Predicate and Propositional Logic - Tutorial 5

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1. Let $\mathcal{S}$ be a countable nonempty family of nonempty finite sets. We say that $\mathcal{S}$ has a *selector* if there exists an injective $f : \mathcal{S} \to \bigcup \mathcal{S}$ such that $f(S) \in S$ for every $S \in \mathcal{S}$. Prove that $\mathcal{S}$ has a selector if and only if every nonempty finite part of $\mathcal{S}$ has a selector.

2. Let φ be the proposition $\neg(p \vee q) \to (\neg p \wedge \neg q)$.

   (a) Transform $\neg\varphi$ into CNF and into set representation (clausal form).

   (b) Find a resolution refutation of $\neg\varphi$; that is, a proof of φ.

3. Find resolution closures $\mathcal{R}(S)$ of the following formulas S.

   (a) $\{\{p, q\}, \{\neg p, \neg q\}, \{\neg p, q\}\}$

   (b) $\{\{p, q\}, \{p, \neg q\}, \{p, \neg q\}\}$

   (c) $\{\{p, \neg q, r\}, \{q, r\}, \{\neg p, r\}, \{q, \neg r\}, \{\neg q\}\}$

4. Find resolution refutations of the following propositions.

   (a) $(p \leftrightarrow (q \to r)) \wedge ((p \leftrightarrow q) \wedge (p \leftrightarrow \neg r))$

   (b) $\neg(((p \to q) \to \neg q) \to \neg q)$

5. Prove by resolution that s is valid in a theory $T = \{\neg p \to \neg q, \neg q \to \neg r, (r \to p) \to s\}$.

6. Show that if $S = \{C_1, C_2\}$ is satisfiable and C is a resolvent of C_1 and C_2, then C is satisfiable as well.

7. Find the *tree of reductions* of a formula $S = \{\{p, r\}, \{q, \neg r\}, \{\neg q\}, \{\neg p, t\}, \{\neg s\}, \{s, \neg t\}\}$.

8. Assume that we have available MgO, H_2, O_2, C and we can perform the following chemical reactions.

   $$\begin{array}{ll} (1) & MgO + H_2 \ \to \ Mg + H_2O \\ (2) & C + O_2 \ \to \ CO_2 \\ (3) & CO_2 + H_2O \ \to \ H_2CO_3 \end{array}$$

   (a) Represent the state of affairs as a proposition in a suitable language and transform it into a set representation.

   (b) Prove by (linear input) resolution that we can produce H_2CO_3.

9. Show that in Hilbert's calculus the following is provable for every formulas φ, ψ, χ.

   (a) $\vdash_H \varphi \to \varphi$

   (b) $T \vdash_H \varphi \to \chi$ where $T = \{\varphi \to \psi,\ \psi \to \chi\}$

   (c) $T \vdash_H \psi \to \chi$ where $T = \{\varphi,\ \psi \to (\varphi \to \chi)\}$

Example questions for the midterm test.

1. In the presidental elections we have two candidates, Mr. A and Mr. B.

   (i) *Mr. A says: "I will be elected or Mr. B lies."*

   (ii) *Mr. B says: "Mr. A will not be elected or I lie."*

   (iii) *Exactly one candidate will be elected.*

   Let the propositional atoms e_A, e_B represent that *Mr. A (resp. Mr. B) will be elected* and let t_A, t_B represent that *Mr. A (resp. Mr. B) speaks the truth.* Let us denote $\mathbb{P} = \{e_A, e_B, t_A, t_B\}$.

   (a) Write propositions φ_1, φ_2 in the form of equivalence and a proposition φ_3 in CNF expressing (in this order) (i), (ii), (iii), all over the language $\mathbb{P}$. *(20p)*

   (b) Let $T = \{\varphi_1, \varphi_2, \varphi_3\}$. Prove by tableau method that $T \models e_B$. *(40p).*

   (c) Give an example of a proposition over $\mathbb{P}$ that is independent in theory T, or show that such proposition does not exist. *(20p).*

   (d) Find a theory S over $\{e_A, e_B\}$ such that T is a conservative extension of S. *(20p)*

2. *(Pigeonhole principle).* Let $n \geq 2$ be a fixed natural number. Assume that we have n pigeons and $n-1$ pigeonholes. We want to show (by resolution) that the following two statements cannot both be true:

   (i) *Every pigeon sits in some pigeonhole,*

   (ii) *there is no pigeonhole with more than one pigeon sitting in it.*

   Let $\mathbb{P} = \{p^i_j \mid 1 \leq i \leq n, 1 \leq j \leq n-1\}$ be a set of propositional variables, where p^i_j represents that *"the i-th pigeon sits in the j-th pigeonhole".*

   (a) Write propositions φ_i and ψ_j over $\mathbb{P}$ expressing that *"the i-th pigeon sits in some pigeonhole"* and *"in the j-th pigeonhole sits not more than one pigeon"*, respectively, where $1 \leq i \leq n$, $1 \leq j \leq n-1$. Using the propositions φ_i and ψ_j construct a theory T_n expressing (i) and (ii). *(20p)*

   (b) Now let $n = 3$ and $T' = T_3 \cup \{p^1_1\}$, that is, we additionally assume that *"the 1st pigeon sits in the 1st pigeonhole".* Convert T' to set representation. *(20p)*

   (c) Show that $T' \vdash_R \square$. Draw the resolution refutation in the form of a resolution tree. *(40p)*

   (d) Let $T^* = T' \setminus \{\psi_2\}$ be a theory over $\mathbb{P}$. Is the theory T' a conservative extension of the theory T^*? Justify your answer. *(20p)*

3. Let $T = \{(\neg p \wedge q) \rightarrow r,\ (q \rightarrow r) \leftrightarrow p\}$ be a theory over the language $\mathbb{P} = \{p, q, r\}$.

   (a) Use the tableau method to find all models of the theory T. *(40p)*

   (b) Axiomatize $M^{\mathbb{P}}(T)$ by a proposition in DNF and also by a proposition in CNF. *(20p)*

   (c) Is T an extension of the theory $S = \{q \rightarrow p\}$ over the language $\{p, q\}$? Is T a conservative extension of S? Justify your answers. *(20p)*

   (d) Determine the number of mutually inequivalent propositions over $\mathbb{P}$ that are independent in both S and T. Justify your answer. *(20p)*