# Propositional and Predicate Logic - VI

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# Hilbert's calculus

- basic connectives: $\neg$, $\rightarrow$ (others can be defined from them)
- *logical axioms* (schemes of axioms):

$$
\begin{array}{rc}
(i) & \varphi \rightarrow (\psi \rightarrow \varphi) \\
(ii) & (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)) \\
(iii) & (\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)
\end{array}
$$

  where $\varphi$, $\psi$, $\chi$ are any propositions (of a given language).
- *a rule of inference*:

$$\frac{\varphi,\ \varphi \rightarrow \psi}{\psi} \qquad \text{(modus ponens)}$$

A *proof* (in *Hilbert-style*) of a formula $\varphi$ from a theory $T$ is a finite sequence $\varphi_0, \ldots, \varphi_n = \varphi$ of formulas such that for every $i \leq n$

- $\varphi_i$ is a logical axiom or $\varphi_i \in T$ (an axiom of the theory), or
- $\varphi_i$ can be inferred from the previous formulas applying a rule of inference.

*Remark* *Choice of axioms and inference rules differs in various Hilbert-style proof systems.*

# Example and soundness

A formula $\varphi$ is *provable* from $T$ if it has a proof from $T$, denoted by $T \vdash_H \varphi$.
If $T = \emptyset$, we write $\vdash_H \varphi$. E.g. for $T = \{\neg\varphi\}$ we have $T \vdash_H \varphi \to \psi$ for every $\psi$.

| | | |
|---|---|---|
| $1)$ | $\neg\varphi$ | an axiom of $T$ |
| $2)$ | $\neg\varphi \to (\neg\psi \to \neg\varphi)$ | a logical axiom $(i)$ |
| $3)$ | $\neg\psi \to \neg\varphi$ | by modus ponens from 1), 2) |
| $4)$ | $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$ | a logical axiom $(iii)$ |
| $5)$ | $\varphi \to \psi$ | by modus ponens from 3), 4) |

**Theorem** *For every theory $T$ and formula $\varphi$, $T \vdash_H \varphi \;\Rightarrow\; T \models \varphi$.*

*Proof*

- If $\varphi$ is an axiom (logical or from $T$), then $T \models \varphi$ (l. axioms are tautologies),
- if $T \models \varphi$ and $T \models \varphi \to \psi$, then $T \models \psi$, i.e. modus ponens is sound,
- thus every formula in a proof from $T$ is valid in $T$. □

*Remark* *The completeness holds as well, i.e. $T \models \varphi \Rightarrow T \vdash_H \varphi$.*

# Predicate logic

*Deals with statements about objects, their properties and relations.*

*"She is intelligent and her father knows the rector."* $I(x) \wedge K(f(x), r)$

- $x$ is a variable, representing an object,
- $r$ is a constant symbol, representing a particular object,
- $f$ is a function symbol, representing a function,
- $I$, $K$ are relation (predicate) symbols, representing relations (the property of *"being intelligent"* and the relation *"to know"*).

*"Everybody has a father."* $(\forall x)(\exists y)(y = f(x))$

- $(\forall x)$ is the universal quantifier (*for every* $x$),
- $(\exists y)$ is the existential quantifier (*there exists* $y$),
- $=$ is a (binary) relation symbol, representing the identity relation.

# Language

A first-order language consists of

- variables $x, y, z, \dots, x_0, x_1, \dots$ (countable many), the set of all variables is denoted by $\mathrm{Var}$,
- function symbols $f, g, h, \dots$, including constant symbols $c, d, \dots$, which are nullary function symbols,
- relation (predicate) symbols $P, Q, R, \dots$, eventually the symbol $=$ (equality) as a special relation symbol,
- quantifiers $(\forall x)$, $(\exists x)$ for every variable $x \in \mathrm{Var}$,
- logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
- parentheses ( , )

Every function and relation symbol $S$ has an associated *arity* $\mathrm{ar}(S) \in \mathbb{N}$.

*Remark Compared to propositional logic we have no (explicit) propositional variables, but they can be introduced as nullary relation symbols.*

# Signatures

- *Symbols of logic* are variables, quantifiers, connectives and parentheses.
- *Non-logical symbols* are function and relation symbols except the equality symbol. The equality is (usually) considered separately.
- A *signature* is a pair $\langle \mathcal{R}, \mathcal{F} \rangle$ of disjoint sets of relation and function symbols with associated arities, whereas none of them is the equality symbol. A signature lists all non-logical symbols.
- A *language* is determined by a signature $L = \langle \mathcal{R}, \mathcal{F} \rangle$ and by specifying whether it is a language with equality or not. A language must contain at least one relation symbol (non-logical or the equality).

*Remark* *The meaning of symbols in a language is not assigned, e.g. the symbol $+$ does not have to represent the standard addition.*

# Examples of languages

*We describe a language by a list of all non-logical symbols with eventual clarification of arity and whether they are relation or function symbols.*

The following examples of languages are all with equality.

- $L = \langle\ \rangle$ is the language of pure equality,
- $L = \langle c_i \rangle_{i \in \mathbb{N}}$ is the language of countable many constants,
- $L = \langle \leq \rangle$ is the language of orderings,
- $L = \langle E \rangle$ is the language of the graph theory,
- $L = \langle +, -, 0 \rangle$ is the language of the group theory,
- $L = \langle +, -, \cdot, 0, 1 \rangle$ is the language of the field theory,
- $L = \langle -, \wedge, \vee, 0, 1 \rangle$ is the language of Boolean algebras,
- $L = \langle S, +, \cdot, 0, \leq \rangle$ is the language of arithmetic,

where $c_i$, $0$, $1$ are constant symbols, $S$, $-$ are unary function symbols, $+$, $\cdot$ , $\wedge$, $\vee$ are binary function symbols, $E$, $\leq$ are binary relation symbols.

# Terms

*Are expressions representing values of (composed) functions.*

*Terms* of a language $L$ are defined inductively by

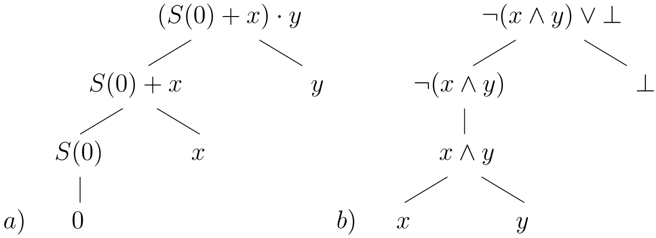
$(i)$ every variable or constant symbol in $L$ is a term,

$(ii)$ if $f$ is a function symbol in $L$ of arity $n > 0$ and $t_1, \ldots, t_n$ are terms, then also the expression $f(t_1, \ldots, t_n)$ is a term,

$(iii)$ every term is formed by a finite number of steps $(i)$, $(ii)$.

- A *ground term* is a term with no variables.
- The set of all terms of a language $L$ is denoted by $\mathrm{Term}_L$.
- A term that is a part of another term $t$ is called a *subterm* of $t$.
- The structure of terms can be represented by their formation trees.
- For binary function symbols we often use infix notation, e.g. we write $(x + y)$ instead of $+(x, y)$.

# Examples of terms




$a)$ The formation tree of the term $(S(0) + x) \cdot y$ of the language of arithmetic.

$b)$ Propositional formulas only with connectives $\neg$, $\wedge$, $\vee$, eventually with constants $\top$, $\bot$ can be viewed as terms of the language of Boolean algebras.

# Atomic formulas

*Are the simplest formulas.*

- An *atomic formula* of a language $L$ is an expression $R(t_1, \ldots, t_n)$ where $R$ is an $n$-ary relation symbol in $L$ and $t_1, \ldots, t_n$ are terms of $L$.
- The set of all atomic formulas of a language $L$ is denoted by $\mathrm{AFm}_L$.
- The structure of an atomic formula can be represented by a formation tree from the formation subtrees of its terms.
- For binary relation symbols we often use infix notation, e.g. $t_1 = t_2$ instead of $=(t_1, t_2)$ or $t_1 \leq t_2$ instead of $\leq(t_1, t_2)$.
- *Examples of atomic formulas*

$$K(f(x), r), \quad x \cdot y \leq (S(0) + x) \cdot y, \quad \neg(x \wedge y) \vee \bot = \bot.$$

# Formula

*Formulas* of a language $L$ are defined inductively by

$(i)$ every atomic formula is a formula,

$(ii)$ if $\varphi$, $\psi$ are formulas, then also the following expressions are formulas

$$(\neg\varphi)\ ,(\varphi \wedge \psi)\ ,(\varphi \vee \psi)\ ,(\varphi \rightarrow \psi)\ ,(\varphi \leftrightarrow \psi),$$

$(iii)$ if $\varphi$ is a formula and $x$ is a variable, then also the expressions $((\forall x)\varphi)$ and $((\exists x)\varphi)$ are formulas.

$(iv)$ every formula is formed by a finite number of steps $(i)$, $(ii)$, $(iii)$.

- The set of all formulas of a language $L$ is denoted by $\mathrm{Fm}_L$.
- A formula that is a part of another formula $\varphi$ is called a *subformula* of $\varphi$.
- The structure of formulas can be represented by their formation trees.

# Conventions

- After introducing *priorities* for binary function symbols e.g. $+$ , $\cdot$ we are in infix notation allowed to omit parentheses that are around a subterm formed by a symbol of higher priority, e.g. $x \cdot y + z$ instead of $(x \cdot y) + z$.
- After introducing *priorities* for connectives and quantifiers we are allowed to omit parentheses that are around subformulas formed by connectives of higher priority.

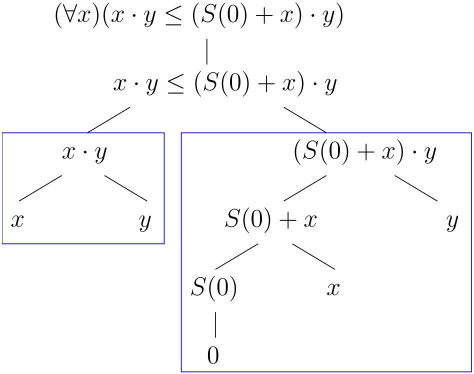
$$(1)\ \ \rightarrow,\ \leftrightarrow \qquad (2)\ \ \wedge,\ \vee \qquad (3)\ \ \neg,\ (\forall x),\ (\exists x)$$

- They can be always omitted around subformulas formed by $\neg$, $(\forall x)$, $(\exists x)$.
- We may also omit parentheses in $(\forall x)$ and $(\exists x)$ for every $x \in \mathrm{Var}$.
- The outer parentheses may be omitted as well.

$$(((\neg((\forall x)R(x))) \wedge ((\exists y)P(y))) \rightarrow (\neg(((\forall x)R(x)) \vee (\neg((\exists y)P(y))))))$$

$$\neg(\forall x)R(x) \wedge (\exists y)P(y) \rightarrow \neg((\forall x)R(x) \vee \neg(\exists y)P(y))$$

# An example of a formula




The formation tree of the formula $(\forall x)(x \cdot y \leq (S(0) + x) \cdot y)$.

# Occurrences of variables

Let $\varphi$ be a formula and $x$ be a variable.

- An *occurrence* of $x$ in $\varphi$ is a leaf labeled by $x$ in the formation tree of $\varphi$.
- An occurrence of $x$ in $\varphi$ is *bound* if it is in some subformula $\psi$ that starts with $(\forall x)$ or $(\exists x)$. An occurrence of $x$ in $\varphi$ is *free* if it is not bound.
- A variable $x$ is *free* in $\varphi$ if it has at least one free occurrence in $\varphi$. It is *bound* in $\varphi$ if it has at least one bound occurrence in $\varphi$.
- A variable $x$ can be both free and bound in $\varphi$. For example in

$$(\forall x)(\exists y)(x \leq y) \vee x \leq z.$$

- We write $\varphi(x_1, \ldots, x_n)$ to denote that $x_1, \ldots, x_n$ are all free variables in the formula $\varphi$. *($\varphi$ states something about these variables.)*

*Remark* *We will see that the truth value of a formula (in a given interpretation of symbols) depends only on the assignment of free variables.*

# Open and closed formulas

- A formula is *open* if it is without quantifiers. For the set $\mathrm{OFm}_L$ of all open formulas in a language $L$ it holds that $\mathrm{AFm}_L \subsetneq \mathrm{OFm}_L \subsetneq \mathrm{Fm}_L$.
- A formula is *closed* (a *sentence*) if it has no free variable; that is, all occurrences of variables are bound.
- A formula can be both open and closed. In this case, all its terms are ground terms.

| | |
|---|---|
| $x + y \leq 0$ | *open*, $\varphi(x, y)$ |
| $(\forall x)(\forall y)(x + y \leq 0)$ | *a sentence*, |
| $(\forall x)(x + y \leq 0)$ | *neither open nor a sentence*, $\varphi(y)$ |
| $1 + 0 \leq 0$ | *open sentence* |

*Remark* *We will see that in a fixed interpretation of symbols a sentence has a fixed truth value; that is, it does not depend on the assignment of variables.*

# Instances

*After substituting a term $t$ for a free variable $x$ in a formula $\varphi$, we would expect that the new formula (newly) says about $t$ "the same" as $\varphi$ did about $x$.*

| | | |
|---|---|---|
| $\varphi(x)$ | $(\exists y)(x + y = 1)$ | *"there is an element $1 - x$"* |
| *for $t = 1$ we can $\varphi(x/t)$* | $(\exists y)(1 + y = 1)$ | *"there is an element $1 - 1$"* |
| *for $t = y$ we cannot* | $(\exists y)(y + y = 1)$ | *"1 is divisible by 2"* |

- A term $t$ is *substitutable* for a variable $x$ in a formula $\varphi$ if substituting $t$ for all free occurrences of $x$ in $\varphi$ does not introduce a new bound occurrence of a variable from $t$.
- Then we denote the obtained formula $\varphi(x/t)$ and we call it an *instance* of the formula $\varphi$ after a *substitution* of a term $t$ for a variable $x$.
- $t$ is not substitutable for $x$ in $\varphi$ if and only if $x$ has a free occurrence in some subformula that starts with $(\forall y)$ or $(\exists y)$ for some variable $y$ in $t$.
- Ground terms are always substitutable.

# Variants

*Quantified variables can be (under certain conditions) renamed so that we obtain an equivalent formula.*

Let $(Qx)\psi$ be a subformula of $\varphi$ where $Q$ means $\forall$ or $\exists$ and $y$ is a variable such that the following conditions hold.

1) $y$ is substitutable for $x$ in $\psi$, and
2) $y$ does not have a free occurrence in $\psi$.

Then by replacing the subformula $(Qx)\psi$ with $(Qy)\psi(x/y)$ we obtain a *variant* of $\varphi$ *in subformula* $(Qx)\psi$. After variation of one or more subformulas in $\varphi$ we obtain a *variant* of $\varphi$. *For example,*

| | |
|---|---|
| $(\exists x)(\forall y)(x \leq y)$ | *is a formula* $\varphi$, |
| $(\exists u)(\forall v)(u \leq v)$ | *is a variant of* $\varphi$, |
| $(\exists y)(\forall y)(y \leq y)$ | *is not a variant of* $\varphi$, $1)$ *does not hold*, |
| $(\exists x)(\forall x)(x \leq x)$ | *is not a variant of* $\varphi$, $2)$ *does not hold*. |

# Structures

- $\underline{S} = \langle S, \leq \rangle$ is an ordered set where $\leq$ is reflexive, antisymmetric, transitive binary relation on $S$,
- $G = \langle V, E \rangle$ is an undirected graph without loops where $V$ is the set of *vertices* and $E$ is irreflexive, symmetric binary relation on $V$ *(adjacency)*,
- $\underline{\mathbb{Z}_p} = \langle \mathbb{Z}_p, +, -, 0 \rangle$ is the additive group of integers modulo $p$,
- $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, -, \cdot, 0, 1 \rangle$ is the field of rational numbers,
- $\underline{\mathcal{P}(X)} = \langle \mathcal{P}(X), -, \cap, \cup, \emptyset, X \rangle$ is the set algebra over $X$,
- $\underline{\mathbb{N}} = \langle \mathbb{N}, S, +, \cdot, 0, \leq \rangle$ is the standard model of arithmetic,
- finite automata and other models of computation,
- relational databases, ...

# A structure for a language

Let $L = \langle \mathcal{R}, \mathcal{F} \rangle$ be a signature of a language and $A$ be a nonempty set.

- A *realization* (*interpretation*) of a *relation symbol* $R \in \mathcal{R}$ on $A$ is any relation $R^A \subseteq A^{\mathrm{ar}(R)}$. A realization of $=$ on $A$ is the relation $Id_A$ (identity).
- A *realization* (*interpretation*) of a *function symbol* $f \in \mathcal{F}$ on $A$ is any function $f^A \colon A^{\mathrm{ar}(f)} \to A$. Thus a realization of a constant symbol is some element of $A$.

A *structure* for the language $L$ (*$L$-structure*) is a triple $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$, where

- $A$ is nonempty set, called the *domain* of the structure $\mathcal{A}$,
- $\mathcal{R}^A = \langle R^A \mid R \in \mathcal{R} \rangle$ is a collection of realizations of relation symbols,
- $\mathcal{F}^A = \langle f^A \mid f \in \mathcal{F} \rangle$ is a collection of realizations of function symbols.

A structure for the language $L$ is also called a *model of the language $L$*. The class of all models of $L$ is denoted by $M(L)$. *Examples for $L = \langle \leq \rangle$ are*

$$\langle \mathbb{N}, \leq \rangle,\ \langle \mathbb{Q}, > \rangle,\ \langle X, E \rangle,\ \langle \mathcal{P}(X), \subseteq \rangle.$$

# Value of terms

Let $t$ be a term of $L = \langle \mathcal{R}, \mathcal{F} \rangle$ and $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ be an $L$-structure.

- A *variable assignment* over the domain $A$ is a function $e\colon \mathrm{Var} \to A$.
- The *value* $t^A[e]$ of the term $t$ in the structure $\mathcal{A}$ with respect to the assignment $e$ is defined by

$$x^A[e] = e(x) \quad \text{for every } x \in \mathrm{Var},$$
$$(f(t_1, \ldots, t_n))^A[e] = f^A(t_1^A[e], \ldots, t_n^A[e]) \quad \text{for every } f \in \mathcal{F}.$$

- In particular, for a constant symbol $c$ we have $c^A[e] = c^A$.
- If $t$ is a ground term, its value in $\mathcal{A}$ is independent on the assignment $e$.
- The value of $t$ in $\mathcal{A}$ depends only on the assignment of variables in $t$.

*For example, the value of the term* $x + 1$ *in the structure* $\mathcal{N} = \langle \mathbb{N}, +, 1 \rangle$ *with respect to the assignment* $e$ *with* $e(x) = 2$ *is* $(x + 1)^N[e] = 3$.

# Values of atomic formulas

Let $\varphi$ be an atomic formula of $L = \langle \mathcal{R}, \mathcal{F} \rangle$ in the form $R(t_0, \dots, t_{n-1})$, $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ be an $L$-structure, and $e$ be a variable assignment over $A$.

- The *value* $H_{at}^{A}(\varphi)[e]$ of the formula $\varphi$ in the structure $\mathcal{A}$ with respect to $e$ is

$$H_{at}^{A}(R(t_1, \dots, t_n))[e] = \begin{cases} 1 & \text{if } (t_1^A[e], \dots, t_n^A[e]) \in R^A, \\ 0 & \text{otherwise.} \end{cases}$$

  where $=^A$ is $\mathrm{Id}_A$; that is, $H_{at}^{A}(t_1 = t_2)[e] = 1$ if $t_1^A[e] = t_2^A[e]$, and $H_{at}^{A}(t_1 = t_2)[e] = 0$ otherwise.
- If $\varphi$ is a sentence; that is, all its terms are ground, then its value in $\mathcal{A}$ is independent on the assignment $e$.
- The value of $\varphi$ in $\mathcal{A}$ depends only on the assignment of variables in $\varphi$.

*For example, the value of $\varphi$ in form $x + 1 \leq 1$ in $\mathcal{N} = \langle \mathbb{N}, +, 1, \leq \rangle$ with respect to the assignment $e$ is $H_{at}^{N}(\varphi)[e] = 1$ if and only if $e(x) = 0$.*

# Values of formulas

The *value* $H^A(\varphi)[e]$ of the formula $\varphi$ in the structure $\mathcal{A}$ with respect to $e$ is

$$
\begin{aligned}
H^A(\varphi)[e] &= H^A_{at}(\varphi)[e] \text{ if } \varphi \text{ is atomic,}\\
H^A(\neg\varphi)[e] &= -_1(H^A(\varphi)[e])\\
H^A(\varphi \wedge \psi)[e] &= \wedge_1(H^A(\varphi)[e], H^A(\psi)[e])\\
H^A(\varphi \vee \psi)[e] &= \vee_1(H^A(\varphi)[e], H^A(\psi)[e])\\
H^A(\varphi \rightarrow \psi)[e] &= \rightarrow_1(H^A(\varphi)[e], H^A(\psi)[e])\\
H^A(\varphi \leftrightarrow \psi)[e] &= \leftrightarrow_1(H^A(\varphi)[e], H^A(\psi)[e])\\
H^A((\forall x)\varphi)[e] &= \min_{a \in A}(H^A(\varphi)[e(x/a)])\\
H^A((\exists x)\varphi)[e] &= \max_{a \in A}(H^A(\varphi)[e(x/a)])
\end{aligned}
$$

where $-_1$, $\wedge_1$, $\vee_1$, $\rightarrow_1$, $\leftrightarrow_1$ are the Boolean functions given by the tables and $e(x/a)$ for $a \in A$ denotes the assignment obtained from $e$ by setting $e(x) = a$.

*Observation* $H^A(\varphi)[e]$ *depends only on the assignment of free variables in* $\varphi$.

# Satisfiability with respect to assignments

The structure $\mathcal{A}$ satisfies the formula $\varphi$ with assignment $e$ if $H^A(\varphi)[e] = 1$. Then we write $\mathcal{A} \models \varphi[e]$, and $\mathcal{A} \not\models \varphi[e]$ otherwise. It holds that

$$\begin{array}{lcl}
\mathcal{A} \models \neg\varphi[e] & \Leftrightarrow & \mathcal{A} \not\models \varphi[e] \\
\mathcal{A} \models (\varphi \wedge \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \text{ and } \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\varphi \vee \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \text{ or } \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\varphi \rightarrow \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \text{ implies } \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\varphi \leftrightarrow \psi)[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e] \text{ if and only if } \mathcal{A} \models \psi[e] \\
\mathcal{A} \models (\forall x)\varphi[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e(x/a)] \text{ for every } a \in A \\
\mathcal{A} \models (\exists x)\varphi[e] & \Leftrightarrow & \mathcal{A} \models \varphi[e(x/a)] \text{ for some } a \in A
\end{array}$$

*Observation* *Let term $t$ be substitutable for $x$ in $\varphi$ and $\psi$ be a variant of $\varphi$. Then for every structure $\mathcal{A}$ and assignment $e$*

1) $\mathcal{A} \models \varphi(x/t)[e]$ *if and only if* $\mathcal{A} \models \varphi[e(x/a)]$ *where* $a = t^A[e]$,
2) $\mathcal{A} \models \varphi[e]$ *if and only if* $\mathcal{A} \models \psi[e]$.

# Validity in a structure

Let $\varphi$ be a formula of a language $L$ and $\mathcal{A}$ be an $L$-structure.

- $\varphi$ is *valid* (*true*) *in the structure* $\mathcal{A}$, denoted by $\mathcal{A} \models \varphi$, if $\mathcal{A} \models \varphi[e]$ for every $e\colon \mathrm{Var} \to A$. We say that $\mathcal{A}$ *satisfies* $\varphi$. Otherwise, we write $\mathcal{A} \not\models \varphi$.
- $\varphi$ is *contradictory in* $\mathcal{A}$ if $\mathcal{A} \models \neg\varphi$; that is, $\mathcal{A} \not\models \varphi[e]$ for every $e\colon \mathrm{Var} \to A$.
- For every formulas $\varphi$, $\psi$, variable $x$, and structure $\mathcal{A}$

$$
\begin{array}{llcl}
(1) & \mathcal{A} \models \varphi & \Rightarrow & \mathcal{A} \not\models \neg\varphi \\
(2) & \mathcal{A} \models \varphi \wedge \psi & \Leftrightarrow & \mathcal{A} \models \varphi \text{ and } \mathcal{A} \models \psi \\
(3) & \mathcal{A} \models \varphi \vee \psi & \Leftarrow & \mathcal{A} \models \varphi \text{ or } \mathcal{A} \models \psi \\
(4) & \mathcal{A} \models \varphi & \Leftrightarrow & \mathcal{A} \models (\forall x)\varphi
\end{array}
$$

- If $\varphi$ is a sentence, it is valid or contradictory in $\mathcal{A}$, and thus (1) holds also in $\Leftarrow$. If moreover $\psi$ is a sentence, also $(3)$ holds in $\Rightarrow$.
- By $(4)$, $\mathcal{A} \models \varphi$ if and only if $\mathcal{A} \models \psi$ where $\psi$ is a *universal closure* of $\varphi$, i.e. a formula $(\forall x_1)\cdots(\forall x_n)\varphi$ where $x_1, \ldots, x_n$ are all free variables in $\varphi$.

# Validity in a theory

- A *theory* of language $L$ is any set $T$ of formulas of $L$ (so called *axioms*).
- A *model of a theory* $T$ is an $L$-structure $\mathcal{A}$ such that $\mathcal{A} \models \varphi$ for every $\varphi \in T$. Then we write $\mathcal{A} \models T$ and we say that $\mathcal{A}$ *satisfies* $T$.
- The *class of models* of a theory $T$ is $M(T) = \{\mathcal{A} \in M(L) \mid \mathcal{A} \models T\}$.
- A formula $\varphi$ is *valid in $T$* (*true in $T$*), denoted by $T \models \varphi$, if $\mathcal{A} \models \varphi$ for every model $\mathcal{A}$ of $T$. Otherwise, we write $T \not\models \varphi$.
- $\varphi$ is *contradictory in $T$* if $T \models \neg\varphi$, i.e. $\varphi$ is contradictory in all models of $T$.
- $\varphi$ is *independent in $T$* if it is neither valid nor contradictory in $T$.
- If $T = \emptyset$, we have $M(T) = M(L)$ and we omit $T$, eventually we say *"in logic"*. Then $\models \varphi$ means that $\varphi$ is (*universally*) *valid* (a *tautology*).
- A *consequence* of $T$ is the set $\theta^L(T)$ of all sentences of $L$ valid in $T$, i.e.

$$\theta^L(T) = \{\varphi \in \mathrm{Fm}_L \mid T \models \varphi \text{ and } \varphi \text{ is a sentence}\}.$$

# Example of a theory

A *theory of orderings* $T$ in language $L = \langle \leq \rangle$ with equality has axioms

$$x \leq x \quad \text{(reflexivity)}$$

$$x \leq y \ \wedge \ y \leq x \ \rightarrow \ x = y \quad \text{(antisymmetry)}$$

$$x \leq y \ \wedge \ y \leq z \ \rightarrow \ x \leq z \quad \text{(transitivity)}$$

Models of $T$ are $L$-structures $\langle S, \leq_S \rangle$, so called ordered sets, that satisfy the axioms of $T$, for example $\mathcal{A} = \langle \mathbb{N}, \leq \rangle$ or $\mathcal{B} = \langle \mathcal{P}(X), \subseteq \rangle$ for $X = \{0, 1, 2\}$.

- *A formula* $\varphi\colon x \leq y \vee y \leq x$ *is valid in* $\mathcal{A}$ *but not in* $\mathcal{B}$ *since* $\mathcal{B} \not\models \varphi[e]$ *for the assignment* $e(x) = \{0\}, e(y) = \{1\}$*, thus* $\varphi$ *is independent in* $T$*.*
- *A sentence* $\psi\colon (\exists x)(\forall y)(y \leq x)$ *is valid in* $\mathcal{B}$ *and contradictory in* $\mathcal{A}$*, hence it is independent in* $T$ *as well. We write* $\mathcal{B} \models \psi$*,* $\mathcal{A} \models \neg\psi$*.*
- *A formula* $\chi\colon (x \leq y \wedge y \leq z \wedge z \leq x) \rightarrow (x = y \wedge y = z)$ *is valid in* $T$*, denoted by* $T \models \chi$*, the same holds for its universal closure.*