Predicate and Propositional Logic - Tutorial 3

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1. Applying the implication graph determine whether the following proposition in 2-CNF is satisfiable or not; and if yes, find a satisfying assignment.

$$(p_0 \vee p_2) \wedge (p_0 \vee \neg p_3) \wedge (p_1 \vee \neg p_3) \wedge (p_1 \vee \neg p_4) \wedge (p_2 \vee \neg p_4) \wedge (p_0 \vee \neg p_5) \wedge$$
$$(p_1 \vee \neg p_5) \wedge (p_2 \vee \neg p_5) \wedge (\neg p_1 \vee \neg p_6) \wedge (p_4 \vee p_6) \wedge (p_5 \vee p_6) \wedge p_1$$

2. Applying unit propagation determine whether the following Horn formula is satisfiable; and if yes, find a satisfying assignment.

$$(\neg p_1 \vee \neg p_3 \vee p_2) \wedge (\neg p_1 \vee p_2) \wedge p_1 \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge$$
$$(\neg p_2 \vee \neg p_4 \vee p_1) \wedge (p_4 \vee \neg p_3 \vee \neg p_2) \wedge (\neg p_4 \vee p_5)$$

3. Find both DNF and CNF representations of the Boolean function $\mathrm{maj}\colon \{0,1\}^3 \to \{0,1\}$ defined as the majority of the three (truth) values.

4. Let $\mathrm{maj}_n\colon (\{0,1\}^n)^3 \to \{0,1\}^n$ be the coordinate-wise majority function; that is, for example

$$\mathrm{maj}_4((0,1,0,1),(1,1,0,0),(1,1,0,0)) = (1,1,0,0)$$

We say that a set $K \subseteq \{0,1\}^n$ is a *median* set if it is closed under maj_n.

(a) Show that for every 2-CNF proposition φ it holds that $M(\varphi)$ is a median set.

(b)* Show that for every median set $K \subseteq \{0,1\}^n$ there exists a 2-CNF proposition φ over n variables such that $M(\varphi) = K$.

5. Consider an infinite theory $T = \{p_i \to (p_{i+1} \vee q_{i+1}), q_i \to (p_{i+1} \vee q_{i+1}) \mid i \in \mathbb{N}\}$ over $\mathrm{var}(T)$.

(a) Which propositions in the form $p_i \to p_j$ are logical consequences of T?

(b) Which propositions in the form $p_i \to (p_j \vee q_j)$ are logical consequences of T?

(c) Determine all models of the theory T.

6. Prove or disprove (or find the correct relation) that for every theory T and propositions φ, ψ over $\mathbb{P}$ it holds

(a) $T \models \varphi$, if and only if $T \not\models \neg\varphi$

(b) $T \models \varphi$ and $T \models \psi$, if and only if $T \models \varphi \wedge \psi$

(c) $T \models \varphi$ or $T \models \psi$, if and only if $T \models \varphi \vee \psi$

(d) $T \models \varphi \to \psi$ and $T \models \psi \to \chi$, if and only if $T \models \varphi \to \chi$

7. Prove or disprove (or find the correct relation). For every theories T and S over $\mathbb{P}$

(a) $S \subseteq T \Rightarrow \theta^{\mathbb{P}}(T) \subseteq \theta^{\mathbb{P}}(S)$

(b) $\theta^{\mathbb{P}}(S \cup T) = \theta^{\mathbb{P}}(S) \cup \theta^{\mathbb{P}}(T)$

(c) $\theta^{\mathbb{P}}(S \cap T) = \theta^{\mathbb{P}}(S) \cap \theta^{\mathbb{P}}(T)$

8. Let $|\mathbb{P}| = n$ and $\varphi \in \mathrm{VF}_{\mathbb{P}}$ with $|M(\varphi)| = m$.

(a) What is the number of nonequivalent propositions ψ such that $\varphi \models \psi$ or $\psi \models \varphi$?

(b) What is the number of nonequivalent theories over $\mathbb{P}$ in which φ is valid? What is the number of nonequivalent *complete* theories over $\mathbb{P}$ in which φ is valid?

(c) What is the number of nonequivalent theories T over $\mathbb{P}$ such that $T \cup \{\varphi\}$ is satisfiable?

(d) Let, moreover, $\{\varphi, \psi\}$ be an unsatisfiable theory with $|M(\psi)| = p$. What is the number of nonequivalent propositions χ such that $\varphi \vee \psi \models \chi$? What is the number of nonequivalent theories in which $\varphi \vee \psi$ is valid?

9. Let $T = \{q \to (\neg p \to r),\ \neg r \to (\neg p \wedge q),\ (s \to r) \to p\}$ be a theory over the language $\mathbb{P} = \{p, q, r, s\}$.
   (a) Axiomatize the theory T by a proposition in CNF.
   (b) Find all models of the theory T.
   (c) Is the theory T an extension of the theory $S = \{q \leftrightarrow \neg r\}$ over the language $\{q, r\}$? Is T a conservative extension of S? Justify.
   (d) Determine the number of mutually inequivalent propositions in the language $\mathbb{P}$ that are *contradictory* in both the theories T and S. Justify.

10. Let $T = \{p \vee q \to r, \neg(p \to \neg s)\}$ be a theory over the propositional language $\mathbb{P} = \{p, q, r, s\}$.
    (a) Is the proposition $q \to p$ valid in the theory T? Is it contradictory? Is it independent? Justify.
    (b) Find all models of the theory T.
    (c) Find a theory S over the language $\mathbb{P}' = \{p, q, r\}$ such that T is a conservative extension of S. Axiomatize S by a proposition in CNF. Justify why S has the desired property.
    (d) Determine the number of mutually inequivalent propositions φ over $\mathbb{P}$ such that φ is valid in T and independent in S.

11. Let $T = \{p,\ \neg q \to \neg r,\ \neg q \to \neg s,\ r \to p,\ \neg s \to \neg p\}$ be a theory over the language $\mathbb{P} = \{p, q, r, s\}$.
    (a) Using the implication graph, show that T is satisfiable.
    (b) Find all models of the theory T and axiomatize $M^{\mathbb{P}}(T)$ by a proposition in CNF.
    (c) Determine, and justify, the number of mutually
        (i) inequivalent propositions over $\mathbb{P}$ that are independent in T,
        (ii) T-inequivalent propositions over $\mathbb{P}$ which are independent in T.

12. Let $T = \{(r \to p) \to \neg q,\ \neg q \to p, \neg(r \wedge q), r \to \neg s\}$ be a theory in the language $\mathbb{P} = \{p, q, r, s\}$.
    (a) Axiomatize $M^{\mathbb{P}}(T)$ by a proposition in CNF.
    (b) Is the theory T a conservative extension of some theory over the language $\{p, q, r\}$? Justify.
    (c) Determine the number of mutually inequivalent noncontradictory extensions of the theory T over the language $\{p, q, r, s, t\}$. Justify.

13. Let $T = \{p \to \neg q \wedge r, q \vee r, (q \wedge s) \leftrightarrow r\}$ be a theory over the language $\mathbb{P} = \{p, q, r, s\}$.
    (a) Is the proposition $q \to p$ valid in the theory T? Is it contradictory? Is it independent? Justify your answers.
    (b) Axiomatize $M(T)$ by a proposition in CNF.
    (c) Determine the number of mutually inequivalent theories S over $\mathbb{P}' = \{r, s\}$ such that T is a conservative extension of S. How many of them are complete?