Propositional and Predicate Logic - IV

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WS 2023/2024

Formal proof systems

We formalize precisely the notion of proof as a syntactical procedure.

In *(standard)* formal proof systems,

- a proof is a finite object, it can be built from axioms of a given theory,
- $T \vdash \varphi$ denotes that φ is *provable* from a theory T,
- if a formula has a proof, it can be found *"algorithmically"*,
  (If T is *"given algorithmically"*.)

We usually require that a formal proof system is

- *sound*, i.e. every formula provable from a theory T is also valid in T,
- *complete*, i.e. every formula valid in T is also provable from T.

Examples of formal proof systems (calculi): tableaux methods, *Hilbert systems, Gentzen systems, natural deduction systems*.

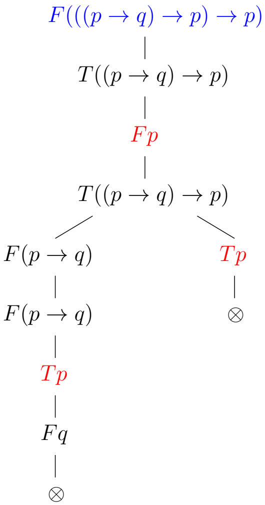
Tableau method - introduction

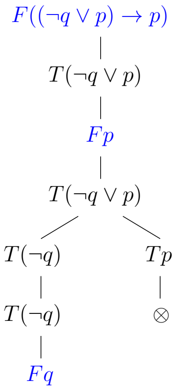
We assume that the language is fixed and countable, i.e. the set $\mathbb{P}$ of propositional letters is countable. Then every theory over $\mathbb{P}$ is countable.

Main features of the tableau method *(informally)*

- a tableau for a formula φ is a binary labeled tree representing systematic search for *counterexample* to φ, i.e. a model of theory is which φ is false,
- a formula is proved if every branch in tableau 'fails', i.e counterexample was not found. In this case the (systematic) tableau will be finite,
- if a counterexample exists, there will be a branch in a (finished) tableau that provides us with this counterexample, but this branch can be infinite.

Introductory examples





Explanation to examples

Nodes in tableaux are labeled by *entries*. An entry is a formula with a *sign* T / F representing an assumption that the formula is true / false in some model. If this assumption is correct, then it is correct also for all the entries in some branch below that came from this entry.

In both examples we have finished (systematic) tableaux from no axioms.

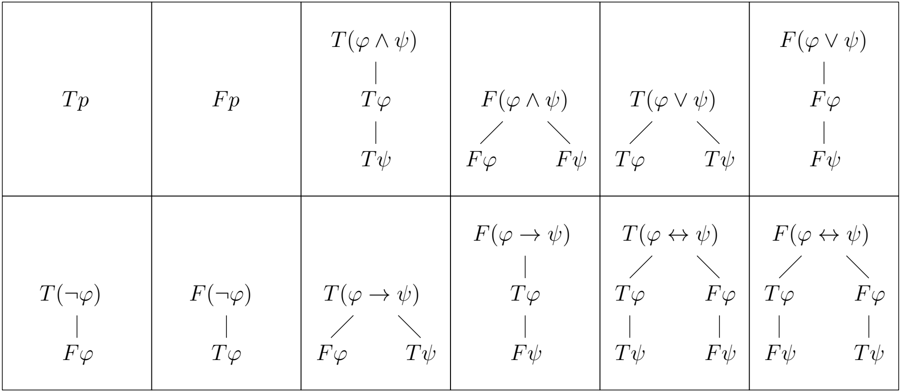
- On the left, there is a *tableau proof* for $((p \to q) \to p) \to p$. All branches *"failed"*, denoted by $\otimes$, as each contains a pair $T\varphi$, $F\varphi$ for some φ *(counterexample was not found)*. Thus the formula is provable, written by

$$\vdash ((p \to q) \to p) \to p$$

- On the right, there is a (finished) tableau for $(\neg q \vee p) \to p$. The left branch did not *"fail"* and is finished (all its entries were considered) *(it provides us with a counterexample $v(p) = v(q) = 0$)*.

Atomic tableaux

An *atomic tableau* is one of the following trees (labeled by entries), where p is any propositional letter and φ, ψ are any propositions.



All tableaux will be formally defined with atomic tableaux and rules how to expand them.

Tableaux

A *finite tableau* is a binary tree labeled with entries described (inductively) by

(i) every atomic tableau is a finite tableau,

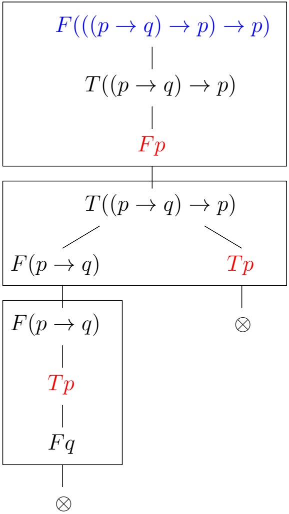
(ii) if P is an entry on a branch V in a finite tableau τ and τ' is obtained from τ by appending the atomic tableaux for P at the end of branch V, then τ' is also a finite tableau,

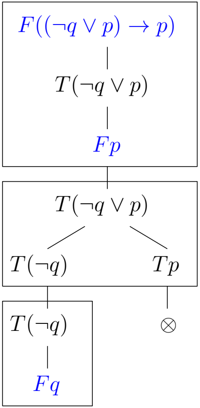
(iii) every finite tableau is formed by a finite number of steps (i), (ii).

A *tableau* is a sequence $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ (finite or infinite) of finite tableaux such that τ_{n+1} is formed from τ_n by an application of (ii), formally $\tau = \cup\tau_n$.

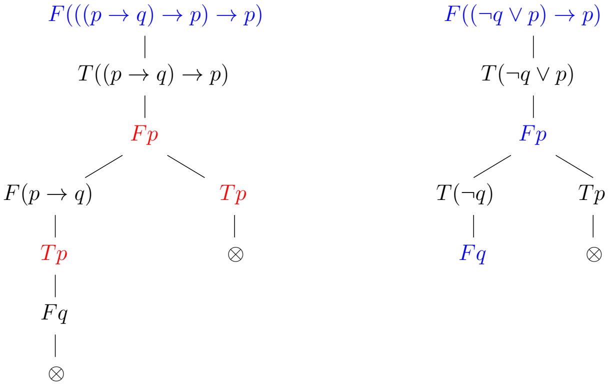
Remark It is not specified how to choose the entry P and the branch V for expansion. This will be specified in systematic tableaux.

Construction of tableaux





Convention



We will not write the entry that is expanded again on the branch.

Remark They will actually be needed later in predicate tableau method.

Tableau proofs

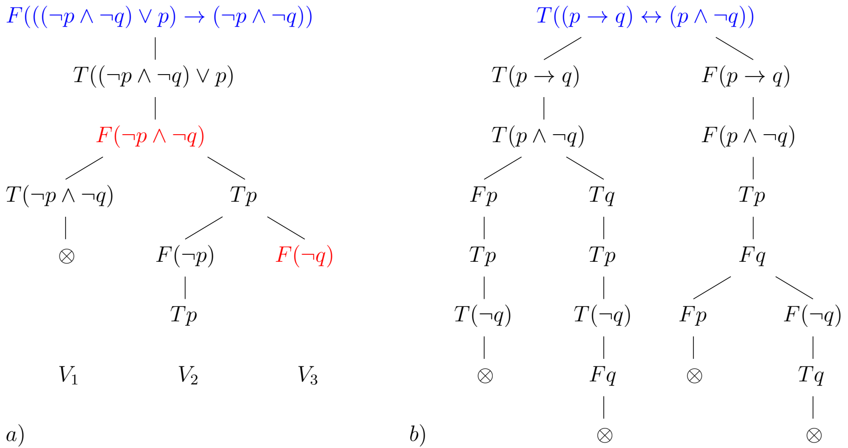
Let P be an entry on a branch V in a tableau τ. We say that

- the entry P is *reduced* on V if it occurs on V as a root of an atomic tableau, i.e. it was already expanded on V during the construction of τ,
- the branch V is *contradictory* if it contains entries $T\varphi$ and $F\varphi$ for some proposition φ, otherwise V is *noncontradictory*. The branch V is *finished* if it is contradictory or every entry on V is already reduced on V,
- the tableau τ is *finished* if every branch in τ is finished, and τ is *contradictory* if every branch in τ is contradictory.

A *tableau proof* (*proof by tableau*) of φ is a contradictory tableau with the root entry $F\varphi$. φ is *(tableau) provable*, denoted by $\vdash \varphi$, if it has a tableau proof.

Similarly, a *refutation* of φ by *tableau* is a contradictory tableau with the root entry $T\varphi$. φ is *(tableau) refutable* if it has a refutation by tableau, i.e. $\vdash \neg\varphi$.

Examples



a) $F(\neg p \wedge \neg q)$ not reduced on V_1, V_1 contradictory, V_2 finished, V_3 unfinished,

b) a (tableau) refutation of φ: $(p \to q) \leftrightarrow (p \wedge \neg q)$, i.e. $\vdash \neg\varphi$.

Tableau from a theory

How to add axioms of a given theory into a proof?

A *finite tableau from a theory* T is generalized tableau with an additional rule

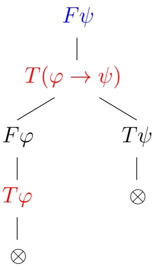
(ii)' if V is a branch of a finite tableau (from T) and $\varphi \in T$, then by appending $T\varphi$ at the end of V we obtain a finite tableau from T.

We generalize other definitions by appending *"from T"*.

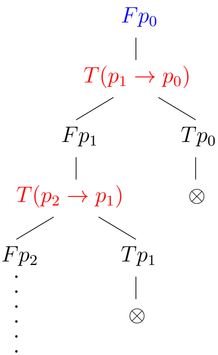
- a *tableau from T* is a sequence $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ of finite tableaux from T such that τ_{n+1} is formed from τ_n applying (ii) or (ii)', formally $\tau = \cup \tau_n$,
- a *tableau proof* of φ *from T* is a contradictory tableaux from T with $F\varphi$ in the root. $T \vdash \varphi$ denotes that φ is *(tableau) provable from T*.
- a *refutation* of φ by a *tableau from T* is a contradictory tableau from T with the root entry $T\varphi$.

Unlike in previous definitions, a branch V of a tableau from T is *finished*, if it is contradictory, or every entry on V is already reduced on V and, moreover, V contains $T\varphi$ for every $\varphi \in T$.

Examples of tableaux from theories



$a)$



$b)$

a) A tableau proof of ψ from $T = \{\varphi, \varphi \to \psi\}$, so $T \vdash \psi$.

b) A finished tableau with the root Fp_0 from $T = \{p_{n+1} \to p_n \mid n \in \mathbb{N}\}$.
All branches are finished, the leftmost branch is noncontradictory and infinite. It provides us with the (only one) model of T in which p_0 is false.

Systematic tableaux

We describe a systematic construction that leads to a finished tableau.

Let R be an entry and $T = \{\varphi_0, \varphi_1, \dots\}$ be a (possibly infinite) theory.

(1) We take the atomic tableau for R as τ_0. Till possible, proceed as follows.

(2) Let P be the leftmost entry in the smallest level as possible of the tableau τ_n s.t. P is not reduced on some noncontradictory branch through P.

(3) Let τ_n' be the tableau obtained from τ_n by appending the atomic tableau for P to every noncontradictory branch through P. (If P does not exists, we take $\tau_n' = \tau_n$.)

(4) Let τ_{n+1} be the tableau obtained from τ_n' by appending $T\varphi_n$ to every noncontradictory branch that does not contain $T\varphi_n$ yet. (If φ_n does not exists, we take $\tau_{n+1} = \tau_n'$.)

The *systematic tableau* from T for the entry R is the result of the above construction, i.e. $\tau = \cup\tau_n$.

Systematic tableau - being finished

Proposition *Every systematic tableau is finished.*

Proof Let $\tau = \cup \tau_n$ be a systematic tableau from $T = \{\varphi_0, \varphi_1, \ldots\}$ with root R.

- If a branch is noncontradictory in τ, its prefix in every τ_n is noncontradictory as well.
- If an entry P in unreduced on some branch in τ, it is unreduced on its prefix in every τ_n as well (assuming P occurs on this prefix).
- There are only finitely many entries in τ in levels up to the level of P.
- Thus, if P was unreduced on some noncontradictory branch in τ, it would be considered in some step (2) and reduced by step (3).
- By step (4) every $\varphi_n \in T$ will be (no later than) in τ_{n+1} on every noncontradictory branch.
- Hence the systematic tableau τ has all branches finished. □

Finiteness of proofs

Proposition *For every contradictory tableau $\tau = \cup\tau_n$ there is some n such that τ_n is a contradictory finite tableau.*

- *Proof* Let S be the set of nodes in τ that have no pair of contradictory entries $T\varphi$, $F\varphi$ amongst their predecessors.
- If S was infinite, then by König's lemma, the subtree of τ induced by S would contain an infinite brach, and thus τ would not be contradictory.
- Since S is finite, for some m all nodes of S belong to levels up to m.
- Thus every node in level $m+1$ has a pair of contradictory entries amongst its predecessors.
- Let n be such that τ_n agrees with τ at least up to the level $m+1$.
- Then every branch in τ_n is contradictory. □

Corollary *If a systematic tableau (from a theory) is a proof, it is finite.*

Proof In its construction, only noncontradictory branches are extended. □

Soundness

We say that an entry P *agrees* with an assignment v, if P is $T\varphi$ and $\overline{v}(\varphi) = 1$, or if P is $F\varphi$ and $\overline{v}(\varphi) = 0$. A branch V *agrees* with v, if every entry on V agrees with v.

Lemma *Let v be a model of a theory T that agrees with the root entry of a tableau $\tau = \cup\tau_n$ from T. Then τ contains a branch that agrees with v.*

Proof By induction we find a sequence $V_0, V_1, \ldots$ so that for every n, V_n is a branch in τ_n agreeing with v and V_n is contained in V_{n+1}.

- By considering all atomic tableaux we verify that base of induction holds.
- If τ_{n+1} is obtained from τ_n without extending V_n, we put $V_{n+1} = V_n$.
- If τ_{n+1} is obtained from τ_n by appending $T\varphi$ to V_n for some $\varphi \in T$, let V_{n+1} be this branch. Since v is a model of φ, V_{n+1} agrees with v.
- Otherwise τ_{n+1} is obtained from τ_n by appending the atomic tableau for some entry P on V_n to the end of V_n. Since P agrees with v and atomic tableaux are verified, V_n can be extended to V_{n+1} as required. □

Theorem on soundness

We will show that the tableau method in propositional logic is sound.

Theorem *For every theory T and proposition φ, if φ is tableau provable from T, then φ is valid in T, i.e. $T \vdash \varphi \;\Rightarrow\; T \models \varphi$.*

Proof

- Let φ be tableau provable from a theory T, i.e. there is a contradictory tableau τ from T with the root entry $F\varphi$.
- Suppose for a contradiction that φ is not valid in T, i.e. there exists a model v of the theory T if which φ is false (a counterexample).
- Since the root entry $F\varphi$ agrees with v, by the previous lemma, there is a branch in the tableau τ that agrees with v.
- But this is impossible, since every branch of τ is contradictory, i.e. it contains a pair of entries $T\psi$, $F\psi$ for some ψ. $\square$

Completeness

A noncontradictory branch in a finished tableau gives us a counterexample.

Lemma *Let V be a noncontradictory branch of a finished tableau τ. Then V agrees with the following assignment v.*

$$v(p) = \begin{cases} 1 & \text{if } Tp \text{ occurs on } V \\ 0 & \text{otherwise} \end{cases}$$

Proof By induction on the structure of formulas in entries occurring on V.

- For an entry Tp on V, where p is a letter, we have $\overline{v}(p) = 1$ by definition.
- For an entry Fp on V, Tp in not on V since V is noncontradictory, thus $\overline{v}(p) = 0$ by definition of v.
- For an entry $T(\varphi \wedge \psi)$ on V, we have $T\varphi$ and $T\psi$ on V since τ is finished. By induction, we have $\overline{v}(\varphi) = \overline{v}(\psi) = 1$, and thus $\overline{v}(\varphi \wedge \psi) = 1$.
- For an entry $F(\varphi \wedge \psi)$ on V, we have $F\varphi$ or $F\psi$ on V since τ is finished. By induction, we have $\overline{v}(\varphi) = 0$ or $\overline{v}(\psi) = 0$, and thus $\overline{v}(\varphi \wedge \psi) = 0$.
- For other entries similarly as in previous two cases. □

Theorem on completeness

We will show that the tableau method in propositional logic is complete.

Theorem *For every theory T and proposition φ, if φ is valid in T, then φ is tableau provable from T, i.e. $T \models \varphi \Rightarrow T \vdash \varphi$.*

Proof Let φ be valid in T. We will show that an arbitrary finished tableau (e.g. *systematic*) τ from theory T with the root entry $F\varphi$ is contradictory.

- If not, let V be some noncontradictory branch in τ.
- By the previous lemma, there exists an assignment v such that V agrees with v, in particular in the root entry $F\varphi$, i.e. $\overline{v}(\varphi) = 0$.
- Since V is finished, it contains $T\psi$ for every $\psi \in T$.
- Thus v is a model of theory T (since V agrees with v).
- But this contradicts the assumption that φ is valid in T.

Hence the tableau τ is a proof of φ from T. □

Properties of theories

We introduce syntactic variants of previous semantically defined notions.

Let T be a theory over $\mathbb{P}$. If φ is provable from T, we say that φ is a *theorem* of T. The set of theorems of T is denoted by

$$\mathrm{Thm}^{\mathbb{P}}(T) = \{\varphi \in \mathrm{VF}_{\mathbb{P}} \mid T \vdash \varphi\}.$$

We say that a theory T is

- *inconsistent* if $T \vdash \bot$, otherwise T is *consistent*,
- *complete* if it is consistent and every proposition is provable or refutable from T, i.e. $T \vdash \varphi$ or $T \vdash \neg\varphi$ for every $\varphi \in \mathrm{VF}_{\mathbb{P}}$,
- *extension* of a theory T' over $\mathbb{P}'$ if $\mathbb{P}' \subseteq \mathbb{P}$ and $\mathrm{Thm}^{\mathbb{P}'}(T') \subseteq \mathrm{Thm}^{\mathbb{P}}(T)$; we say that an extension T of a theory T' is *simple* if $\mathbb{P} = \mathbb{P}'$; and *conservative* if $\mathrm{Thm}^{\mathbb{P}'}(T') = \mathrm{Thm}^{\mathbb{P}}(T) \cap \mathrm{VF}_{\mathbb{P}'}$,
- *equivalent* with a theory T' if T is an extension of T' and vice-versa.

Corollaries

From the soundness and completeness of the tableau method it follows that these syntactic definitions agree with their semantic variants.

Corollary *For every theory T and propositions φ, ψ over $\mathbb{P}$,*

- $T \vdash \varphi$ *if and only if* $T \models \varphi$,
- $\mathrm{Thm}^{\mathbb{P}}(T) = \theta^{\mathbb{P}}(T)$,
- T *is inconsistent if and only if* T *is unsatisfiable, i.e. it has no model,*
- T *is complete if and only if* T *is semantically complete, i.e. it has a single model,*
- $T, \varphi \vdash \psi$ *if and only if* $T \vdash \varphi \to \psi$ *(Deduction theorem).*

Remark Deduction theorem can be proved directly by transformations of tableaux.

Theorem on compactness

Theorem *A theory T has a model iff every finite subset of T has a model.*

Proof 1 The implication from left to right is obvious. If T has no model, then it is inconsistent, i.e. $\perp$ is provable by a systematic tableau τ from T. Since τ is finite, $\perp$ is provable from some finite $T' \subseteq T$, i.e. T' has no model. □

Remark *This proof is based on finiteness of proofs, soundness and completeness. We present an alternative proof (applying König's lemma).*

Proof 2 Let $T = \{\varphi_i \mid i \in \mathbb{N}\}$. Consider a tree S on (certain) finite binary strings σ ordered by being a prefix. We put $\sigma \in S$ if and only if there exists an assignment v with prefix σ such that $v \models \varphi_i$ for every $i \leq \mathrm{lth}(\sigma)$.

Observation *S has an infinite branch if and only if T has a model.*

Since $\{\varphi_i \mid i \in n\} \subseteq T$ has a model for every $n \in \mathbb{N}$, every level in S is nonempty. Thus S is infinite and moreover binary, hence by König's lemma, S contains an infinite branch. □

Application of compactness

A graph (V, E) is *k-colorable* if there exists $c\colon V \to \{1, \dots, k\}$ such that $c(u) \neq c(v)$ for every edge $\{u, v\} \in E$.

Theorem *A countably infinite graph $G = (V, E)$ is k-colorable if and only if every finite subgraph of G is k-colorable.*

Proof The implication $\Rightarrow$ is obvious. Assume that every finite subgraph of G is k-colorable. Consider $\mathbb{P} = \{p_{u,i} \mid u \in V, 1 \leq i \leq k\}$ and a theory T with axioms

$$\begin{aligned} p_{u,1} \vee \dots \vee p_{u,k} & \qquad \text{for every } u \in V, \\ \neg(p_{u,i} \wedge p_{u,j}) & \qquad \text{for every } u \in V, i < j \leq k, \\ \neg(p_{u,i} \wedge p_{v,i}) & \qquad \text{for every } \{u, v\} \in E, i \leq k. \end{aligned}$$

Then G is k-colorable if and only if T has a model. By compactness, it suffices to show that every finite $T' \subseteq T$ has a model. Let G' be the subgraph of G induced by vertices u such that $p_{u,i}$ appears in T' for some i. Since G' is k-colorable by the assumption, the theory T' has a model. $\square$