## Predicate and Propositional Logic - Tutorial 2

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1. Find first order formulas (with use of equality) expressing for a fixed $n > 0$ that
   (a) "there exist at least $n$ elements",
   (b) "there exist at most $n$ elements",
   (c) "there exist exactly $n$ elements"

   Is it possible to express with use of (possibly infinite) set of formulas that "there are infinitely many elements"?

2. Find a second-order formula expressing "there exist finitely many elements". Hint:
   (a) Find first-order formulas (with a symbol $f$ for a function) expressing "$f$ is injective", "$f$ is surjective".
   (b) Find a second-order formula expressing "every function that is surjective is also injective".

3. Can we color the integers from 1 to $n$ with two colors such that there is no monochromatic solution of an equation $a + b = c$ with $1 \leq a < b < c \leq n$? Write a proposition $\varphi_n$ for $n = 8$ that is satisfiable if and only if such coloring exists.

4. *(Pigeonhole principle).* Let $n \geq 2$ be a fixed natural number. Assume that we have $n$ pigeons and $n - 1$ pigeonholes. We want to express that
   $(i)$ Every pigeon sits in some pigeonhole,
   $(ii)$ there is no pigeonhole with more than one pigeon sitting in it.

   Let $\mathbb{P} = \{p^i_j \mid 1 \leq i \leq n, 1 \leq j \leq n-1\}$ be a set of propositional variables, where $p^i_j$ represents that *"the $i$-th pigeon sits in the $j$-th pigeonhole"*.

   (a) Write propositions $\varphi_i$ and $\psi_j$ over $\mathbb{P}$ expressing that *"the $i$-th pigeon sits in some pigeonhole"* and *"in the $j$-th pigeonhole sits not more than one pigeon"*, respectively, where $1 \leq i \leq n$, $1 \leq j \leq n-1$. Write a set of propositions expressing $(i)$ and $(ii)$.

5. Three proposals are being discussed in the parliament: *school charges*, *tax increase*, *restriction of smoking in restaurants.*
   $(i)$ *The party A demands that in case the party B or the party C has his demand fulfilled, there will be no school charges or no tax increase.*
   $(ii)$ *The party B wants to restrict smoking if the party C does not have his demand fulfilled or tax do not increase.*
   $(iii)$ *The party C requires that in case the party A has his demand fulfilled, there will be no tax increase and no smoking restriction.*
   $(iv)$ *In the final voting exactly two parties had their demands fulfilled.*

   Let the propositional letters $p$, $q$, $r$ represent (respectively) that the proposals on *school charges, tax increase, smoking restriction* have been passed. Furthermore, let $a$, $b$, $c$ represent (respectively) that each party's demand has been fulfilled and let $\mathbb{P} = \{p, q, r, a, b, c\}$.

   (a) Write propositions $\varphi_1$, $\varphi_2$, $\varphi_3$ in the form of an equivalence and a proposition $\varphi_4$ over $\mathbb{P}$ that express (respectively) $(i)$, $(ii)$, $(iii)$, and $(iv)$.

6. Consider a theory $T = \{\neg q \to (\neg p \vee q),\ \neg p \to q,\ r \to q\}$. Which of the following propositions are valid, contradictory, independent, satisfiable, equivalent in $T$?

   (a) $p$, $q$, $r$, $s$

   (b) $p \vee q$, $p \vee r$, $p \vee s$, $q \vee s$

   (c) $p \wedge q$, $q \wedge s$, $p \to q$, $s \to q$

7. Prove or disprove that the following sets of connectives are adequate.

   (a) $\{\downarrow\}$ where $\downarrow$ is Peirce arrow (NOR)

   (b) $\{\uparrow\}$ where $\uparrow$ is Sheffer stroke (NAND)

   (c) $\{\vee, \to, \leftrightarrow\}$, $\{\vee, \wedge, \to\}$

8. Transform the following propositions into DNF and CNF a) by using truth tables (determining the models), b) by using transformation rules.

   (a) $(\neg p \vee q) \to (\neg q \wedge r)$

   (b) $(\neg p \to (\neg q \to r)) \to p$

   (c) $((p \to \neg q) \to \neg r) \to \neg p$

9. Applying the implication graph determine whether the following proposition in 2-CNF is satisfiable or not; and if yes, find a satisfying assignment.

$$(p_0 \vee p_2) \wedge (p_0 \vee \neg p_3) \wedge (p_1 \vee \neg p_3) \wedge (p_1 \vee \neg p_4) \wedge (p_2 \vee \neg p_4) \wedge (p_0 \vee \neg p_5) \wedge$$
$$(p_1 \vee \neg p_5) \wedge (p_2 \vee \neg p_5) \wedge (\neg p_1 \vee \neg p_6) \wedge (p_4 \vee p_6) \wedge (p_5 \vee p_6) \wedge p_1$$

10. Applying unit propagation determine whether the following Horn formula is satisfiable; and if yes, find a satisfying assignment.

$$(\neg p_1 \vee \neg p_3 \vee p_2) \wedge (\neg p_1 \vee p_2) \wedge p_1 \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge$$
$$(\neg p_2 \vee \neg p_4 \vee p_1) \wedge (p_4 \vee \neg p_3 \vee \neg p_2) \wedge (\neg p_4 \vee p_5)$$

11. Find both DNF and CNF representations of the Boolean function $\mathrm{maj}\colon \{0,1\}^3 \to \{0,1\}$ defined as the majority of the three (truth) values.

12. Let $\mathrm{maj}_n\colon (\{0,1\}^n)^3 \to \{0,1\}^n$ be the coordinate-wise majority function; that is, for example

$$\mathrm{maj}_4((0,1,0,1),(1,1,0,0),(1,1,0,0)) = (1,1,0,0)$$

    We say that a set $K \subseteq \{0,1\}^n$ is a *median* set if it is closed under $\mathrm{maj}_n$.

    (a) Show that for every 2-CNF proposition $\varphi$ it holds that $M(\varphi)$ is a median set.

    (b)* Show that for every median set $K \subseteq \{0,1\}^n$ there exists a 2-CNF proposition $\varphi$ over $n$ variables such that $M(\varphi) = K$.