Example of test problems

January 4, 2024

1. Let $T = \{(\exists x)P(x,x), (\forall x)(\exists y)R(x,y), (\forall u)(\forall v)(\exists x)(\forall y)(R(x,y) \to \neg P(u,v))\}$ be a theory of the language $L = \langle P, R\rangle$ without equality, where P, R are binary relation symbols.
   (a) Using Skolemization, find a theory T' (over a suitably extended language) equisatisfiable with T and axiomatized only by universal sentences.
   (b) Use the tableau method to prove that T' is unsatisfiable.
   (c) Let T'' be a theory consisting of matrices of all the axioms of T'. Find an unsatisfiable conjunction of ground instances of axioms of T''. *Hint: use the tableau from (b).*
   (d) Is the sentence $(\forall x)P(x,x)$ true / contradictory / independent in T? Justify your answers.

2. Let $T = \{(\forall x)(P(x) \to (\exists y)R(x,y)), (\exists y)((\exists x)R(y,x) \to \neg(\forall z)P(z)), (\forall x)P(x)\}$ be a theory in language $L = \langle P, R\rangle$ without equality where P, R is unary resp. binary relation symbol.
   (a) Applying skolemization find a theory T' (in a some extended language) such that T' is equisatisfiable with T and all axioms of T' are universal sentences.
   (b) Prove by tableau method that T' is unsatisfiable.
   (c) Let T'' denote the set of open matrices of axioms of T', so T'' is an open theory equivalent to T'. Find a conjunction of ground instances of axioms of T'' that is unsatisfiable. *Hint: use the tableau from (b).*
   (d) Find some complete extension of T or explain why no such extension exists.

3. We know that:
   (i) Aristotle is Greek and Caesar is Roman and Dido is Carthaginian.
   (ii) No Greek is Roman.
   (iii) No Carthaginian is Greek.
   (iv) Only Carthaginians were born in Carthage.

   Using resolution, we want to prove that:

   (v) There exists someone who was not born in Carthage and who is not Roman.

   In particular:

   (a) Express the statements as <u>sentences</u> $\varphi_1, \dots, \varphi_5$ in the language $L = \langle G, R, C, B, a, c, d\rangle$ without equality, where G, R, C, B are unary relation symbols and $G(x), R(x), C(x)$, $B(x)$ mean that "x is Greek / Roman / Carthaginian" and "x was born in Carthage", respectively, and a, c, d are constant symbols denoting Aristotle, Caesar, and Dido.
   (b) Using Skolemization, find an open theory T (possibly in an extended language) which is unsatisfiable, if and only if $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \models \varphi_5$. Convert T to CNF and write it in set representation.
   (c) Prove by resolution that T is not satisfiable. Draw the resolution refutation in the form of a resolution tree. Write the unification used in every step.
   (d) Find a conjunction of ground instances of axioms of T which is unsatisfiable.
   (e) Do the statements (i) to (iv) imply that "Caesar was not born in Carthage"? Justify your answer.

4. Let T be the following theory in the language $L = \langle R, A \rangle$ without equality, where R is a binary relation symbol and A is a unary relation symbol:

$$\begin{aligned} T = \{ & (\exists x)(\forall y)(R(x,y) \rightarrow R(y,x)), \\ & (\forall x)((\exists y)(R(x,y) \wedge R(y,x)) \rightarrow \neg A(x)), \\ & (\exists y)R(x,y), \\ & \neg(\exists x)\neg A(x)\} \end{aligned}$$

   (a) Using Skolemization, find an open theory T' (over a suitable extension of L) equisatisfiable with T.

   (b) Convert T' to an equivalent theory S in CNF. Write S in set representation.

   (c) Find a resolution refutation of the theory S. Draw the resolution tree and in each step, write the unification used.

   (d) Find an unsatisfiable conjunction of ground instances of axioms of S *Hint: Use the unifications from (c).*

   (e) Does the theory T have a complete simple extension? If yes, give an example. If not, explain why.

5. We know that:

   (i) Parents are older than their children.

   (ii) "Being a parent" is an asymmetric relation.

   (iii) "Being older" is a transitive relation.

   (iv) Tom is a father of Mary, Mary is not older than Bob, Bob is a son of Jane.

   Use the resolution method to show that:

   (v) Tom is older than Bob or Mary is not a mother of Jane.

   In particular:

   (a) Express the statements (i) to (v) by <u>open</u> formulæφ_1 to φ_5 of the language $L = \langle P, O, t, m, b, j \rangle$ without equality, where P, O are binary relation symbols, $P(x,y)$, $O(x,y)$ denotes that "x is a parent of y", "x is older than y' (respectivelly), and t, m, b, j are constant symbols denoting Tom, Mary, Bob, and Jane (respectivelly).

   (b) By transforming to CNF, find an open theory S in set representation which is unsatisfiable, if and only if φ_5 is valid in the theory $T = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$.

   (c) Prove by resolution that S is not satisfiablen. Depict the resolution refutation of S by a resolution tree. At every step, write down the unification used.

   (d) Find an unsatisfiable conjunction of ground instances of axioms of S.

   (e) Is the theory T complete? Justify your answer.