x \cdot y) = f(x) \cdot f(y)$$

- Find Herbrand universe and an example of a Herbrand structure for the following languages.

- $L = \langle P, Q, f, a, b \rangle$  where  $P, Q$  are unary resp. binary relation symbols,  $f$  is a unary function symbol,  $a, b$  are constant symbols.
- $L = \langle P, f, g, a \rangle$  where  $P$  is a binary relation s.,  $f, g$  unary function s.,  $a$  constant symbol.

- Find Herbrand models for the following theories or find unsatisfiable conjunctions of ground instances of their axioms. Assume that the language has constant symbols  $a, b$ .

- $T = \{\neg P(x) \vee Q(f(x), y), \neg Q(x, b), P(a)\}$
- $T = \{\neg P(x) \vee Q(f(x), y), Q(x, b), P(a)\}$
- $T = \{P(x, f(x)), \neg P(x, g(x))\}$
- $T = \{P(x, f(x)), \neg P(x, g(x)), P(g(x), f(y)) \rightarrow P(x, y)\}$

- Transform the following formulas to equisatisfiable formulas in clausal form.

- $(\forall y)(\exists x)P(x, y)$
- $\neg(\forall y)(\exists x)P(x, y)$
- $\neg(\exists x)((P(x) \rightarrow P(a)) \wedge (P(x) \rightarrow P(b)))$

$$(d) (\exists x)(\forall y)(\exists z)(P(x, z) \wedge P(z, y) \rightarrow R(x, y))$$

9. Find (all) resolvents of the following pairs of clauses.

$$(a) \{P(x, y), P(y, z)\}, \{\neg P(u, f(u))\}$$

$$(b) \{P(x, x), \neg R(x, f(x))\}, \{R(x, y), Q(y, z)\}$$

$$(c) \{P(x, y), \neg P(x, x), Q(x, f(x), z)\}, \{\neg Q(f(x), x, z), P(x, z)\}$$

10. Show that the following set of clauses is resolution refutable. Describe the resolution refutation by a resolution tree. In each resolution step write down the unification used and underline resolved literals.

$$(a) \{P(a, x, f(y)), P(a, z, f(h(b))), \neg Q(y, z)\}$$

$$(b) \{\neg Q(h(b), w), H(w, a)\}$$

$$(c) \{\neg P(a, w, f(h(b))), H(x, a)\}$$

$$(d) \{P(a, u, f(h(u))), H(u, a), Q(h(b), b)\}$$

$$(e) \{\neg H(v, a)\}$$

11. We know that

(a) If a brick is on (another) brick, then it is not on the ground.

(b) Every brick is on (another) brick or on the ground.

(c) No brick is on a brick that is on (another) brick.

Express these facts in a first-order language and prove by resolution that if a brick is on another brick, the lower brick is on the ground.

12. We know that

(a) Every barber shaves all who do not shave themselves.

(b) No barber shaves someone who shaves himself.

Express these facts in a first-order language and prove by resolution that no barber exists.

13. Consider the following statements about following people on the network:

(i) *Everyone is either followed by someone or is offline (but not both at the same time).*

(ii) *If a person is offline, (s)he follows noone.*

Prove by resolution it follows that:

(iii) *Everyone who is following someone, is also being followed (not necessarily by the same person).*

More specifically:

(a) Formalize the statements (i), (ii), (iii) as sentences  $\varphi_1, \varphi_2, \varphi_3$  (respectively) of language  $L = \langle F, O \rangle$  without equality where  $F$  is binary relation symbol,  $O$  is unary relation symbol and  $F(x, y), O(x)$  means (resp.) that “a person  $x$  follows a person  $y$ ” and “a person  $x$  is offline”.

(b) Applying skolemisation on the sentences from (a) or their negations find an open theory  $T$  (possibly in extended language) such that  $T$  is unsatisfiable if and only if  $\{\varphi_1, \varphi_2\} \models \varphi_3$ .

(c) Transform the theory  $T$  to an equivalent theory  $T'$  that has only clauses as axioms. Write  $T'$  in clausal form.

(d) Prove by resolution that  $T'$  is unsatisfiable. Present your resolution refutation as a resolution tree. For each step write down the unification used.

(e) Find a conjunction of ground instances of axioms of  $T'$  that is unsatisfiable. *Hint: use unifications from (d).*