Propositional and Predicate Logic - VIII

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Extensions by definition of a relation symbol

Let T be a theory of L, $\psi(x_1, \ldots, x_n)$ be a formula of L in free variables $x_1, \ldots, x_n$ and L' denote the language L with a new n-ary relation symbol R.

The *extension* of T *by definition of R* with the formula ψ is the theory T' of L' obtained from T by adding the axiom

$$R(x_1, \ldots, x_n) \;\leftrightarrow\; \psi(x_1, \ldots, x_n)$$

Observation *Every model of T can be uniquely expanded to a model of T'.*

Corollary *T' is a conservative extension of T.*

Proposition *For every formula φ' of L' there is φ of L s.t. $T' \models \varphi' \leftrightarrow \varphi$.*

Proof Replace each subformula $R(t_1, \ldots, t_n)$ in φ with $\psi'(x_1/t_1, \ldots, x_n/t_n)$, where ψ' is a suitable variant of ψ allowing all substitutions. $\square$

For example, the symbol $\leq$ can be defined in arithmetics by the axiom

$$x \leq y \;\;\leftrightarrow\;\; (\exists z)(x + z = y)$$

Extensions by definition of a function symbol

Let T be a theory of a language L and $\psi(x_1, \ldots, x_n, y)$ be a formula of L in free variables $x_1, \ldots, x_n, y$ such that

$$T \models (\exists y)\psi(x_1, \ldots, x_n, y) \qquad \text{(existence)}$$

$$T \models \psi(x_1, \ldots, x_n, y) \ \wedge \ \psi(x_1, \ldots, x_n, z) \ \rightarrow \ y = z \qquad \text{(uniqueness)}$$

Let L' denote the language L with a new n-ary function symbol f.

The *extension* of T *by definition of f* with the formula ψ is the theory T' of L' obtained from T by adding the axiom

$$f(x_1, \ldots, x_n) = y \ \leftrightarrow \ \psi(x_1, \ldots, x_n, y)$$

Remark *In particular, if ψ is $t(x_1, \ldots, x_n) = y$ where t is a term and $x_1, \ldots, x_n$ are the variables in t, both the conditions of existence and uniqueness hold.*

For example binary $-$ can be defined using $+$ and unary $-$ by the axiom

$$x - y = z \ \leftrightarrow \ x + (-y) = z$$

Extensions by definition of a function symbol (cont.)

Observation *Every model of T can be uniquely expanded to a model of T'.*

Corollary *T' is a conservative extension of T.*

Proposition *For every formula φ' of L' there is φ of L s.t. $T' \models \varphi' \leftrightarrow \varphi$.*

Proof It suffices to consider φ' with a single occurrence of f. If φ' has more, we may proceed inductively. Let φ^* denote the formula obtained from φ' by replacing the term $f(t_1, \ldots, t_n)$ with a new variable z. Let φ be the formula

$$(\exists z)(\varphi^* \ \wedge \ \psi'(x_1/t_1, \ldots, x_n/t_n, y/z)),$$

where ψ' is a suitable variant of ψ allowing all substitutions.

Let $\mathcal{A}$ be a model of T', e be an assignment, and $a = f^A(t_1, \ldots, t_n)[e]$. By the two conditions, $\mathcal{A} \models \psi'(x_1/t_1, \ldots, x_n/t_n, y/z)[e]$ if and only if $e(z) = a$. Thus

$$\mathcal{A} \models \varphi[e] \quad \Leftrightarrow \quad \mathcal{A} \models \varphi^*[e(z/a)] \quad \Leftrightarrow \quad \mathcal{A} \models \varphi'[e]$$

for every assignment e, i.e. $\mathcal{A} \models \varphi' \leftrightarrow \varphi$ and so $T' \models \varphi' \leftrightarrow \varphi$. $\square$

Extensions by definitions

A theory T' of L' is called an *extension* of a theory T of L *by definitions* if it is obtained from T by successive definitions of relation and function symbols.

Corollary *Let T' be an extension of a theory T by definitions. Then*

- *every model of T can be uniquely expanded to a model of T',*
- *T' is a conservative extension of T,*
- *for every formula φ' of L' there is a formula φ of L such that $T' \models \varphi' \leftrightarrow \varphi$.*

For example, in $T = \{(\exists y)(x + y = 0), (x + y = 0) \land (x + z = 0) \rightarrow y = z\}$ of $L = \langle +, 0, \leq \rangle$ with equality we can define $<$ and unary $-$ by the axioms

$$\begin{aligned} -x = y \quad &\leftrightarrow \quad x + y = 0 \\ x < y \quad &\leftrightarrow \quad x \leq y \ \land \ \neg(x = y) \end{aligned}$$

Then the formula $-x < y$ is equivalent in this extension to a formula

$$(\exists z)((z \leq y \ \land \ \neg(z = y)) \ \land \ x + z = 0).$$

Definable sets

We interested in which sets can be defined within a given structure.

- *A set defined by a formula* $\varphi(x_1, \dots, x_n)$ *in structure* $\mathcal{A}$ is the set

$$\varphi^{\mathcal{A}}(x_1, \dots, x_n) = \{(a_1, \dots, a_n) \in A^n \mid \mathcal{A} \models \varphi[e(x_1/a_1, \dots, x_n/a_n)]\}.$$

  Shortly, $\varphi^{\mathcal{A}}(\overline{x}) = \{\overline{a} \in A^{|\overline{x}|} \mid \mathcal{A} \models \varphi[e(\overline{x}/\overline{a})]\}$, where $|\overline{x}| = n$.

- *A set defined by a formula* $\varphi(\overline{x}, \overline{y})$ *with parameters* $\overline{b} \in A^{|\overline{y}|}$ in $\mathcal{A}$ is

$$\varphi^{\mathcal{A}, \overline{b}}(\overline{x}, \overline{y}) = \{\overline{a} \in A^{|\overline{x}|} \mid \mathcal{A} \models \varphi[e(\overline{x}/\overline{a}, \overline{y}/\overline{b})]\}.$$

  Example: $E(x, y)^{\mathcal{G}, b}$ *is the set of neighbors of a vertex* b *in a graph* $\mathcal{G}$.

- For a structure $\mathcal{A}$, a set $B \subseteq A$, and $n \in \mathbb{N}$ let $\mathrm{Df}^n(\mathcal{A}, B)$ denote the class of definable sets $D \subseteq A^n$ in the structure $\mathcal{A}$ with parameters from B.

Observation $\mathrm{Df}^n(\mathcal{A}, B)$ *is closed under complements, union, intersection and it contains* $\emptyset$, A^n. *Thus it forms a subalgebra of the set algebra* $\underline{\mathcal{P}}(A^n)$.

Example - database queries

Movie	*name*	*director*	*actor*
	Lidé z Maringotek	M. Frič	J. Tříska
	Po strništi bos	J. Svěrák	Z. Svěrák
	Po strništi bos	J. Svěrák	J. Tříska
	...	...	...

Program	*cinema*	*name*	*time*
	Světozor	Po strništi bos	13:15
	Mat	Po strništi bos	16:15
	Mat	Lidé z Maringotek	18:30
	...	...	...

Where and when can I see a movie with J. Tříska?

select *Program.cinema, Program.time* **from** *Movie, Program*
where *Movie.name = Program.name* **and** *actor* = 'J. Tříska';

Equivalently, it is the set $\varphi^{\mathcal{D}}(x, y)$ defined by the formula $\varphi(x, y)$

$$(\exists n)(\exists d)(P(x, n, y) \wedge M(n, d, \text{'J. Tříska'}))$$

in the structure $\mathcal{D} = \langle D, \mathit{Movie}, \mathit{Program}, c^D \rangle_{c \in D}$ of $L = \langle M, P, c \rangle_{c \in D}$, where $D = \{$'Po strništi bos', 'J. Tříska', 'Mat', '13:15', $\dots\}$ and $c^D = c$ for any $c \in D$.

Boolean algebras

The theory of *Boolean algebras* has the language $L = \langle -, \wedge, \vee, 0, 1 \rangle$ with equality and the following axioms.

$x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (associativity of $\wedge$)
$x \vee (y \vee z) = (x \vee y) \vee z$ (associativity of $\vee$)
$x \wedge y = y \wedge x$ (commutativity of $\wedge$)
$x \vee y = y \vee x$ (commutativity of $\vee$)
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (distributivity of $\wedge$ over $\vee$)
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ (distributivity of $\vee$ over $\wedge$)
$x \wedge (x \vee y) = x, \quad x \vee (x \wedge y) = x$ (absorption)
$x \vee (-x) = 1, \quad x \wedge (-x) = 0$ (complementation)
$0 \neq 1$ (non-triviality)

The smallest model is $\underline{2} = \langle \{0, 1\}, -_1, \wedge_1, \vee_1, 0, 1 \rangle$. Finite Boolean algebras are (up to isomorphism) $\langle \{0, 1\}^n, -_n, \wedge_n, \vee_n, 0_n, 1_n \rangle$ for $n \in \mathbb{N}^+$, where the operations *(on binary n-tuples)* are the coordinate-wise operations of $\underline{2}$.

Relations of propositional and predicate logic

- Propositional formulas over connectives $\neg$, $\wedge$, $\vee$ (eventually with $\top$, $\bot$) can be viewed as Boolean terms. Then the truth value of φ in a given assignment is the value of the term in the Boolean algebra $\underline{2}$.
- Lindenbaum-Tarski algebra over $\mathbb{P}$ is Boolean algebra (also for $\mathbb{P}$ infinite).
- If we represent atomic subformulas in an open formula φ (without equality) with propositional letters, we obtain a proposition that is valid if and only if φ is valid.
- Propositional logic can be introduced as a fragment of predicate logic using nullary relation symbols (*syntax*) and nullary relations (*semantics*) since $A^0 = \{\emptyset\} = 1$, so $R^A \subseteq A^0$ is either $R^A = \emptyset = 0$ or $R^A = \{\emptyset\} = 1$.

Tableau method in propositional logic - a review

- A tableau is a binary tree that represents a search for a *counterexample*.
- Nodes are labeled by entries, i.e. formulas with a sign T / F that represents an assumption that the formula is true / false in some model.
- If this assumption is correct, then it is correct also for all the entries in some branch below that came from this entry.
- A branch is contradictory (it fails) if it contains $T\psi$, $F\psi$ for some ψ.
- A proof of formula φ is a contradictory tableau with root $F\varphi$, i.e. a tableau in which every branch is contradictory. If φ has a proof, it is valid.
- If a counterexample exists, there will be a branch in a finished tableau that provides us with this counterexample, but this branch can be infinite.
- We can construct a systematic tableau that is always finished.
- If φ is valid, the systematic tableau for φ is contradictory, i.e. it is a proof of φ; and in this case, it is also finite.

Tableau method in predicate logic - what is different

- Formulas in entries will always be sentences (closed formulas), i.e. formulas without free variables.
- We add new atomic tableaux for quantifiers.
- In these tableaux we substitute ground terms for quantified variables following certain rules.
- We extend the language by new (auxiliary) constant symbols (countably many) to represent *"witnesses"* of entries $T(\exists x)\varphi(x)$ and $F(\forall x)\varphi(x)$.
- In a finished noncontradictory branch containing an entry $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$ we have instances $T\varphi(x/t)$ resp. $F\varphi(x/t)$ for every ground term t (of the extended language).

Assumptions

1) *The formula φ that we want to prove (or refute) is a sentence.* If not, we can replace φ with its universal closure φ', since for every theory T,

$$T \models \varphi \quad \text{if and only if} \quad T \models \varphi'.$$

2) *We prove from a theory in a closed form, i.e. every axiom is a sentence.* By replacing every axiom ψ with its universal closure ψ' we obtain an equivalent theory since for every structure $\mathcal{A}$ (of the given language L),

$$\mathcal{A} \models \psi \quad \text{if and only if} \quad \mathcal{A} \models \psi'.$$

3) *The language L is countable.* Then every theory of L is countable. We denote by L_C the extension of L by new constant symbols $c_0, c_1, \ldots$ (countably many). Then there are countably many ground terms of L_C. Let t_i denote the i-th ground term (in some fixed enumeration).

4) *First, we assume that the language is without equality.*

Tableaux in predicate logic - examples

$F((\exists x)\neg P(x) \rightarrow \neg(\forall x)P(x))$

$T(\exists x)\neg P(x)$

$F(\neg(\forall x)P(x))$

$T(\forall x)P(x)$

$T(\neg P(c))$ $\quad c$ new

$FP(c)$

$T(\forall x)P(x)$

$TP(c)$

$\otimes$

$F(\neg(\forall x)P(x) \rightarrow (\exists x)\neg P(x))$

$T(\neg(\forall x)P(x))$

$F(\exists x)\neg P(x)$

$F(\forall x)P(x)$

$FP(d)$ $\quad d$ new

$F(\exists x)\neg P(x)$

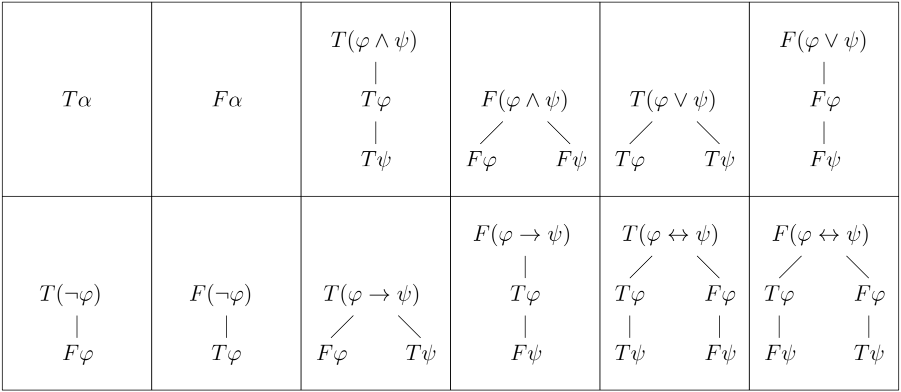
$F(\neg P(d))$

$TP(d)$

$\otimes$

Atomic tableaux - previous

An *atomic tableau* is one of the following trees (labeled by entries), where α is any atomic sentence and φ, ψ are any sentences, all of language L_C.



$T\alpha$	$F\alpha$	$T(\varphi \wedge \psi)$ $T\varphi$ $T\psi$	$F(\varphi \wedge \psi)$ $F\varphi$ $F\psi$	$T(\varphi \vee \psi)$ $T\varphi$ $T\psi$	$F(\varphi \vee \psi)$ $F\varphi$ $F\psi$
$T(\neg\varphi)$ $F\varphi$	$F(\neg\varphi)$ $T\varphi$	$T(\varphi \rightarrow \psi)$ $F\varphi$ $T\psi$	$F(\varphi \rightarrow \psi)$ $T\varphi$ $F\psi$	$T(\varphi \leftrightarrow \psi)$ $T\varphi$ $F\varphi$ $T\psi$ $F\psi$	$F(\varphi \leftrightarrow \psi)$ $T\varphi$ $F\varphi$ $F\psi$ $T\psi$

Atomic tableaux - new

Atomic tableaux are also the following trees (labeled by entries), where φ is any formula of the language L_C with a free variable x, t is any ground term of L_C and c is a new constant symbol from $L_C \setminus L$.

$\sharp$	$*$	$*$	$\sharp$
$T(\forall x)\varphi(x)$ $\vert$ $T\varphi(x/t)$	$F(\forall x)\varphi(x)$ $\vert$ $F\varphi(x/c)$	$T(\exists x)\varphi(x)$ $\vert$ $T\varphi(x/c)$	$F(\exists x)\varphi(x)$ $\vert$ $F\varphi(x/t)$
for any ground term t of L_C	for a *new* constant c	for a *new* constant c	for any ground term t of L_C

Remark *The constant symbol c represents a "witness" of the entry $T(\exists x)\varphi(x)$ or $F(\forall x)\varphi(x)$. Since we need that no prior demands are put on c, we specify (in the definition of a tableau) which constant symbols c may be used.*

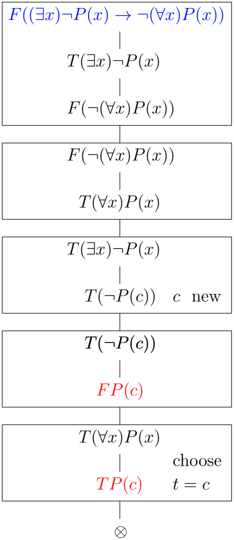
Tableau

A *finite tableau* from a theory T is a binary tree labeled with entries described

- (i) every atomic tableau is a finite tableau from T, whereas in case $(*)$ we may use any constant symbol $c \in L_C \setminus L$,
- (ii) if P is an entry on a branch V in a finite tableau from T, then by adjoining the atomic tableau for P at the end of branch V we obtain (again) a finite tableau from T, whereas in case $(*)$ we may use only a constant symbol $c \in L_C \setminus L$ that does not appear on V,
- (iii) if V is a branch in a finite tableau from T and $\varphi \in T$, then by adjoining $T\varphi$ at the end of branch V we obtain (again) a finite tableau from T.
- (iv) every finite tableau from T is formed by finitely many steps (i), (ii), (iii).

A *tableau* from T is a sequence $\tau_0, \tau_1, \dots, \tau_n, \dots$ of finite tableaux from T such that τ_{n+1} is formed from τ_n by (ii) or (iii), formally $\tau = \cup\tau_n$.

Construction of tableaux



$F((\exists x)\neg P(x) \rightarrow \neg(\forall x)P(x))$

$T(\exists x)\neg P(x)$

$F(\neg(\forall x)P(x))$

$F(\neg(\forall x)P(x))$

$T(\forall x)P(x)$

$T(\exists x)\neg P(x)$

$T(\neg P(c))$ c new

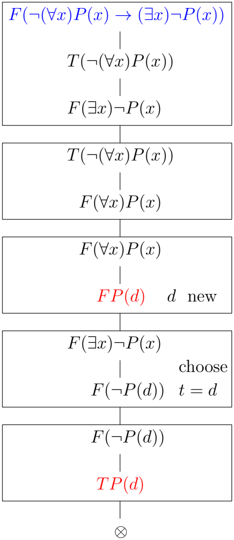
$T(\neg P(c))$

$FP(c)$

$T(\forall x)P(x)$

$TP(c)$ choose $t = c$

$\otimes$



$F(\neg(\forall x)P(x) \rightarrow (\exists x)\neg P(x))$

$T(\neg(\forall x)P(x))$

$F(\exists x)\neg P(x)$

$T(\neg(\forall x)P(x))$

$F(\forall x)P(x)$

$F(\forall x)P(x)$

$FP(d)$ d new

$F(\exists x)\neg P(x)$

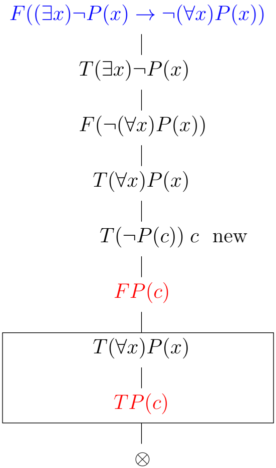
$F(\neg P(d))$ choose $t = d$

$F(\neg P(d))$

$TP(d)$

$\otimes$

Convention



$F((\exists x)\neg P(x) \to \neg(\forall x)P(x))$

$T(\exists x)\neg P(x)$

$F(\neg(\forall x)P(x))$

$T(\forall x)P(x)$

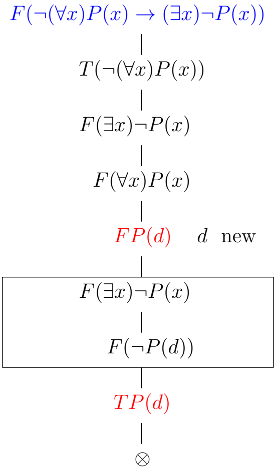
$T(\neg P(c))$ c new

$FP(c)$

$T(\forall x)P(x)$

$TP(c)$

$\otimes$



$F(\neg(\forall x)P(x) \to (\exists x)\neg P(x))$

$T(\neg(\forall x)P(x))$

$F(\exists x)\neg P(x)$

$F(\forall x)P(x)$

$FP(d)$ d new

$F(\exists x)\neg P(x)$

$F(\neg P(d))$

$TP(d)$

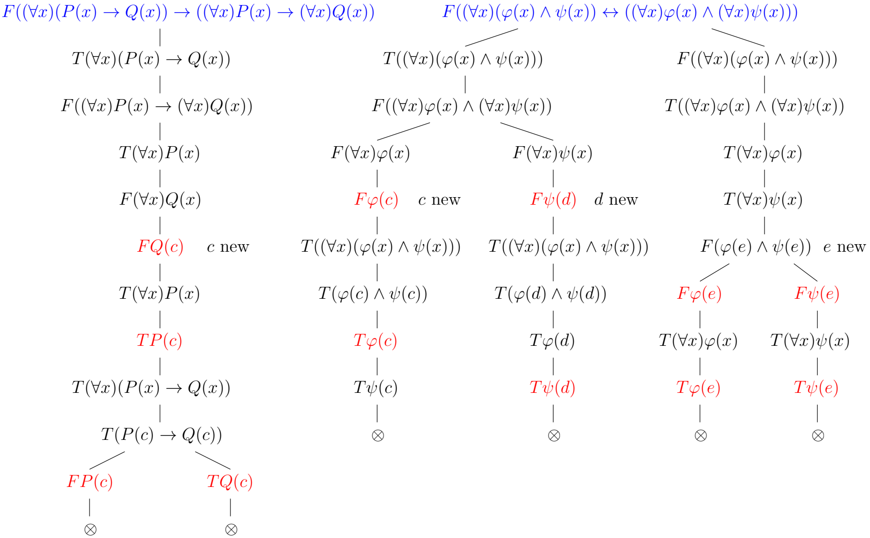
$\otimes$

We will not write the entry that is expanded again on the branch, except in cases when the entry is in the form of $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$.

Tableau proof

- A branch V in a tableau τ is *contradictory* if it contains entries $T\varphi$ and $F\varphi$ for some sentence φ, otherwise V is *noncontradictory*.
- A tableau τ is *contradictory* if every branch in τ is contradictory.
- A *tableau proof* (*proof by tableau*) of a sentence φ from a theory T is a contradictory tableau from T with $F\varphi$ in the root.
- A sentence φ is *(tableau) provable* from T, denoted by $T \vdash \varphi$, if it has a tableau proof from T.
- A *refutation* of a sentence φ by *tableau* from a theory T is a contradictory tableau from T with the root entry $T\varphi$.
- A sentence φ is *(tableau) refutable* from T if it has a refutation by tableau from T, i.e. $T \vdash \neg\varphi$.

Examples



$F((\forall x)(P(x) \to Q(x)) \to ((\forall x)P(x) \to (\forall x)Q(x)))$

$T(\forall x)(P(x) \to Q(x))$

$F((\forall x)P(x) \to (\forall x)Q(x))$

$T(\forall x)P(x)$

$F(\forall x)Q(x)$

$FQ(c)$ c new

$T(\forall x)P(x)$

$TP(c)$

$T(\forall x)(P(x) \to Q(x))$

$T(P(c) \to Q(c))$

$FP(c)$ $TQ(c)$

$\otimes$ $\otimes$

$F((\forall x)(\varphi(x) \wedge \psi(x)) \leftrightarrow ((\forall x)\varphi(x) \wedge (\forall x)\psi(x)))$

$T((\forall x)(\varphi(x) \wedge \psi(x)))$ $F((\forall x)(\varphi(x) \wedge \psi(x)))$

$F((\forall x)\varphi(x) \wedge (\forall x)\psi(x))$ $T((\forall x)\varphi(x) \wedge (\forall x)\psi(x))$

$F(\forall x)\varphi(x)$ $F(\forall x)\psi(x)$ $T(\forall x)\varphi(x)$

$F\varphi(c)$ c new $F\psi(d)$ d new $T(\forall x)\psi(x)$

$T((\forall x)(\varphi(x) \wedge \psi(x)))$ $T((\forall x)(\varphi(x) \wedge \psi(x)))$ $F(\varphi(e) \wedge \psi(e))$ e new

$T(\varphi(c) \wedge \psi(c))$ $T(\varphi(d) \wedge \psi(d))$ $F\varphi(e)$ $F\psi(e)$

$T\varphi(c)$ $T\varphi(d)$ $T(\forall x)\varphi(x)$ $T(\forall x)\psi(x)$

$T\psi(c)$ $T\psi(d)$ $T\varphi(e)$ $T\psi(e)$

$\otimes$ $\otimes$ $\otimes$ $\otimes$

Finished tableau

A finished noncontradictory branch should provide us with a counterexample.

An occurrence of an entry P in a node v of a tableau τ is *i-th* if v has exactly $i - 1$ predecessors labeled by P; and is *reduced* on a branch V through v if

$a)$ P is neither in form of $T(\forall x)\varphi(x)$ nor $F(\exists x)\varphi(x)$ and P occurs on V as a root of an atomic tableau, i.e. it was already expanded on V, or

$b)$ P is in form of $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$, P has an $(i+1)$-th occurrence on V, and V contains an entry $T\varphi(x/t_i)$ resp. $F\varphi(x/t_i)$ where t_i is the i-th ground term (of the language L_C).

Let V be a branch in a tableau τ from a theory T. We say that

- V is *finished* if it is contradictory, or every occurrence of an entry on V is reduced on V and, moreover, V contains $T\varphi$ for every $\varphi \in T$,
- τ is *finished* if every branch in τ is finished.

Systematic tableau - construction

Let R be an entry and $T = \{\varphi_0, \varphi_1, \dots\}$ be a (possibly infinite) theory.

(1) We take the atomic tableau for R as τ_0. In case $(*)$ we choose any $c \in L_C \setminus L$, in case $(\sharp)$ we take t_1 for t. Till possible, proceed as follows.

(2) Let v be the leftmost node in the smallest level as possible in tableau τ_n containing an occurrence of an entry P that is not reduced on some noncontradictory branch through v. (If v does not exist, we take $\tau_n' = \tau_n$.)

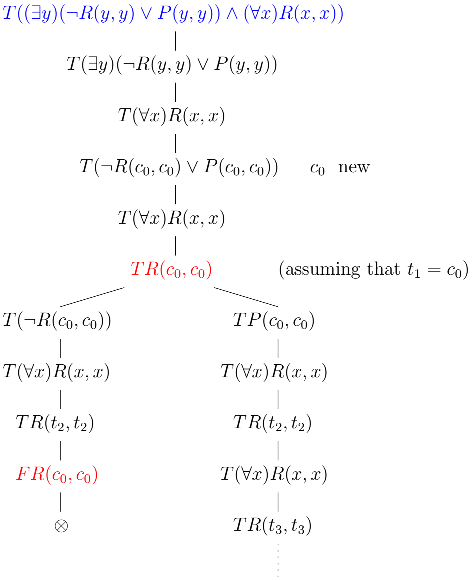
$(3a)$ If P is neither $T(\forall x)\varphi(x)$ nor $F(\exists x)\varphi(x)$, let τ_n' be the tableau obtained from τ_n by adjoining the atomic tableau for P to every noncontradictory branch through v. In case $(*)$ we choose c_i for the smallest possible i.

$(3b)$ If P is $T(\forall x)\varphi(x)$ or $F(\exists x)\varphi(x)$ and it has i-th occurrence in v, let τ_n' be the tableau obtained from τ_n by adjoining atomic tableau for P to every noncontradictory branch through v, where we take the term t_i for t.

(4) Let τ_{n+1} be the tableau obtained from τ_n' by adjoining $T\varphi_n$ to every noncontradictory branch that does not contain $T\varphi_n$ yet. (If φ_n does not exist, we take $\tau_{n+1} = \tau_n'$.)

The *systematic tableau* for R from T is the result $\tau = \cup\tau_n$ of this construction.

Systematic tableau - an example



$T((\exists y)(\neg R(y,y) \vee P(y,y)) \wedge (\forall x)R(x,x))$

$T(\exists y)(\neg R(y,y) \vee P(y,y))$

$T(\forall x)R(x,x)$

$T(\neg R(c_0,c_0) \vee P(c_0,c_0))$ c_0 new

$T(\forall x)R(x,x)$

$TR(c_0,c_0)$ (assuming that $t_1 = c_0$)

$T(\neg R(c_0,c_0))$

$T(\forall x)R(x,x)$

$TR(t_2,t_2)$

$FR(c_0,c_0)$

$\otimes$

$TP(c_0,c_0)$

$T(\forall x)R(x,x)$

$TR(t_2,t_2)$

$T(\forall x)R(x,x)$

$TR(t_3,t_3)$

$\vdots$

Systematic tableau - being finished

Proposition *Every systematic tableau is finished.*

Proof Let $\tau = \cup\tau_n$ be a systematic tableau from $T = \{\varphi_0, \varphi_1, \dots\}$ with root R and let P be an entry in a node v of the tableau τ.

- There are only finitely many entries in τ in levels up to the level of v.
- If the occurrence of P in v was unreduced on some noncontradictory branch in τ, it would be found in some step (2) and reduced by $(3a)$, $(3b)$.
- By step (4) every $\varphi_n \in T$ will be (no later than) in τ_{n+1} on every noncontradictory branch.
- Hence the systematic tableau τ has all branches finished. $\square$

Proposition *If a systematic tableau τ is a proof (from a theory T), it is finite.*

Proof Suppose that τ is infinite. Then by König's lemma, τ contains an infinite branch. This branch is noncontradictory since in the construction only noncontradictory branches are prolonged. But this contradicts the assumption that τ is a contradictory tableau. $\square$

Equality

Axioms of equality for a language L with equality are

(*i*) $x = x$

(*ii*) $x_1 = y_1 \ \wedge \ \cdots \ \wedge \ x_n = y_n \ \ \rightarrow \ \ f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$
for each n-ary function symbol f of the language L.

(*iii*) $x_1 = y_1 \ \wedge \ \cdots \ \wedge \ x_n = y_n \ \ \rightarrow \ \ (R(x_1, \ldots, x_n) \rightarrow R(y_1, \ldots, y_n))$
for each n-ary relation symbol R of the language L including $=$.

A *tableau proof* from a theory T in a language L *with equality* is a tableau proof from T^* where T^* denotes the extension of T by adding axioms of equality for L *(resp. their universal closures).*

Remark *In context of logic programming the equality often has other meaning than in mathematics (identity). For example in Prolog, $t_1 = t_2$ means that t_1 and t_2 are unifiable.*

Congruence and quotient structure

Let $\sim$ be an equivalence on A, $f : A^n \to A$, and $R \subseteq A^n$ for $n \in \mathbb{N}$. Then $\sim$ is

- a *congruence for the function* f if for every $x_1, \dots, x_n, y_1, \dots, y_n \in A$
$$x_1 \sim y_1 \ \wedge \ \cdots \ \wedge \ x_n \sim y_n \quad \Rightarrow \quad f(x_1, \dots, x_n) \sim f(y_1, \dots, y_n),$$
- a *congruence for the relation* R if for every $x_1, \dots, x_n, y_1, \dots, y_n \in A$
$$x_1 \sim y_1 \ \wedge \ \cdots \ \wedge \ x_n \sim y_n \quad \Rightarrow \quad (R(x_1, \dots, x_n) \Leftrightarrow R(y_1, \dots, y_n)).$$

Let an equivalence $\sim$ on A be a congruence for every function and relation in a structure $\mathcal{A} = \langle A, \mathcal{F}^A, \mathcal{R}^A \rangle$ of language $L = \langle \mathcal{F}, \mathcal{R} \rangle$. Then the *quotient* (*structure*) of $\mathcal{A}$ by $\sim$ is the structure $\mathcal{A}/\!\sim\, = \langle A/\!\sim, \mathcal{F}^{A/\sim}, \mathcal{R}^{A/\sim} \rangle$ where

$$f^{A/\sim}([x_1]_\sim, \dots, [x_n]_\sim) = [f^A(x_1, \dots, x_n)]_\sim$$
$$R^{A/\sim}([x_1]_\sim, \dots, [x_n]_\sim) \Leftrightarrow R^A(x_1, \dots, x_n)$$

for each $f \in \mathcal{F}$, $R \in \mathcal{R}$, and $x_1, \dots, x_n \in A$, i.e. the functions and relations are defined from $\mathcal{A}$ using representatives.

Example: $\underline{\mathbb{Z}}_p$ *is the quotient of* $\underline{\mathbb{Z}} = \langle \mathbb{Z}, +, -, 0 \rangle$ *by the congruence modulo* p.

Role of axioms of equality

Let $\mathcal{A}$ be a structure of a language L in which the equality is interpreted as a relation $=^A$ satisfying the axioms of equality for L, i.e. not necessarily the identity relation.

1) From axioms (i) and (iii) it follows that the relation $=^A$ is an equivalence.
2) Axioms (ii) and (iii) express that the relation $=^A$ is a congruence for every function and relation in $\mathcal{A}$.
3) If $\mathcal{A} \models T^*$ then also $(\mathcal{A}/{=^A}) \models T^*$ where $\mathcal{A}/{=^A}$ is the quotient of $\mathcal{A}$ by $=^A$. Moreover, the equality is interpreted in $\mathcal{A}/{=^A}$ as the identity relation.

On the other hand, in every model in which the equality is interpreted as the identity relation, all axioms of equality evidently hold.