

Predicate and Propositional Logic - Tutorial 7

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- Which of the variable occurrences are free/bound in the following formulas? Find variants of these formulas without variables that have both free and bound occurrence.

(a) $(\exists x)(\forall y)P(y, z) \vee (y = 0)$

(b) $(\exists x)(P(x) \wedge (\forall x)Q(x)) \vee (x = 0)$

(c) $(\exists x)(x > y) \wedge (\exists y)(y > x)$

- Let φ denote the formula $(\forall x)((x = z) \vee (\exists y)(f(x) = y) \vee (\forall z)(y = f(z)))$. Which of the following terms are substitutable into φ ?

(a) the term z for the variable x , the term y for the variable x ,

(b) the term z for the variable y , the term $2 * y$ for the variable y ,

(c) the term x for the variable z , the term y for the variable z ,

- Are the following formulas variants of the formula $(\forall x)(x < y \vee (\exists z)(z = y \wedge z \neq x))$?

(a) $(\forall z)(z < y \vee (\exists z)(z = y \wedge z \neq z))$

(b) $(\forall y)(y < y \vee (\exists z)(z = y \wedge z \neq y))$

(c) $(\forall u)(u < y \vee (\exists z)(z = y \wedge z \neq u))$

- Let $\mathcal{A} = (\{a, b, c, d\}, \triangleright^A)$ be a structure for the language with a single binary relation symbol \triangleright , where $\triangleright^A = \{(a, c), (b, c), (c, c), (c, d)\}$. Which of the following formulas are valid in \mathcal{A} ?

(a) $x \triangleright y$

(b) $(\exists x)(\forall y)(y \triangleright x)$

(c) $(\exists x)(\forall y)((y \triangleright x) \rightarrow (x \triangleright x))$

(d) $(\forall x)(\forall y)(\exists z)((x \triangleright z) \wedge (z \triangleright y))$

(e) $(\forall x)(\exists y)((x \triangleright z) \vee (z \triangleright y))$

- For every formula φ from the previous exercise find a structure \mathcal{B} (if it exists) such that $\mathcal{B} \models \varphi$ if and only if $\mathcal{A} \not\models \varphi$.

- Are the following sentences valid / contradictory / independent (in logic)?

(a) $(\exists x)(\forall y)(P(x) \vee \neg P(y))$

(b) $(\forall x)(P(x) \rightarrow Q(f(x))) \wedge (\forall x)P(x) \wedge (\exists x)\neg Q(x)$

(c) $(\forall x)(P(x) \vee Q(x)) \rightarrow ((\forall x)(P(x) \vee (\forall x)Q(x)))$

(d) $(\forall x)(P(x) \rightarrow Q(x)) \rightarrow ((\exists x)P(x) \rightarrow (\exists x)Q(x))$

(e) $(\exists x)(\forall y)P(x, y) \rightarrow (\forall y)(\exists x)P(x, y)$

- Prove (semantically) the following claims. For every structure \mathcal{A} , formula φ , and sentence ψ ,

(a) $\mathcal{A} \models (\psi \rightarrow (\exists x)\varphi) \Leftrightarrow \mathcal{A} \models (\exists x)(\psi \rightarrow \varphi)$

(b) $\mathcal{A} \models (\psi \rightarrow (\forall x)\varphi) \Leftrightarrow \mathcal{A} \models (\forall x)(\psi \rightarrow \varphi)$

(c) $\mathcal{A} \models ((\exists x)\varphi \rightarrow \psi) \Leftrightarrow \mathcal{A} \models (\forall x)(\varphi \rightarrow \psi)$

(d) $\mathcal{A} \models ((\forall x)\varphi \rightarrow \psi) \Leftrightarrow \mathcal{A} \models (\exists x)(\varphi \rightarrow \psi)$

Does this hold also for every formula ψ with a free variable x ? And for every formula ψ in which x is not free?