Propositional and Predicate Logic - VII

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Validity in a theory

- A *theory* of a language L is any set T of formulas of L (so called *axioms*).
- A *model of a theory* T is an L-structure $\mathcal{A}$ such that $\mathcal{A} \models \varphi$ for every $\varphi \in T$. Then we write $\mathcal{A} \models T$ and we say that $\mathcal{A}$ *satisfies* T.
- The *class of models* of a theory T is $M(T) = \{\mathcal{A} \in M(L) \mid \mathcal{A} \models T\}$.
- A formula φ is *valid in T* (*true in T*), denoted by $T \models \varphi$, if $\mathcal{A} \models \varphi$ for every model $\mathcal{A}$ of T. Otherwise, we write $T \not\models \varphi$.
- φ is *contradictory in T* if $T \models \neg\varphi$, i.e. φ is contradictory in all models of T.
- φ is *independent in T* if it is neither valid nor contradictory in T.
- If $T = \emptyset$, we have $M(T) = M(L)$ and we omit T, eventually we say *"in logic"*. Then $\models \varphi$ means that φ is (*logically*) *valid* (a *tautology*).
- A *consequence* of T is the set $\theta^L(T)$ of all sentences of L valid in T, i.e.

$$\theta^L(T) = \{\varphi \in \mathrm{Fm}_L \mid T \models \varphi \text{ and } \varphi \text{ is a sentence}\}.$$

Example of a theory

The *theory of orderings* T of the language $L = \langle \leq \rangle$ with equality has axioms

$$x \leq x \qquad \text{(reflexivity)}$$
$$x \leq y \ \wedge \ y \leq x \ \rightarrow \ x = y \qquad \text{(antisymmetry)}$$
$$x \leq y \ \wedge \ y \leq z \ \rightarrow \ x \leq z \qquad \text{(transitivity)}$$

Models of T are L-structures $\langle S, \leq_S \rangle$, so called ordered sets, that satisfy the axioms of T, for example $\mathcal{A} = \langle \mathbb{N}, \leq \rangle$ or $\mathcal{B} = \langle \mathcal{P}(X), \subseteq \rangle$ for $X = \{0, 1, 2\}$.

- *The formula* $\varphi\colon x \leq y \vee y \leq x$ *is valid in* $\mathcal{A}$ *but not in* $\mathcal{B}$ *since* $\mathcal{B} \not\models \varphi[e]$ *for the assignment* $e(x) = \{0\}, e(y) = \{1\}$*, thus* φ *is independent in* T.
- *The sentence* $\psi\colon (\exists x)(\forall y)(y \leq x)$ *is valid in* $\mathcal{B}$ *and contradictory in* $\mathcal{A}$*, hence it is independent in* T *as well. We write* $\mathcal{B} \models \psi$, $\mathcal{A} \models \neg\psi$.
- *The formula* $\chi\colon (x \leq y \wedge y \leq z \wedge z \leq x) \rightarrow (x = y \wedge y = z)$ *is valid in* T*, denoted by* $T \models \chi$*, the same holds for its universal closure.*

Unsatisfiability and validity

The problem of validity in a theory can be transformed to the problem of satisfiability of (another) theory.

Proposition *For every theory T and sentence φ (of the same language)*

$$T, \neg\varphi \text{ is unsatisfiable} \quad \Leftrightarrow \quad T \models \varphi.$$

Proof By definitions, it is equivalent that

(1) $T, \neg\varphi$ is unsatisfiable (i.e. it has no model),

(2) $\neg\varphi$ is not valid in any model of T,

(3) φ is valid in every model of T,

(4) $T \models \varphi$. $\square$

Remark *The assumption that φ is a sentence is necessary for $(2) \Rightarrow (3)$.*

For example, the theory $\{P(c), \neg P(x)\}$ is unsatisfiable, but $P(c) \not\models P(x)$, where P is a unary relation symbol and c is a constant symbol.

Basic algebraic theories

- theory of *groups* in the language $L = \langle +, -, 0 \rangle$ with equality has axioms

  $x + (y + z) = (x + y) + z$ (associativity of $+$)

  $0 + x = x = x + 0$ (0 is neutral to $+$)

  $x + (-x) = 0 = (-x) + x$ ($-x$ is inverse of x)

- theory of *Abelian groups* has moreover ax. $x + y = y + x$ (commutativity)
- theory of *rings* in $L = \langle +, -, \cdot, 0, 1 \rangle$ with equality has moreover axioms

  $1 \cdot x = x = x \cdot 1$ (1 is neutral to $\cdot$)

  $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity of $\cdot$)

  $x \cdot (y + z) = x \cdot y + x \cdot z, \ (x + y) \cdot z = x \cdot z + y \cdot z$ (distributivity)

- theory of *commutative rings* has moreover ax. $x \cdot y = y \cdot x$ (commutativity)
- theory of *fields* in the same language has additional axioms

  $x \neq 0 \to (\exists y)(x \cdot y = 1)$ (existence of inverses to $\cdot$)

  $0 \neq 1$ (nontriviality)

Properties of theories

A theory T of a language L is *(semantically)*

- *inconsistent* if $T \models \bot$, otherwise T is *consistent* (*satisfiable*),
- *complete* if it is consistent and every sentence of L is valid in T or contradictory in T,
- an *extension* of a theory T' of language L' if $L' \subseteq L$ and $\theta^{L'}(T') \subseteq \theta^{L}(T)$, we say that an extension T of a theory T' is *simple* if $L = L'$; and *conservative* if $\theta^{L'}(T') = \theta^{L}(T) \cap \mathrm{Fm}_{L'}$,
- *equivalent* with a theory T' if T is an extension of T' and vice-versa,

Structures $\mathcal{A}$, $\mathcal{B}$ for a language L are *elementarily equivalent*, denoted by $\mathcal{A} \equiv \mathcal{B}$, if they satisfy the same sentences of L.

Observation *Let T and T' be theories of a language L. T is (semantically)*

(1) *consistent if and only if it has a model,*

(2) *complete iff it has a single model, up to elementarily equivalence,*

(3) *an extension of T' if and only if $M(T) \subseteq M(T')$,*

(4) *equivalent with T' if and only if $M(T) = M(T')$.*

Substructures

Let $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ and $\mathcal{B} = \langle B, \mathcal{R}^B, \mathcal{F}^B \rangle$ be structures for $L = \langle \mathcal{R}, \mathcal{F} \rangle$.

We say that $\mathcal{B}$ is an (induced) *substructure* of $\mathcal{A}$, denoted by $\mathcal{B} \subseteq \mathcal{A}$, if

(i) $B \subseteq A$,

(ii) $R^B = R^A \cap B^{\mathrm{ar}(R)}$ for every $R \in \mathcal{R}$,

(iii) $f^B = f^A \cap (B^{\mathrm{ar}(f)} \times B)$; that is, $f^B = f^A \restriction B^{\mathrm{ar}(f)}$, for every $f \in \mathcal{F}$.

A set $C \subseteq A$ is a domain of some substructure of $\mathcal{A}$ if and only if C is closed under all functions of $\mathcal{A}$. Then the respective substructure, denoted by $\mathcal{A} \restriction C$, is said to be the *restriction* of the structure $\mathcal{A}$ to C.

- A set $C \subseteq A$ is *closed* under a function $f \colon A^n \to A$ if $f(x_1, \ldots, x_n) \in C$ for every $x_1, \ldots, x_n \in C$.

Example: $\underline{\mathbb{Z}} = \langle \mathbb{Z}, +, \cdot, 0 \rangle$ *is a substructure of* $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, \cdot, 0 \rangle$ *and* $\underline{\mathbb{Z}} = \underline{\mathbb{Q}} \restriction \mathbb{Z}$*.*
Furthermore, $\underline{\mathbb{N}} = \langle \mathbb{N}, +, \cdot, 0 \rangle$ *is their substructure and* $\underline{\mathbb{N}} = \underline{\mathbb{Q}} \restriction \mathbb{N} = \underline{\mathbb{Z}} \restriction \mathbb{N}$*.*

Validity in a substructure

Let $\mathcal{B}$ be a substructure of a structure $\mathcal{A}$ for a (fixed) language L.

Proposition *For every open formula φ and assignment $e\colon \mathrm{Var} \to B$,*

$$\mathcal{A} \models \varphi[e] \quad \textit{if and only if} \quad \mathcal{B} \models \varphi[e].$$

Proof For atomic φ it follows from the definition of the truth value with respect to an assignment. Otherwise by induction on the structure of the formula. □

Corollary *For every open formula φ and structure $\mathcal{A}$,*

$$\mathcal{A} \models \varphi \quad \textit{if and only if} \quad \mathcal{B} \models \varphi \textit{ for every substructure } \mathcal{B} \subseteq \mathcal{A}.$$

- A theory T is *open* if all axioms of T are open.

Corollary *Every substructure of a model of an open theory T is a model of T.*

For example, every substructure of a graph, i.e. a model of theory of graphs, is a graph, called a subgraph. Similarly subgroups, Boolean subalgebras, etc.

Generated substructure, expansion, reduct

Let $\mathcal{A} = \langle A, \mathcal{R}^A, \mathcal{F}^A \rangle$ be a structure and $X \subseteq A$. Let B be the smallest subset of A containing X that is closed under all functions of the structure $\mathcal{A}$ (including constants). Then the structure $\mathcal{A} \restriction B$ is denoted by $\mathcal{A}\langle X \rangle$ and is called the substructure of $\mathcal{A}$ *generated* by the set X.

Example: for $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, \cdot, 0 \rangle$, $\underline{\mathbb{Z}} = \langle \mathbb{Z}, +, \cdot, 0 \rangle$, $\underline{\mathbb{N}} = \langle \mathbb{N}, +, \cdot, 0 \rangle$ *it is* $\underline{\mathbb{Q}}\langle \{1\} \rangle = \underline{\mathbb{N}}$, $\underline{\mathbb{Q}}\langle \{-1\} \rangle = \underline{\mathbb{Z}}$, *and* $\underline{\mathbb{Q}}\langle \{2\} \rangle$ *is the substructure on all even natural numbers.*

Let $\mathcal{A}$ be a structure for a language L and $L' \subseteq L$. By omitting realizations of symbols that are not in L' we obtain from $\mathcal{A}$ a structure $\mathcal{A}'$ called the *reduct* of $\mathcal{A}$ to the language L'. Conversely, $\mathcal{A}$ is an *expansion* of $\mathcal{A}'$ into L.

For example, $\langle \mathbb{N}, + \rangle$ *is a reduct of* $\langle \mathbb{N}, +, \cdot, 0 \rangle$. *On the other hand, the structure* $\langle \mathbb{N}, +, c_i \rangle_{i \in \mathbb{N}}$ *with* $c_i = i$ *for every* $i \in \mathbb{N}$ *is the expansion of* $\langle \mathbb{N}, + \rangle$ *by names of elements from* $\mathbb{N}$.

Theorem on constants

Theorem *Let φ be a formula in a language L with free variables $x_1, \ldots, x_n$ and let T be a theory in L. Let L' be the extension of L with new constant symbols $c_1, \ldots, c_n$ and let T' denote the theory T in L'. Then*

$$T \models \varphi \quad \text{if and only if} \quad T' \models \varphi(x_1/c_1, \ldots, x_n/c_n).$$

Proof ($\Rightarrow$) If $\mathcal{A}'$ is a model of T', let $\mathcal{A}$ be the reduct of $\mathcal{A}'$ to L. Since $\mathcal{A} \models \varphi[e]$ for every assignment e, we have in particular

$$\mathcal{A} \models \varphi[e(x_1/c_1^{A'}, \ldots, x_n/c_n^{A'})], \quad \text{i.e. } \mathcal{A}' \models \varphi(x_1/c_1, \ldots, x_n/c_n).$$

($\Leftarrow$) If $\mathcal{A}$ is a model of T and e an assignment, let $\mathcal{A}'$ be the expansion of A into L' by setting $c_i^{A'} = e(x_i)$ for every i. Since $\mathcal{A}' \models \varphi(x_1/c_1, \ldots, x_n/c_n)[e']$ for every assignment e', we have

$$\mathcal{A}' \models \varphi[e(x_1/c_1^{A'}, \ldots, x_n/c_n^{A'})], \quad \text{i.e. } \mathcal{A} \models \varphi[e]. \quad \square$$

Extensions of theories

Proposition *Let T be a theory of L and T' be a theory of L' where $L \subseteq L'$.*

(i) *T' is an extension of T if and only if the reduct $\mathcal{A}$ of every model $\mathcal{A}'$ of T' to the language L is a model of T,*

(ii) *T' is a conservative extension of T if T' is an extension of T and every model $\mathcal{A}$ of T can be expanded to the language L' on a model $\mathcal{A}'$ of T'.*

Proof

$(i)a)$ If T' is an extension of T and φ is any axiom of T, then $T' \models \varphi$. Thus $\mathcal{A}' \models \varphi$ and also $\mathcal{A} \models \varphi$, which implies that $\mathcal{A}$ is a model of T.

$(i)b)$ If $\mathcal{A}$ is a model of T and $T \models \varphi$ where φ is of L, then $\mathcal{A} \models \varphi$ and also $\mathcal{A}' \models \varphi$. This implies that $T' \models \varphi$ and thus T' is an extension of T.

(ii) If $T' \models \varphi$ where φ is of L and $\mathcal{A}$ is a model of T, then in its expansion $\mathcal{A}'$ that models T' we have $\mathcal{A}' \models \varphi$. Thus also $\mathcal{A} \models \varphi$, and hence $T \models \varphi$. Therefore T' is conservative. □

Extensions by definition of a relation symbol

Let T be a theory of L, $\psi(x_1, \dots, x_n)$ be a formula of L in free variables $x_1, \dots, x_n$ and L' denote the language L with a new n-ary relation symbol R.

The *extension* of T *by definition of R* with the formula ψ is the theory T' of L' obtained from T by adding the axiom

$$R(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n)$$

Observation *Every model of T can be uniquely expanded to a model of T'.*

Corollary *T' is a conservative extension of T.*

Proposition *For every formula φ' of L' there is φ of L s.t. $T' \models \varphi' \leftrightarrow \varphi$.*

Proof Replace each subformula $R(t_1, \dots, t_n)$ in φ with $\psi'(x_1/t_1, \dots, x_n/t_n)$, where ψ' is a suitable variant of ψ allowing all substitutions. □

For example, the symbol $\leq$ can be defined in arithmetics by the axiom

$$x \leq y \leftrightarrow (\exists z)(x + z = y)$$

Extensions by definition of a function symbol

Let T be a theory of a language L and $\psi(x_1, \ldots, x_n, y)$ be a formula of L in free variables $x_1, \ldots, x_n, y$ such that

$$T \models (\exists y)\psi(x_1, \ldots, x_n, y) \qquad \text{(existence)}$$

$$T \models \psi(x_1, \ldots, x_n, y) \ \wedge\ \psi(x_1, \ldots, x_n, z) \ \rightarrow\ y = z \qquad \text{(uniqueness)}$$

Let L' denote the language L with a new n-ary function symbol f.

The *extension* of T *by definition of f* with the formula ψ is the theory T' of L' obtained from T by adding the axiom

$$f(x_1, \ldots, x_n) = y \ \leftrightarrow\ \psi(x_1, \ldots, x_n, y)$$

Remark *In particular, if ψ is $t(x_1, \ldots, x_n) = y$ where t is a term and $x_1, \ldots, x_n$ are the variables in t, both the conditions of existence and uniqueness hold.*

For example binary $-$ can be defined using $+$ and unary $-$ by the axiom

$$x - y = z \ \leftrightarrow\ x + (-y) = z$$

Extensions by definition of a function symbol (cont.)

Observation *Every model of T can be uniquely expanded to a model of T'.*

Corollary *T' is a conservative extension of T.*

Proposition *For every formula φ' of L' there is φ of L s.t. $T' \models \varphi' \leftrightarrow \varphi$.*

Proof It suffices to consider φ' with a single occurrence of f. If φ' has more, we may proceed inductively. Let φ^* denote the formula obtained from φ' by replacing the term $f(t_1, \ldots, t_n)$ with a new variable z. Let φ be the formula

$$(\exists z)(\varphi^* \ \wedge \ \psi'(x_1/t_1, \ldots, x_n/t_n, y/z)),$$

where ψ' is a suitable variant of ψ allowing all substitutions.

Let $\mathcal{A}$ be a model of T', e be an assignment, and $a = f^A(t_1, \ldots, t_n)[e]$. By the two conditions, $\mathcal{A} \models \psi'(x_1/t_1, \ldots, x_n/t_n, y/z)[e]$ if and only if $e(z) = a$. Thus

$$\mathcal{A} \models \varphi[e] \quad \Leftrightarrow \quad \mathcal{A} \models \varphi^*[e(z/a)] \quad \Leftrightarrow \quad \mathcal{A} \models \varphi'[e]$$

for every assignment e, i.e. $\mathcal{A} \models \varphi' \leftrightarrow \varphi$ and so $T' \models \varphi' \leftrightarrow \varphi$. □

Extensions by definitions

A theory T' of L' is called an *extension* of a theory T of L *by definitions* if it is obtained from T by successive definitions of relation and function symbols.

Corollary *Let T' be an extension of a theory T by definitions. Then*

- *every model of T can be uniquely expanded to a model of T',*
- *T' is a conservative extension of T,*
- *for every formula φ' of L' there is a formula φ of L such that $T' \models \varphi' \leftrightarrow \varphi$.*

For example, in $T = \{(\exists y)(x+y=0), (x+y=0) \wedge (x+z=0) \rightarrow y=z\}$ of $L = \langle +, 0, \leq \rangle$ with equality we can define $<$ and unary $-$ by the axioms

$$-x = y \quad \leftrightarrow \quad x + y = 0$$
$$x < y \quad \leftrightarrow \quad x \leq y \ \wedge \ \neg(x = y)$$

Then the formula $-x < y$ is equivalent in this extension to a formula

$$(\exists z)((z \leq y \ \wedge \ \neg(z = y)) \ \wedge \ x + z = 0).$$

Definable sets

We interested in which sets can be defined within a given structure.

- *A set defined by a formula* $\varphi(x_1, \ldots, x_n)$ *in structure* $\mathcal{A}$ is the set

$$\varphi^{\mathcal{A}}(x_1, \ldots, x_n) = \{(a_1, \ldots, a_n) \in A^n \mid \mathcal{A} \models \varphi[e(x_1/a_1, \ldots, x_n/a_n)]\}.$$

  Shortly, $\varphi^{\mathcal{A}}(\overline{x}) = \{\overline{a} \in A^{|\overline{x}|} \mid \mathcal{A} \models \varphi[e(\overline{x}/\overline{a})]\}$, where $|\overline{x}| = n$.

- *A set defined by a formula* $\varphi(\overline{x}, \overline{y})$ *with parameters* $\overline{b} \in A^{|\overline{y}|}$ in $\mathcal{A}$ is

$$\varphi^{\mathcal{A},\overline{b}}(\overline{x}, \overline{y}) = \{\overline{a} \in A^{|\overline{x}|} \mid \mathcal{A} \models \varphi[e(\overline{x}/\overline{a}, \overline{y}/\overline{b})]\}.$$

  Example: $E(x,y)^{\mathcal{G},b}$ *is the set of neighbors of a vertex* b *in a graph* $\mathcal{G}$.

- For a structure $\mathcal{A}$, a set $B \subseteq A$, and $n \in \mathbb{N}$ let $\mathrm{Df}^n(\mathcal{A}, B)$ denote the class of definable sets $D \subseteq A^n$ in the structure $\mathcal{A}$ with parameters from B.

Observation $\mathrm{Df}^n(\mathcal{A}, B)$ *is closed under complements, union, intersection and it contains* $\emptyset$, A^n. *Thus it forms a subalgebra of the set algebra* $\underline{\mathcal{P}}(A^n)$.

Example - database queries

Movie	*name*	*director*	*actor*
	Lidé z Maringotek	M. Frič	J. Tříska
	Po strništi bos	J. Svěrák	Z. Svěrák
	Po strništi bos	J. Svěrák	J. Tříska
	...	...	...

Program	*cinema*	*name*	*time*
	Světozor	Po strništi bos	13:15
	Mat	Po strništi bos	16:15
	Mat	Lidé z Maringotek	18:30
	...	...	...

Where and when can I see a movie with J. Tříska?

select *Program.cinema, Program.time* **from** *Movie, Program*
where *Movie.name = Program.name* **and** *actor* = 'J. Tříska';

Equivalently, it is the set $\varphi^{\mathcal{D}}(x, y)$ defined by the formula $\varphi(x, y)$

$$(\exists n)(\exists d)(P(x, n, y) \wedge M(n, d, \text{'J. Tříska'}))$$

in the structure $\mathcal{D} = \langle D, Movie, Program, c^D\rangle_{c \in D}$ of $L = \langle M, P, c\rangle_{c \in D}$, where $D = \{\text{'Po strništi bos', 'J. Tříska', 'Mat', '13:15'}, \ldots\}$ and $c^D = c$ for any $c \in D$.

Boolean algebras

The theory of *Boolean algebras* has the language $L = \langle -, \wedge, \vee, 0, 1 \rangle$ with equality and the following axioms.

$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	(asociativity of $\wedge$)
$x \vee (y \vee z) = (x \vee y) \vee z$	(asociativity of $\vee$)
$x \wedge y = y \wedge x$	(commutativity of $\wedge$)
$x \vee y = y \vee x$	(commutativity of $\vee$)
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	(distributivity of $\wedge$ over $\vee$)
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	(distributivity of $\vee$ over $\wedge$)
$x \wedge (x \vee y) = x, \quad x \vee (x \wedge y) = x$	(absorption)
$x \vee (-x) = 1, \quad x \wedge (-x) = 0$	(complementation)
$0 \neq 1$	(non-triviality)

The smallest model is $\underline{2} = \langle \{0, 1\}, -_1, \wedge_1, \vee_1, 0, 1 \rangle$. Finite Boolean algebras are (up to isomorphism) $\langle \{0, 1\}^n, -_n, \wedge_n, \vee_n, 0_n, 1_n \rangle$ for $n \in \mathbb{N}^+$, where the operations *(on binary n-tuples)* are the coordinate-wise operations of $\underline{2}$.

Relations of propositional and predicate logic

- Propositional formulas over connectives $\neg$, $\wedge$, $\vee$ (eventually with $\top$, $\bot$) can be viewed as Boolean terms. Then the truth value of φ in a given assignment is the value of the term in the Boolean algebra $\underline{2}$.
- Lindenbaum-Tarski algebra over $\mathbb{P}$ is Boolean algebra (also for $\mathbb{P}$ infinite).
- If we represent atomic subformulas in an open formula φ (without equality) with propositional letters, we obtain a proposition that is valid if and only if φ is valid.
- Propositional logic can be introduced as a fragment of predicate logic using nullary relation symbols (*syntax*) and nullary relations (*semantics*) since $A^0 = \{\emptyset\} = 1$, so $R^A \subseteq A^0$ is either $R^A = \emptyset = 0$ or $R^A = \{\emptyset\} = 1$.