Predicate and Propositional Logic - Tutorial 9

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1. Let $\underline{\mathbb{Q}} = \langle \mathbb{Q}, +, -, \cdot, 0, 1\rangle$ be the structure of rational numbers with standard operations (thus forming a *field*) and let T be the theory of groups (see previous tutorials).
   (a) Is there a reduct of $\underline{\mathbb{Q}}$ that is a model of T?
   (b) Can we expand the reduct $\langle \mathbb{Q}, \cdot, 1\rangle$ to a model of T?
   (c) Does $\underline{\mathbb{Q}}$ contain a substructure that is not elementary equivalent to $\underline{\mathbb{Q}}$?
   (d) Let $Th(\underline{\mathbb{Q}})$ denote the set of all sentences valid in $\underline{\mathbb{Q}}$. Is $Th(\underline{\mathbb{Q}})$ a complete theory?

2. Let $T = \{x = c_1 \vee x = c_2 \vee x = c_3\}$ be a theory of $L = \langle c_1, c_2, c_3\rangle$ with equality.
   (a) Is T (semantically) consistent?
   (b) Are all models of T elementarily equivalent? That is, is T (semantically) complete?
   (c) Find all simple complete extensions of T.
   (d) Is a theory $T' = T \cup \{x = c_1 \vee x = c_4\}$ of the language $L = \langle c_1, c_2, c_3, c_4\rangle$ an extension of T? Is T' a simple extension of T? Is T' a conservative extension of T?

3. Let T' be the extension of $T = \{(\exists y)(x + y = 0), (x + y = 0) \wedge (x + z = 0) \rightarrow y = z\}$ in $L = \langle +, 0, \leq\rangle$ with equality by definitions of $<$ and unary $-$ with axioms

$$
\begin{aligned}
-x = y &\quad\leftrightarrow\quad x + y = 0\\
x < y &\quad\leftrightarrow\quad x \leq y \ \wedge\ \neg(x = y)
\end{aligned}
$$

   Find formulas of L that are equivalent in T' to the following formulas.
   (a) $x + (-x) = 0$
   (b) $x + (-y) < x$
   (c) $-(x + y) < -x$

4. Consider the following database as a relational structure $\mathcal{D} = \langle D, Movies, Program, c^D\rangle_{c\in D}$ of language $L = \langle F, P, c\rangle_{c\in D}$ with equality where $D = \{$'Po strništi bos', 'J. Tříska', 'Mat', '13:15', $\dots\}$ and $c^D = c$ for every $c \in D$. Write formulas that define in $\mathcal{D}$ tables of
   (a) movies in which a director is acting,
   (b) cinemas and times where and when one can see a movie in which a director is acting,
   (c) directors that act in movies that are on program in the cinema Mat,
   (d) actors or directors whose movie is not on a program in any cinema.

Movie	*name*	*director*	*actor*
	Lidé z Maringotek	M. Frič	J. Tříska
	Po strništi bos	J. Svěrák	Z. Svěrák
	Po strništi bos	J. Svěrák	J. Tříska
	...	...	...

Program	*cinema*	*name*	*time*
	Světozor	Po strništi bos	13:15
	Mat	Po strništi bos	16:15
	Mat	Lidé z Maringotek	18:30
	...	...	...

5. Let $L = \langle F\rangle$ be a language with equality where F is a binary function symbol. Write formulas that define (without parameters) the following sets in the following structures:
   (a) the interval $(0, \infty)$ in $\mathcal{A} = \langle \mathbb{R}, \cdot\rangle$ where $\cdot$ is the standard multiplication of real numbers,
   (b) the set $\{(x, 1/x) \mid x \neq 0\}$ in the same structure $\mathcal{A}$,
   (c) the set of all at most one-element subsets of $\mathbb{N}$ in $\mathcal{B} = \langle \mathcal{P}(\mathbb{N}), \cup\rangle$.

6. Assume that

   (a) all guilty persons are liars,
   (b) at least one of the accused is also a witness,
   (c) no witness lies.

   Prove by tableau method that not all accused are guilty.

7. Let $L(x, y)$ represent that *"there is a flight from x to y"* and let $S(x, y)$ represent that *"there is a connection from x to y"*. Assume that

   (a) From Prague you can fly to Bratislava, London and New York, and from New York to Paris,
   (b) $(\forall x)(\forall y)(L(x, y) \rightarrow L(y, x))$,
   (c) $(\forall x)(\forall y)(L(x, y) \rightarrow S(x, y))$,
   (d) $(\forall x)(\forall y)(\forall z)(S(x, y) \wedge L(y, z) \rightarrow S(x, z))$.

   Prove by tableau method that there is a connection from Bratislava to Paris.

8. Let φ, ψ be sentences or formulas in a free variable x, denoted by $\varphi(x)$, $\psi(x)$. Find tableau proofs of the following formulas.

   (a) $(\exists x)(\varphi(x) \vee \psi(x)) \leftrightarrow (\exists x)\varphi(x) \vee (\exists x)\psi(x)$,
   (b) $(\forall x)(\varphi(x) \wedge \psi(x)) \leftrightarrow (\forall x)\varphi(x) \wedge (\forall x)\psi(x)$,
   (c) $(\varphi \vee (\forall x)\psi(x)) \rightarrow (\forall x)(\varphi \vee \psi(x))$ where x is not free in φ,
   (d) $(\varphi \wedge (\exists x)\psi(x)) \rightarrow (\exists x)(\varphi \wedge \psi(x))$ where x is not free in φ.
   (e) $(\exists x)(\varphi \rightarrow \psi(x)) \rightarrow (\varphi \rightarrow (\exists x)\psi(x))$ where x is not free in φ,
   (f) $(\exists x)(\varphi \wedge \psi(x)) \rightarrow (\varphi \wedge (\exists x)\psi(x))$ where x is not free in φ,
   (g) $(\exists x)(\varphi(x) \rightarrow \psi) \rightarrow ((\forall x)\varphi(x) \rightarrow \psi)$ where x is not free in ψ,
   (h) $((\exists x)\varphi(x) \rightarrow \psi) \rightarrow (\forall x)(\varphi(x) \rightarrow \psi)$ where x is not free in ψ.

9. Let T^* be a theory with axioms of equality. Prove by tableau method that

   (a) $T^* \models x = y \;\rightarrow\; y = x$ (symmetry of $=$)
   (b) $T^* \models (x = y \;\wedge\; y = z) \;\rightarrow\; x = z$ (transitivity of $=$)

   Hint: To show (a) apply the axiom of equality (iii) for $x_1 = x$, $x_2 = x$, $y_1 = y$ a $y_2 = x$, to show (b) apply (iii) for $x_1 = x$, $x_2 = y$, $y_1 = x$ a $y_2 = z$.

10. Let L be a language with equality containing a binary relation symbol $\leq$ and let T be a theory of L such that T has an infinite model and the axioms of linear ordering are valid in T. Applying the compactness theorem show that T has a model $\mathcal{A}$ with an *infinite decreasing chain*; that is, there are elements c_i for every $i \in \mathbb{N}$ in A such that

$$\cdots < c_{n+1} < c_n < \cdots < c_0.$$

    (This shows that the notion of *well-ordering* is not definable in a first-order language.)