Charles University

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Doctoral Dissertation

# Subgraphs of Hypercubes - Embeddings with Restrictions or Prescriptions

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*to my beloved family*

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*The ideal situation occurs when the things that we regard as beautiful are also regarded by other people as useful.*

— Donald Knuth

# Contents

# 1 Introduction

## 1.1 Hypercube and graph embeddings

Hypercube ($n$-cube or Boolean lattice) plays an important role in many areas of discrete mathematics and computer science. It represents a binary $n$-dimensional space, subsets over $n$ elements or binary strings of a length $n$.

It is also one of the most versatile and efficient networks for the architecture of parallel computers. It is well suited for the design of parallel algorithms and for simulations of other popular networks. It has many variants and modifications. An excellent survey on hypercubes can be found in [48].

Graph embeddings in the theory of parallel computation are used to study simulations of parallel architectures. Given graphs $G$ and $H$ of a guest and a host network we give mappings $f : V(G) \to V(H)$ and $g : E(G) \to$ paths in $H$ such that for an edge $\{u, v\} \in E(G)$ the path $g(\{u, v\})$ connects $f(u)$ and $f(v)$. The mapping $f$ describes which host vertex will simulate a given guest vertex and the mapping $g$ describes routes in the host network along which messages will traverse. This pair of mappings is called an *embedding* of $G$ into $H$ and the following cost measures are defined.

A *dilation of an edge* $\{u, v\} \in E(G)$ is the length of the path $g(\{u, v\})$ and a *dilation* of the embedding is the maximum of edge dilations. A *load of a vertex* $v \in V(H)$ is the number of vertices of $G$ mapped to $v$ and a *load* of the embedding is the maximum of vertex loads. A *congestion of an edge* $\{u, v\} \in E(H)$ is the number of edges of $E(H)$ mapped to the path going through $\{u, v\}$ and an *edge congestion* of the embedding is the maximum of edge congestions.

The dilation, load and congestion determine the time required to simulate one step of $G$ on $H$. In particular, it has been shown [49] that given an embedding of $G$ in $H$ with dilation $d$, load $l$, and congestion $c$, $H$ can simulate any computation of $G$ with slowdown $O(d+l+c)$. It should be mentioned, that there are also other ways for simulating networks. For example, in [50] an emulation using a redundant computation gives a significant improvement in the number of tolerable faults for reconfiguring of a faulty butterfly network.

The optimal embedding with $load = 1$ and $dilation = 1$ (consequently $congestion = 1$) is called a direct embedding. In fact, a direct embedding of $G$ into $H$ maps $G$ to an isomorphic subgraph of $H$, i.e. it is a monomorphism of $G$ into $H$. It is known that the decision problem of a subgraph isomorphism is NP-complete in general [22].

## 1.2 Restrictions and prescriptions

One of the advantages of the hypercube is its robustness. The regular structure and a rich interconnection topology make it possible to reconfigure its applications in the presence of faults [30, 31].

One approach to fault-tolerance is to study the hypercube without removed vertices or edges that are considered faulty [44, 50, 64]. This is applicable also for situations when part of the hypercube is busy. Fault-tolerant embeddings that avoid restricted vertices or edges have been found in [7, 8, 21, 58].

The opposite of restrictions are prescriptions. Given a set of prescribed vertices or edges we require that the embedding covers all of them. This can be useful for example for connecting parts of the previous computation together or for following a certain pattern of communication within the network.

The covering embedding can be used to find also an avoiding embedding and vice versa. And of course, the including and excluding conditions can be combined.

## 1.3 Subgraphs of hypercube

Hypercube of dimension $n$ has a regular degree $n$ which makes it complicated or practically impossible to construct hypercube machines of high dimensions. Another disadvantage is the exponentially growing number of vertices $2^n$.

These reasons led to an introduction of special subgraphs of hypercube such as Fibonacci cube [35] and its various modifications. They are much sparser than hypercubes and the number of vertices increases more slowly than in the hypercubes, but as well as hypercubes, Fibonacci cubes have a self-similar recursive structure useful for the design of parallel and distributed algorithms. These graphs were also suggested for fault-tolerant embeddings but no bound on the number of tolerable faults was given.

Another subgraphs of hypercube that received a considerable amount of attention [61] are Hamiltonian paths and cycles.

## 1.4 Summary of the presented results

We study the following problems. How many vertices can be removed from hypercube under the condition that there exists a subgraph isomorphic to Fibonacci cube? How many edges of hypercube can be prescribed and under what conditions so that there is a Hamiltonian path passing through all of them?

In Chapter 2 we give an extensive survey of known results about Fibonacci cube and its modifications.

#### Fibonacci cube excluding faulty vertices - same dimension

In Chapter 3 we present a construction of a direct embedding of a Fibonacci Cube of dimension $n$ into a faulty hypercube of dimension $n$ with less or equal $2^{\lceil \frac{n}{4} \rceil - 1}$ faults.

In fact, there exists a direct embedding of a Fibonacci Cube of dimension $n$ into a faulty hypercube of dimension $n$ with at most $\frac{2^n}{4f_{n-2}}$ faults ($f_n$ is the $n$-th Fibonacci number). Thus the number $\phi(n)$ of tolerable faults grows exponentially with respect to dimension $n$, $\phi(n) = \Omega(2^{cn})$, for $c = 2 - \log_2(1+\sqrt{5}) \doteq 0.31$.

On the other hand, $\phi(n) = O(2^{dn})$, for $d = (8 - 3\log_2 3)/4 \doteq 0.82$.

As a corollary, there exists a nearly polynomial algorithm constructing a direct embedding of a Fibonacci Cube of dimension $n$ into a faulty hypercube of dimension $n$ (if it exists) provided that faults are given on input by enumeration.

However, the problem is NP-complete, if faults are given on input with an asterisk convention.

#### Fibonacci cube excluding faulty vertices - general case

In Chapter 4 we consider the problem of determining the minimum number of vertices in $n$-dimensional hypercube whose removal leaves no subgraph isomorphic to $m$-dimensional Fibonacci cube.

The exact values for small $m$ are given and several recursive bounds are established using the symmetry property of Lucas cubes and the technique of labeling.

The relation to the problem of subcube fault-tolerance in hypercube is also determined.

#### Hamiltonian path including prescribed edges

In Chapter 5 we show that given a set $\mathcal{P}$ of at most $2n-4$ prescribed edges ($n \geq 5$) and vertices $u$ and $v$ whose mutual distance is odd, the $n$-dimensional hypercube $Q_n$ contains a hamiltonian path between $u$ and $v$ passing through all edges of $\mathcal{P}$ if and only if the subgraph induced by $\mathcal{P}$ consists of pairwise vertex-disjoint paths, none of them having $u$ or $v$ as internal vertices or both of them as endvertices.

This resolves a problem of R. Caha and V. Koubek [5], who showed that for any $n \geq 3$ there exist vertices $u, v$ and $2n-3$ edges of $Q_n$ not contained in any hamiltonian cycle between $u$ and $v$, but still satisfying the condition above. The proof of the main theorem is based on an inductive construction whose basis for $n = 5$ was verified by a computer search.

Classical results on hamiltonian edge-fault tolerance of hypercubes are obtained as a corollary.

## 1.5 Bibliographic note

The thesis is drawn up as a compilation of three papers that were published or accepted for publication in refereed international journals or conferences.

Chapter 3 is a joint work with R. Caha [4] that started in [25]. A short version was published also in [26]. Here we present a full version with the proofs of Lemma 3.2.1, Lemma 3.4.1 and Theorem 3.5.2 that were omitted in [4].

Chapter 4 is based on [27]. It includes an appendix with the proofs of Lemma 4.4.2 and Lemma 4.4.3.

Chapter 5 is a joint work with T. Dvořák [16].

# 2 Fibonacci-like Cubes

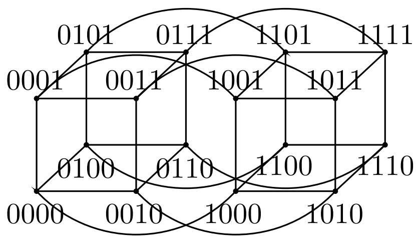
When Fibonacci cube was introduced as a new model for an interconnection network, it soon became increasingly popular. Many modifications appeared and their properties and suitability for parallel algorithms were investigated.

Here is a survey of known results on Fibonacci-like cubes, as to my knowledge. We start with an extensive list of these graphs and then we present their most interesting graph properties. We finish with results about their applications as models for parallel architectures.

## 2.1 Definitions

### Hypercube

Binary hypercube $Q_n$ of dimension $n$ is a graph whose vertex set consists of all binary strings of length $n$ and two vertices are adjacent if and only if they differ in exactly one bit. It will be useful to define $V(Q_0) = \{\lambda\}$ where $\lambda$ denotes empty string. See hypercube $Q_4$ as an example:




### Fibonacci Cube

Fibonacci cube $FC_n$ of dimension $n$ is defined[1] recursively [35, 37] as a subgraph of $Q_n$ induced on vertices

$$V(FC_n) = \{0u; u \in V(FC_{n-1})\} \cup \{10v; v \in V(FC_{n-2})\} \quad \text{for } n \geq 2,$$

and $V(FC_n) = V(Q_n)$ for $n \leq 1$. Since we work with induced subgraphs of $Q_n$, write shortly $FC_n$ instead of $V(FC_n)$ so we have $FC_n = 0FC_{n-1} \cup 10FC_{n-2}$. Observe that $FC_n$ consists of all vertices without two consecutive 1's.

[1] This is equivalent to the more complicated original definition that uses Fibonacci codes and Zeckendorf's theorem.

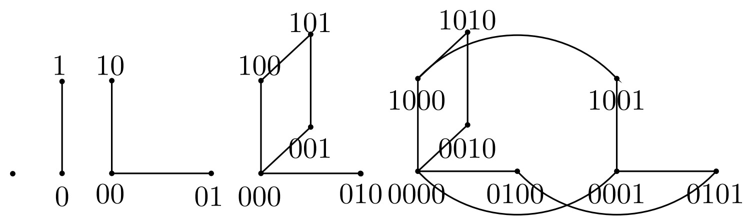



Figure 2.1: $FC_0$, $FC_1$, $FC_2$, $FC_3$, $FC_4$.

### Lucas Cube

Lucas cube [52] of dimension $n$ is a subgraph of $Q_n$ induced on vertices

$$LC_n = 0FC_{n-1} \cup 10FC_{n-3}0 \quad \text{for } n \geq 3,$$

and $LC_2 = FC_2$, $LC_1 = \{0\}$. Observe that $LC_n$ consists of all vertices without two consecutive 1's where the first and last bits are considered to be consecutive. See section 4.2 for more on Lucas cubes.

### Generalized Fibonacci Cube

Generalized Fibonacci cube [10] of order $k \geq 2$ and dimension[2] $n$ is a subgraph of $Q_n$ induced on vertices

$$GFC^k_{n+k} = \bigcup_{i=1}^{k} 1^{i-1}0GFC^k_{n+k-i} \quad \text{for } n \geq k,$$

and $GFC^k_{n+k} = Q_n$ for $n < k$. So $FC_n$ is $GFC^2_{n+2}$.

### Fibonacci ($p$,$r$)-cube

Fibonacci ($p$,$r$)-cube [18] of dimension $n$ with parameters $p$, $r \geq 1$ is a subgraph of $Q_n$ induced on vertices

$$C_n = \bigcup_{i=0}^{r} (10^{p-1})^i 0C_{n-ip-1},$$

where $C_0 = \{\lambda\}$ and $C_i = \emptyset$ for $i < 0$. So $FC_n$ is (1,1)-cube of dimension $n$.

### Extended Fibonacci Cube

Extended Fibonacci cube [66] of dimension $n$ with parameter $k$ is a subgraph of $Q_n$ induced on vertices

$$EFC^k_n = 0EFC^k_{n-1} \cup 10EFC^k_{n-2} \quad \text{for } n \geq k+2,$$

and $EFC^k_{k+1} = Q_{k+1}$, $EFC^k_k = Q_k$. So $FC_n$ is $EFC^0_n$.

[2] A subscript $n+k$ in $GFC^k_{n+k}$ could be reduced to $n$ which would better reflect that it is a subgraph of $Q_n$. Here I preserve the notation from [10].

### Enhanced Fibonacci Cube

Enhanced Fibonacci cube [57] of dimension $n$ is a subgraph of $Q_n$ induced on

$$NFC_n = 00NFC_{n-2} \cup 10NFC_{n-2} \cup 0100NFC_{n-4} \cup 0101NFC_{n-4} \text{ for } n \geq 4,$$

and $NFC_n = FC_n$ for $n \leq 3$. Observe that $FC_n \subseteq NFC_n$. Enhanced Fibonacci cube is Hamiltonian for $n \geq 6$ [57].

### Widened Fibonacci Cube

Widened Fibonacci cube [1] of dimension $n \geq 4$ is a subgraph of $Q_n$ induced on

$$W_n = 00FC_{n-2} \cup 10FC_{n-2} \cup 0100FC_{n-4} \cup 0101FC_{n-4}.$$

This is similar but not the same as Enhanced Fibonacci cubes. Observe that $FC_n \subseteq W_n$ for $n \geq 4$. Widened Fibonacci Cube is Hamiltonian for $n \geq 4$ [1].

### Postal Network

Postal Network [67] of dimension $n$ with a parameter $\lambda \geq 1$ is a subgraph of $Q_n$ induced on vertices

$$PN_n = 0PN_{n-1} \cup 10^{(\lambda-1)}PN_{(n-\lambda)} \quad \text{for } n > \lambda,$$

and $PN_n = \{u \in Q_n \mid w(u) \leq 1\}$ for $n \leq \lambda$. Weight $w(u)$ of a vertex $u$ is the number of 1's in $u$. So $FC_n$ is a Postal Network of dimension $n$ with $\lambda = 2$.

### Linear Recursive Network

Linear Recursive Network [36] is given by a generator string $A$. Define set $J$ of positions in the string $1A1$ on which there are 1's and prefixes $A_j$ of $1A1$ of length $j$ with the last ($j$-th) bit converted to 0. For example, if $A = 010$ then $J = \{1, 3, 5\}$, $A_1 = 0$, $A_3 = 100$ and $A_5 = 10100$. Linear Recursive Network of dimension $n$ given by $A$ is a subgraph of $Q_n$ induced on vertices

$$LRN_n = \bigcup_{j \in J} A_j LRN_{(n-j)},$$

where $LRN_0 = \{\lambda\}$ and for $n < j$ define $A_j LRN_{(n-j)} = \{B\}$, where $B$ is a prefix of $A_j$ of length $n$. $FC_n$ is $LRN_n$ with empty generator string $A$, because then $J = \{0, 1\}$, $A_1 = 0$, $A_2 = 10$.

### Hierarchical Fibonacci Cube

Hierarchical Fibonacci cube [45] consists of Fibonacci number of Fibonacci cubes that are connected also diagonally: vertex $a$ in cube $b$ with vertex $b$ in cube $a$ for $a \neq b$, and vertex $a$ in cube $a$ to vertex $\overline{a}$ in cube $\overline{a}$, provided that $\overline{a} \in FC_n$, i.e. complement of $a$ contains no two consecutive 1's.

## 2.2 Graph properties

Fibonacci-like cubes posses several interesting graph properties that have been studied from a wider perspective, not only as an interconnection network.

The **number of vertices** of $FC_n$ is $f_{n+2}$, the $(n+2)$-th Fibonacci number[3], i.e. $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$. The number of vertices of $LC_n$ is $l_n$, the $n$-th Lucas number, i.e. $l_0 = 2$, $l_1 = 1$, $l_{n+2} = l_{n+1} + l_n$. The **number of edges** of $FC_n$ is $(2(n+1)f_{n+2} - (n+2)f_{n+1})/5$ [35].

The number of **symmetric vertices** of Fibonacci cube is $f_{\lfloor n/2 \rfloor - (-1)^n + 2}$ [51]. Vertex $u$ is symmetric if $u_i = u_{n-i+1}$ for $i = 1, \ldots, n$.

**Radius** of Fibonacci cube is $\lceil n/2 \rceil$ [51]. Radius of a graph is the minimal eccentricity of its vertices, where eccentricity of a vertex in a connected graph is the maximal distance to other vertices. The vertices with eccentricity equal to radius are called centers. Fibonacci cube has one **center** $0^n$ for $n$ even, whereas for $n$ odd, it has also a second center $0^{\lfloor n/2 \rfloor}10^{\lfloor n/2 \rfloor}$ [51].

**Independence number** of Fibonacci cube is $max(e_n, o_n)$ [51], where $e_n$ (resp. $o_n$) denotes the number of vertices of Fibonacci cube with even (resp. odd) number of 1's. Independence number of a graph $G$ is the maximal size of an independent vertex set. Set $S$ of vertices is independent if there is no edge $e \in E(G)$ with $e \subseteq S$.

Fibonacci and Lucas cubes are **median graphs** [46]. A connected graph is called a median graph if every triple of vertices has a unique median. Median of vertices $u$, $v$, $w$ is a vertex that simultaneously lies on a shortest ($u$,$v$)-path, a shortest ($u$,$w$)-path, and a shortest ($v$,$w$)-path. As a result, several enumerative properties are obtained. For example, the **number of squares** $S(FC_n)$ in Fibonacci cube is

$$S(FC_n) = -\frac{3n}{25} f_{n+1} + \left( \frac{n^2}{10} + \frac{3n}{50} - \frac{1}{25} \right) f_n.$$

**Decycling number** of Fibonacci cubes has been studied in [19]. Decycling number $\nabla(G)$ of a graph $G$ is the smallest number of vertices that can be deleted from $G$ such that the resultant graph contains no cycle. It was shown that

$$\nabla(FC_{n-1}) + \nabla(FC_{n-2}) \leq \nabla(FC_n) \leq \nabla(FC_{n-1}) + \left\lfloor \frac{f_n}{2} \right\rfloor.$$

Fibonacci cube is **$\Theta$-graceful graph** [3]. Its vertices can be labeled with a bijection $l : V(FC_n) \to \{0, \ldots, f_{n+2} - 1\}$ such that for any two edges $\{a, b\}$, $\{c, d\}$ we have that

$$|l(a) - l(b)| = |l(c) - l(d)| \text{ if and only if } \{a, b\}, \{c, d\} \text{ are parallel.}$$

Edges $\{a, b\}$, $\{c, d\}$ in hypercube graphs are called parallel if $\Delta(a, b) = \Delta(c, d)$ where $\Delta(x, y)$ denotes a symmetric difference of $x$ and $y$.

**Observability** of Fibonacci cube and Lucas cube is $n$ [14]. Observability of a graph $G$

[3] In some papers on Fibonacci cubes it is defined that Fibonacci sequence starts with $f_0 = 1$, $f_1 = 2$ so $|V(FC_n)| = f_n$.

is the minimum number of colors to be assigned to the edges of $G$ so that the coloring is proper and vertex-distinguishing. Coloring of edges is proper if any two adjacent edges have different colors and vertex-distinguishing if distinct vertices obtain distinct color sets of their incident edges. Observability of Extended Fibonacci cube is $n+1$ for $k = 1$, 2, and for $k \geq 3$ a bound on observability that is sharp in some cases in known [59]. Also observability of Extended Lucas cube is studied in [60].

**Fibonacci semilattice** defined on vertex set of Fibonacci cube in [42] is strictly atomic, simplicial and its characteristic polynomial is known [42]. Similar properties were proved also for **Lucas semilattice** [52].

Fibonacci cubes are **resonance graphs of fibonaccenes** [47]. Fibonaccenes appear in a chemical graph theory studying specific molecules consisting of hexagons, called hexagonal chains. Resonance graph $R(G)$ reflects a structure of perfect matchings of a hexagonal chain $G$. Its vertex set consists of perfect matchings and there is an edge between two of them if and only if their symmetric difference is the edge set of a hexagon of $G$.

Extended Fibonacci cubes and Widened Fibonacci cube, except some initial dimensions, satisfy the property that every edge belongs to cycles of any even length [2]. This property is called **edge-bipancyclicity**.

## 2.3 Applications

### Simulations of other architectures

Fibonacci cube is much sparser than hypercube. Recall that there are $f_{n+2}$ vertices in $FC_n$ and $(2(n+1)f_{n+2} - (n+2)f_{n+1})/5$ edges comparing to $2^n$ vertices and $n2^{n-1}$ edges in $Q_n$. But as well as hypercubes, Fibonacci cubes can be used for simulations of many simple parallel architectures.

For example arrays, rings, 2-D meshes can be effectively simulated on Fibonacci cubes [6, 12]. Fibonacci cube can also simulate trees and X-trees almost as efficiently as the hypercube [13].

Some simulations need Hamiltonian cycle that exists in Fibonacci cube of dimension $n$ only when $n = 3k$ where $k$ denotes an integer, otherwise there is a cycle including all vertices except one [38, 11].

It can be stated that the other Fibonacci-like cubes preserve these abilities. Actually, they sometimes offer better results. For example, Enhanced Fibonacci cubes are Hamiltonian for all $n \geq 6$ [57]. Binary trees are embedded into Extended Fibonacci cubes in [65].

### Parallel computations

A self-similar recursive structure make Fibonacci cube useful for the design of parallel and distributed algorithms. Especially when they are based on Fibonacci trees, meshes or tori.

One of basic paradigms of parallel algorithms that arises in many applications is a prefix computation [48]. If we have an associative binary operator $*$ on a semigroup $(G, *)$ and $S : a_1, \ldots, a_N$ a sequence of $N$ elements, we define $i$-th prefix to be the product $a_1 * a_2 * \cdots * a_i$. The prefix problem is to compute all prefixes in parallel.

It was shown that it can be run in $O(\log N)$ time on Generalized Fibonacci cubes [40, 41], on Extended Fibonacci cubes [53], and on Enhanced Fibonacci cubes [54]. Also a parallel tree contraction, another basic algorithm, can be run in logarithmic time on Generalized Fibonacci cubes [40, 41].

In [18] a new class of discrete orthogonal transform is defined based on Fibonacci $(p,r)$-recursions and it is shown that Fibonacci $(p,r)$-cubes can be used for hardware implementation of these transforms.

#### Parallel communications

The fundamental part of any parallel computation is a data communication. In graph model without a global memory we have tasks of sending a message from one to one vertex (routing), one to all (broadcast) or from one to many vertices (multicast).

Routing between two vertices can use two paths, one for each direction. In Fibonacci cube they can be chosen such that they are parallel [20].

Efficient communication must be done along short paths. For Fibonacci cube there are routing and broadcasting algorithms along the shortest possible paths that run in asymptotically optimal time [63]. For Generalized Fibonacci cubes routing and broadcasting algorithms were proposed in [39].

For Enhanced Fibonacci cube several efficient communication algorithms are given in [56]. Routing is done in optimal time and optimal traffic whereas broadcasting is done in optimal traffic but optimal time only for $n$ even. Also two heuristic algorithms for multicast are provided.

VLSI layout of Enhanced Fibonacci cubes is designed in [55].

#### Fault-tolerance

Fibonacci cube can be seen as a hypercube with removed faulty vertices. However, these vertices are special - they all have two consecutive 1's. What if we allow any vertex to be faulty and look for a subgraph isomorphic to Fibonacci cube that contains no faults? How many faults can we tolerate in the worst case?

This question was studied in [33], where a direct embedding of $\lfloor \frac{n}{2} \rfloor$-th order Generalized Fibonacci cube of dimension $n + \lfloor \frac{n}{2} \rfloor$ into a hypercube of dimension $n$ with no more than three faults was given.

# 3 Embedding Fibonacci Cubes into Hypercubes with $\Omega(2^{cn})$ Faulty Nodes

## Introduction

In this chapter, we consider the following problem: How many nodes in a hypercube of dimension $n$ can be faulty under condition that a Fibonacci Cube of dimension $n$ is isomorphic to a subgraph of the hypercube? The question was raised in [35] (a modification in [66]), but no bound was given. In [33], a direct embedding of $\lfloor \frac{n}{2} \rfloor$-th order Generalized Fibonacci Cube of dimension $n + \lfloor \frac{n}{2} \rfloor$ into a hypercube of dimension $n$ with no more than three faults was given.

We present an effective construction of a direct embedding of a Fibonacci Cube of dimension $n$ into a faulty hypercube of dimension $n$ with less or equal $2^{\lceil \frac{n}{4} \rceil - 1}$ faults and we prove that there exists a direct embedding of a Fibonacci Cube of dimension $n$ into a faulty hypercube of dimension $n$ with at most $\frac{2^n}{4f_{n-2}}$ faults. Thus the number $\phi(n)$ of tolerable faults grows exponentially with respect to dimension $n$, $\phi(n) = \Omega(2^{cn})$, for $c = 2 - \log_2(1 + \sqrt{5}) \doteq 0.31$.

We also give an upper bound $\phi(n) = O(2^{dn})$, for $d = 2 - \frac{3}{4}\log_2 3 \doteq 0.82$.

As a corollary, there exists a nearly polynomial algorithm deciding whether the existence of a direct embedding (and in the affirmative case constructing one) of a Fibonacci Cube of dimension $n$ into a faulty hypercube of dimension $n$ provided that faults are given on input by enumeration. Last but not least, we inquire into an embedding problem under circumstances that the faulty nodes are given on input with an asterisk convention. We show that this problem is NP-complete.

## 3.1 Preliminaries and Notations

The *hypercube* $Q_n$ of dimension $n$ is the network with $2^n$ nodes labeled with binary strings of length $n$ with an edge between two nodes whenever their labels differ exactly by one bit. A *direct embedding* of a graph $G = (V, E)$ into a graph $H = (V', E')$ is an injective mapping $g : V \to V'$ preserving edges: $\{u, v\} \in E \Rightarrow \{g(u), g(v)\} \in E'$. Thus $G$ is isomorphic to a subgraph of $H$.

We assume that there are only faulty nodes in $Q_n$ and no faulty link. A *faulty hypercube* with a set of faults $F$ (we write $Q_n \setminus F$) is the induced subgraph of $Q_n = (V, E)$ on the set $V \setminus F$.

The original definition of a Fibonacci Cube ([35]) was based on *Zeckendorf's theorem* [24] and *Fibonacci codes*. We use an equivalent recursive definition.

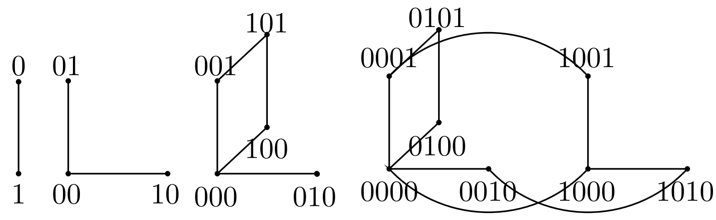



Figure 3.1: $FC_1$, $FC_2$, $FC_3$, $FC_4$.

**Definition 3.1.1.** *For $i = 0, \ldots, n$ define sets $V_i$ of binary strings of length $i$*

$$\begin{aligned} V_0 &= \{\lambda\}, \quad V_1 = \{0, 1\}, \\ V_i &= \{0\alpha \mid \alpha \in V_{i-1}\} \cup \{10\alpha \mid \alpha \in V_{i-2}\}, \quad \text{for } i > 1. \end{aligned}$$

A *Fibonacci Cube* $FC_n$ of dimension $n$ is the induced subgraph of $Q_n$ on the set $V_n$. For a simplicity, let $v \in FC_n$ (resp. $v \in Q_n$) denote that $v$ is a vertex of the Fibonacci Cube (resp. the hypercube) of dimension $n$.

We give a characterization of nodes from Fibonacci Cubes, see [37].

**Lemma 3.1.1.** *Let $v = v_n \ldots v_1$ be a binary string. Then*

$$v \in FC_n \quad \text{if and only if} \quad v_i = 1 \Rightarrow v_{i-1} = 0 \text{ for all } i = n, \ldots, 2.$$

Finally, we introduce some definitions concerning the group $Aut(Q_n)$ of all automorphisms of $Q_n$.

**Definition 3.1.2.** *For a permutation $\pi$ of $\{1, \ldots, n\}$ and subset $A \subseteq \{1, \ldots, n\}$ let $f_\pi$ and $h_A$ denote the bijection of the hypercube $Q_n$ such that*

$$\begin{aligned} f_\pi(v_n \ldots v_1) &= (v_{\pi(n)} \ldots v_{\pi(1)}), \\ h_A(v_n \ldots v_1) &= (w_n \ldots w_1), \quad \text{where } w_i = \begin{cases} v_i & \text{if } i \notin A, \\ \overline{v_i} & \text{if } i \in A. \end{cases} \end{aligned}$$

*Then $f_\pi$ and $h_A$ are automorphisms of $Q_n$ and set*

$$\begin{aligned} R(Q_n) &= \{f_\pi \mid \pi \text{ is a permutation of } \{1, \ldots, n\}\}, \\ S(Q_n) &= \{h_A \mid A \subseteq \{1, \ldots, n\}\}. \end{aligned}$$

The following lemma is a well known fact ([48]).

**Lemma 3.1.2.** *$R(Q_n)$ and $S(Q_n)$ are subgroups of $Aut(Q_n)$ and for every automorphism $g \in Aut(Q_n)$ there exist exactly one $f_\pi \in R(Q_n)$ and $h_A \in S(Q_n)$ with $g = f_\pi \circ h_A$.*

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\phi(n)$ | 0 | 1 | 1 | 3 | 5 | $\geq 7$ |

Figure 3.2: Values of $\phi(n)$ for a small dimension.

## 3.2 Characterization of Embeddings

First we characterize all direct embeddings of $FC_n$ into $Q_n \setminus F$. The following lemma states that any direct embedding can be uniquely extended to an automorphism of $Q_n$.

**Lemma 3.2.1.** *Let $g$ be a direct embedding of $FC_n$ into $Q_n \setminus F$. There exist exactly one $f_\pi \in R(Q_n)$ and $h_A \in S(Q_n)$ such that $g$ is the domain restriction of hypercube automorphism $g' = f_\pi \circ h_A$ on $FC_n$.*

*Proof.* In $Q_n \setminus F$ denote a vertex $u$ and its neighbors $w^j$ for $j = 1, \ldots, n$ such that $u = g(0 \ldots 0)$ and $w^j = g(0^{n-j}10^{j-1})$. Since $g$ is a direct embedding, $w^j$ are distinct and cover all neighbors of $u$. Let $z_j$ be the only position in which $u$ and $w^j$ differ. Define a permutation $\pi$ of $\{1, \ldots, n\}$ and subset $A \subseteq \{1, \ldots, n\}$ such that $\pi(j) = z_j$ and $j \in A$ if and only if $u_{\pi(j)} = 1$ for $j = 1, \ldots, n$.

Clearly, we have that $g(0 \ldots 0) = u = f_\pi \circ h_A(0 \ldots 0)$ and $g(0^{n-j}10^{j-1}) = w^j = f_\pi \circ h_A(0^{n-j}10^{j-1})$. We will show by induction on number $p$ of 1's in vertex $v = v_n, \ldots, v_1 \in FC_n$ that $g(v) = f_\pi \circ h_A(v)$ also for $p \geq 2$.

Let $i_1$, $i_2$ be such that $v_{i_1} = v_{i_2} = 1$. Denote $a$ (resp. $b$) neighbor of $v$ that differ in positions $i_1$ (resp. $i_2$) and $c$ denote the vertex in distance 2 that differs in both, i.e. $c_{i_1} = c_{i_2} = 0$. Vertices $v, a, c, b$ are all in $FC_n$ and form a 4-cycle. By induction hypothesis we can see that $g(c)$ and $g(a)$ (resp. $g(b)$) differ in the only position $\pi(i_2)$ (resp. $\pi(i_1)$). Since $g$ is a direct embedding it must be that $v$ is mapped to the common neighbor of $g(a)$ and $g(b)$ different from $g(c)$. There is only one such vertex and it differs from $g(a)$ (resp. $g(b)$) in the position $\pi(i_1)$ (resp. $\pi(i_2)$) so $g(v) = f_\pi \circ h_A(v)$ holds. $\square$

In subsequent section we suggest an algorithm that for a suitable set of faults $F \subseteq Q_n$ finds a direct embedding of the Fibonacci Cube of dimension $n$ into the faulty hypercube of dimension $n$ with faults $F$. By Lemma 3.2.1, any embedding $g : FC_n \to Q_n \setminus F$ can be extended to an automorphism of $Q_n$ and thus can be composed of a negation and a permutation of bits in node labels. By Lemma 3.1.1, a vertex of $Q_n$ belongs to $FC_n$ if and only if it has not two consecutive ones. The key idea is to find a set $A$ and a permutation $\pi$ of $\{1, \ldots, n\}$ such that $g = f_\pi \circ h_A$ maps all faults to vertices with two consecutive ones. Then the domain restriction of $g^{-1}$ on $FC_n$ is the required embedding.

Before the algorithm, we define an optimal number of tolerable faults.

**Definition 3.2.1.** *For an integer $n \geq 1$, let $\phi(n)$ denote the greatest integer such that for every set $F$ of vertices of $Q_n$ with $|F| \leq \phi(n)$ there exists a direct embedding $g$ of $FC_n$ into $Q_n \setminus F$.*

Fig. 3.2 was obtained by inspection of all cases and verified by computer ([25]).

## 3.3 Embedding Algorithm

First observe that $|FC_4| = f_6 = 8 = \frac{2^4}{2}$. Let $\overline{FC_4}$ denote the complement of $FC_4$ in $Q_4$. Consider the automorphism $g_0 = f_\pi \circ h_{\{1,2,3,4\}}$ where $\pi(1,2,3,4) = (1,3,2,4)$, then $g_0(FC_4) = \overline{FC_4}$. This observation is exploited in the following algorithm.

Assume that a faulty hypercube $Q_n$ with a set $F$ of faults is given with $n > 5$. Then there exists a graph $G = (V, E)$ isomorphic to $Q_{n-4}$ such that $Q_n = Q_4 \times G$. Set $F_1 = F \cap (FC_4 \times G)$, $F_2 = F \cap (\overline{FC_4 \times G})$ then $F = F_1 \cup F_2$ and $F_1 \cap F_2 = \emptyset$. Thus $|F_1| \geq \frac{|F|}{2}$ or $|F_2| \geq \frac{|F|}{2}$. For every automorphism $g'$ of $G$ define an automorphism $g$ of $Q_n$ such that ($id$ is the identical automorphism)

$$g = \begin{cases} id \times g' & \text{if} \quad |F_2| \geq \frac{|F|}{2}, \\ g_0 \times g' & \text{if} \quad |F_1| > \frac{|F|}{2}. \end{cases}$$

Then $g$ is an automorphism of $Q_n$ such that if $|F_2| \geq \frac{|F|}{2}$ then $g(F_2) \cap FC_n = \emptyset$, if $|F_1| > \frac{|F|}{2}$ then $g(F_1) \cap FC_n = \emptyset$. Therefore if $g'$ maps subcubes containing the less set from $F_1$ and $F_2$ into the complement of $FC_n$ we obtain that $g(F) \cap FC_n = \emptyset$. To construct $g'$ we apply this idea recursively and for dimension $n \leq 6$ we search for $g'$ exhaustively.

**Algorithm 1**

1. Initialize $A = \emptyset$, $i = n$, let $\pi$ be the empty partial permutation.
2. Repeat this step until $i \leq 6$: Let

$$\begin{aligned} F_1 &= \{v \in F \mid (v_i, \ldots, v_{i-3}) \text{ does not contain two consecutive ones}\}, \\ F_2 &= \{v \in F \mid (v_i, \ldots, v_{i-3}) \text{ contains two consecutive ones}\}. \end{aligned}$$

If $|F_2| \geq \frac{|F|}{2}$ then set $A \leftarrow A$, $\pi(j) \leftarrow j$ for $j \in \{i, i-1, i-2, i-3\}$ and $F \leftarrow F_1$. If $|F_1| > \frac{|F|}{2}$ then set $A \leftarrow A \cup \{i, i-1, i-2, i-3\}$, $\pi(i) \leftarrow i$, $\pi(i-1) \leftarrow i-2$, $\pi(i-2) \leftarrow i-1$, $\pi(i-3) \leftarrow i-3$ and $F \leftarrow F_2$. Set $i \leftarrow i-4$.
3. If $F \neq \emptyset$ and $i \leq 6$ then set $F' = \{(v_i, \ldots, v_1) \mid (v_n, \ldots, v_1) \in F\}$. By the exhaustive search we find a permutation $g' = f_{\pi'} \circ h_{A'}$ of $Q_i$ (if it exists) such that $g'(F') \cap FC_i = \emptyset$. Set $A \leftarrow A \cup A'$, $\pi(j) \leftarrow \pi'(j)$ for $j \in \{i, i-1, \ldots, 1\}$.

To analyze Algorithm 1, we introduce the following function.

**Definition 3.3.1.** *If $n \leq 5$ then $p(n) = \phi(n)$, $p(6) = 7$ and for $n > 6$*

$$p(n) = \begin{cases} 3 \cdot 2^{\frac{n-4}{4}} & \text{if } n \equiv 0 \bmod 4, \\ 5 \cdot 2^{\frac{n-5}{4}} & \text{if } n \equiv 1 \bmod 4, \\ 7 \cdot 2^{\frac{n-6}{4}} & \text{if } n \equiv 2 \bmod 4, \\ 2^{\frac{n-3}{4}} & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

**Lemma 3.3.1.** *Let $p$ be the function from Definition 3.3.1. Then*

$$p(n) \geq 2^{\lceil \frac{n}{4} \rceil - 1}, \quad \text{for all } n > 1. \tag{3.1}$$

*Proof.* Clearly, $p(n) = 2p(n-4)$ and by recurrence we obtain (3.1) from Fig. 3.2. $\square$

**Lemma 3.3.2.** *Let $n \geq 1$ and $F \subseteq Q_n$ be a set of faults. If $|F| \leq p(n)$ then Algorithm 1 outputs a set $A$ and a permutation $\pi$ such that*

$$g(F) \cap FC_n = \emptyset, \qquad \text{for } g = (f_\pi \circ h_A).$$

*Proof.* Let $k'$ be the number of repetitions of Step 2 and let $F_k$ denote the set $F$ after the $k$-th repetition (and $F = F_0$ is the original set of faults). Thus $F = \bigcup_{k=0}^{k'} (F \setminus F_k) \cup F_{k'}$. By the observation before Algorithm 1, we deduce that $g(F \setminus F_k) \cap FC_n = \emptyset$, for all $k$. If $|F| \leq p(n)$ then $|F'| \leq \phi(i)$ in Step 3 since $|F_k| < \frac{|F_{k-1}|}{2}$. Thus Algorithm 1 successively finds $g'$ and $g(F_{k'}) \cap FC_n = \emptyset$. $\square$

*Remark* 3.3.1. *By $g(F) \cap FC_n = \emptyset$, we deduce that $g^{-1} : FC_n \to Q_n \setminus F$ is a direct embedding of the Fibonacci Cube $FC_n$ into the faulty hypercube $Q_n \setminus F$ and $\phi(n) \geq p(n)$.*

Now, we will prove a stronger estimate of the function $\phi$ than $\phi(n) \geq p(n)$.

**Theorem 3.3.3.** *For every $n \geq 6$*

$$\phi(n) \geq \frac{2^n}{4f_{n-2}} \geq \frac{\sqrt{5}}{4} 2^{n(2-\log_2(1+\sqrt{5}))} = \Omega(2^{0.31n}).$$

*Proof.* The automorphism group $Aut(Q_n)$ is transitive and for $u, v \in Q_n$ the number of automorphisms $g$ of $Aut(Q_n)$ with $g(u) = v$ is $n!$. Thus for $A, B \subseteq Q_n$ we obtain

$$\sum_{g \in Aut(Q_n)} |g(A) \cap B| = \sum_{u \in A, v \in B} n! = |A|.|B|.n!$$

and hence we deduce that there exists an automorphism $g \in Aut(Q_n)$ with $|g(A) \cap B| \leq \frac{|A|.|B|}{2^n}$ because $|Aut(Q_n)| = n!2^n$.

Assume that $F$ is the set of faults with $|F| \leq \frac{2^n}{4f_{n-2}}$. Set $F' = \{(v_{n-2}, \ldots, v_1) \mid \exists v_n, v_{n-1} \in \{0,1\} \text{ with } (v_n, v_{n-1}, \ldots, v_1) \in F\}$, then $|F'| \leq |F| \leq \frac{2^n}{4f_{n-2}}$ and $F' \subseteq Q_{n-2}$. By the above, there exists an automorphism $g'$ of $Q_{n-2}$ with $|g'(F') \cap FC_{n-2}| \leq \frac{|F'|.|FC_{n-2}|}{2^{n-2}} \leq \frac{2^n}{4f_{n-2}}.\frac{f_{n-2}}{2^{n-2}} = 1$. Thus for any automorphism $g$ of $Q_n$ extending $g'$ we conclude that $|g(F) \cap FC_n| \leq 1$. By Fig. 3.2, $\phi(2) = 1$, and therefore there exists an extension automorphism $g \in Aut(Q_n)$ of $g'$ with $g(F) \cap FC_n = \emptyset$. Hence $\phi(n) \geq \frac{2^n}{4f_{n-2}}$. The remainder follows from the standard estimate of Fibonacci numbers. $\square$

## 3.4 Upper Bound

Let us say that a subset $F \subseteq Q_n$ is *tolerable* if there exists a direct embedding $g$ from $FC_n$ to $Q_n \setminus F$. In other words, if we remove faulty vertices $F$ from hypercube, there still exists a subgraph isomorphic to a Fibonacci Cube.

In this section, we search for a small set of faults that is not tolerable. We use the fact that the Fibonacci Cube of dimension $n$ contains a subgraph isomorphic to the hypercube of

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\phi(n)$ | 0 | 1 | 1 | 3 | 5 | 7 | 7 | 7 | 10 | 14 |
| $n$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $\phi(n)$ | 14 | 14 | 20 | 28 | 28 | 28 | 40 | 56 | 56 | 56 |
| $n$ | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| $\phi(n)$ | 80 | 112 | 112 | 112 | 160 | 224 | 224 | 224 | 320 | 448 |

Figure 3.3: Lower bound of $\phi(n)$ from Theorem 3.3.3 and $\phi(n+1) \geq \phi(n) \geq p(n)$.

dimension $\lceil n/2 \rceil$. To see this, consider a subgraph on vertices $0{*}0{*}\dots0{*}$ (resp. ${*}0{*}0{*}\dots0{*}$). The set of faults that is not tolerable for any embedding of a hypercube of dimension $\lceil n/2 \rceil$ to the hypercube of dimension $n$ is also not tolerable for any embedding of the Fibonacci Cube of dimension $n$. Lemma 3.4.1 constructs such set. It consists of certain levels of hypercube.

**Definition 3.4.1.** *For $0 \leq i \leq n$, let $Q_n^i$ denote the i-th level of a hypercube*

$$Q_n^i = \{u \in Q_n \mid w(u) = i\},$$

*where $w(u)$ is the number of 1's in $u$ (weight).*

**Lemma 3.4.1.** *Let $1 \leq m \leq n$. For every $0 \leq k \leq m$ the set of vertices*

$$F = \bigcup_{l=0}^{\lfloor (n-k)/m \rfloor} Q_n^{k+lm}$$

*is not tolerable for any embedding of $Q_m$ into $Q_n$.*

*Proof.* Let $g : Q_m \to Q_n$ be a direct embedding. Let $i$ be the least number with $Q_n^i \cap g(Q_m) \neq \emptyset$. Take an arbitrary $u \in Q_m$ with $g(u) \in Q_n^i$ and choose an automorphism $f$ of $Q_m$ with $f(0\dots0) = u$ (the number of such automorphisms is $m!$). Then $h = g \circ f$ is a direct embedding of $Q_m$ into $Q_n$ with $h(Q_m) = g(Q_m)$ and $h(0\dots0) \in Q_n^i$. Next, we will show that

$$h(v) \in Q_n^{i+j} \quad \text{for all } 0 \leq j \leq m \text{ and for all } v \in Q_m^j. \tag{3.2}$$

This says that the $j$-th level of $Q_m$ is mapped to the $(i+j)$-th level of $Q_n$. There are $m+1$ levels in $Q_m$. The set $F$ contains the $(k+lm)$-th level for all $l = 0, 1, \dots \lfloor (n-k)/m \rfloor$ so that there are not $m+1$ consecutive levels in $Q_n$ disjoint with $F$. This implies that at least one level of $Q_m$ is mapped to $F$ and hence $F$ is not tolerable for the embedding $h$ nor for the embedding $g$.

We prove 3.2, by induction over $j$. Clearly, $Q_m^0 = \{(0\dots0)\}$ and $h(0\dots0) \in Q_n^i$, by the definition. Since there are edges only between consecutive levels in $Q_n$ we conclude that for $v \in Q_m^1$ either $h(v) \in Q_n^{i-1}$ or $h(v) \in Q_n^{i+1}$. But $Q_n^i$ is the least level intersecting $h(Q_m)$ and thus $h(v) \notin Q_n^{i-1}$.

For $v \in Q_m^j$, $j > 1$ there exist vertices $a \in Q_m^{j-2}$ and $b, c \in Q_m^{j-1}$ such that $a$, $b$, $c$, $v$ form a cycle in $Q_m$. Denote $a' = h(a)$, $b' = h(b)$, $c' = h(c)$. By the induction hypothesis,

$a' \in Q_n^{i+j-2}$ and $b', c' \in Q_n^{i+j-1}$. There is only one vertex $v' \in Q_n$ such that $a'$, $b'$, $c'$, $v'$ form a cycle in $Q_n$ and hence $h(v) = v'$. Since $v' \in Q_n^{i+j}$ the proof of 3.2 is done. □

**Theorem 3.4.2.** *For $n > 1$, $\phi(n) = O(2^{dn})$ where $d = (8 - 3\log_2 3)/4 \doteq 0.82$.*

*Proof.* A Fibonacci Cube of dimension $n$ contains a subgraph isomorphic to the hypercube of dimension $m = \lceil n/2 \rceil$. Set $k = \lceil m/2 \rceil$. Take the set $F$ consisting of $k$-th level and $(k+m)$-th level. By Lemma 3.4.1, $F$ is not tolerable for any direct embedding of a hypercube of dimension $m$ and neither is tolerable for any embedding of a Fibonacci Cube of dimension $n$, hence $\phi(n) < |F|$.

There are four cases: $n = 4k, \dots, 4k-3$. In all cases, we can compute

$$\begin{aligned}
\phi(n) &< |F| \leq 2\binom{4k}{k} = 2\frac{(4k)!}{k!(3k)!} \\
&= \frac{2\sqrt{8\pi k}\left(\frac{4k}{e}\right)^{4k}\Theta(1)}{\sqrt{2\pi k}\left(\frac{k}{e}\right)^{k}\Theta(1).\sqrt{6\pi k}\left(\frac{3k}{e}\right)^{3k}\Theta(1)}, && \text{since } x! = \sqrt{2\pi x}\left(\tfrac{x}{e}\right)^x \Theta(1) \\
&= \frac{4^{4k}k^{4k}e^k e^{3k}}{3^{3k}k^k k^{3k} e^{4k}} O(1), && \text{since } \frac{4\sqrt{2\pi k}\Theta(1)}{\sqrt{2\pi k}\sqrt{6\pi k}\Theta(1)\Theta(1)} = O(1) \\
&= \left(\frac{4^4}{3^3}\right)^k O(1) = 2^{(8-3\log_2 3)k} O(1), && \text{we have } k = \tfrac{n}{4} + O(1) \\
&= O(2^{dn}), && \text{where } d = \tfrac{8 - 3\log_2 3}{4} \doteq 0.82 \ .
\end{aligned}$$

□

## 3.5 NP-Complete Embedding Problems

Algorithm 1 runs in polynomial time $O(n.|F|)$ with respect to size of input if faults are given by enumeration. It constructs an embedding of the Fibonacci Cube into a faulty hypercube if the number of faults is at most $p(n) = O(2^{0.25n})$.

If the number $m$ of faults is greater than $p(n)$, we can use a simple algorithm that search all $n!2^n$ possible embeddings with time complexity $O((nm)^{c' \log n})$ for some constant $c'$. In case that we restrict only on $2^n$ possible embeddings generated by negation of dimensions we obtain even polynomial time complexity $O(nm^5)$.

In this section, we investigate time complexity of the embedding problem under circumstance that faulty subcubes are given on input instead of faulty nodes. This problem can be applied if several tasks requiring different subcubes run at the same time. A subcube solving another task can be considered as faulty for a new task.

There is only one difference to the previous problem: faulty subcubes are given with an asterisk convention instead of enumerating of faulty nodes.

**Definition 3.5.1. FibErrSubQ.** *Given dimension $n$ and a set $F$ of $\{0, 1, *\}$-strings of length $n$ representing faulty subcubes of the hypercube $Q_n$. Is there any direct embedding $g : FC_n \to Q_n \setminus F$ ?*

The following theorem shows that the situation dramatically transforms.

**Theorem 3.5.1.** *FibErrSubQ is NP-complete.*

*Proof.* Consider the NP-complete 3-3-SAT ([22]) problem that is a satisfiability problem of a given formula in CNF such that

a) any clause contains at most 3 variables,

b) any variable is contained in at most 3 clauses.

We define a transformation of 3-3-SAT to FibErrSubQ.

Let $\Phi$ be a formula in CNF satisfying a) and b) with clauses $c_1, \dots, c_m$ and variables $x_1, \dots, x_n$. We can assume that

1) no clause contains both $x$ and $\neg x$, otherwise it is always true,

2) every variable is used in formula $\Phi$ both in positive and negative form, otherwise it can be reduced.

From Conditions 2) and b) it follows that every variable can satisfy at most two clauses.

For each clause $c_j$ we construct a $\{0, 1, *\}$-string $s^j = s^j_1 \dots s^j_m s^j_{m+1} \dots s^j_{m+n}$ of length $m + n$ according to a following scheme:

$$s^j_k = \begin{cases} 1 & \text{if } j = k, \\ * & \text{else} \end{cases} \qquad \text{for } k = 1, \dots, m,$$

$$s^j_{m+i} = \begin{cases} 0 & \text{if } c_j \text{ contains } x_i, \\ 1 & \text{if } c_j \text{ contains } \neg x_i, \\ * & \text{if } c_j \text{ does not contain both } x_i \text{ and } \neg x_i \end{cases} \qquad \text{for } i = 1, \dots, n.$$

The first $m$ dimensions are called the clause dimensions, the rest dimensions are called variable dimensions. Let $F = \{s^j \mid j = 1, \dots, m\}$.

Now we prove that the constructed instance of FibErrSubQ has a direct embedding if and only if the given instance of 3-3-SAT has a satisfying assignment of variables.

Let 3-3-SAT have a satisfying assignment of variables $\sigma : X \to \{0, 1\}$. Denote $w$ a witness function: $w(j) = \min\{i \mid x_i \text{ or } \neg x_i \text{ satisfies } c_j\}$.

Consider a direct embedding $g = f_\pi \circ h_A$ of the Fibonacci Cube of dimension $m + n$ into a faulty hypercube of dimension $m + n$ for $A = \{m + i \mid 1 \le i \le n \text{ and } \sigma(x_i) = 1\}$ and a permutation $\pi$ such that $\pi(j) = \pi(m + w(j)) + 1$ or $\pi(j) = \pi(m + w(j)) - 1$ for all $j = 1, 2, \dots, m$. Since any variable can be a witness of satisfiability for at most two clauses such permutation $\pi$ exists.

An automorphism $g = f_\pi \circ h_A$ maps all faulty subcubes onto subcubes containing two consecutive ones. Consequently, $g^{-1} : FC_{m+n} \to Q_{m+n} \setminus F$ is the direct embedding of the FibErrSubQ problem.

Assume that the constructed instance of the FibErrSubQ problem has a required direct embedding. Then there exist a set $A$ and a permutation $\pi$ so that the automorphism

$g = f_\pi \circ h_A$ maps all faulty vertices onto vertices containing two consecutive ones. Define an assignment of variables $\sigma : X \to \{0,1\}$ as follows

$$\sigma(x_i) = 1 \text{ if } m + i \in A \text{ else } \sigma(x_i) = 0.$$

Now, we explore dimensions that can be mapped so that we obtain two consecutive ones. Due to an asterisk convention it occurs only for pairs of fixed (non-asterisk) dimensions in the whole faulty subcube. Hence for every string $s^j$, there exists a pair $\{k_1, k_2\}$ of distinct non-asterisk dimensions, it means that $s^j_{k_1} \neq *$ and $s^j_{k_2} \neq *$, such that the mapping $g = f_\pi \circ h_A$ maps them to two consecutive ones. Thus $\pi(k_1) = \pi(k_2)+1$ or $\pi(k_1) = \pi(k_2)-1$ and $k_1 \in A$ if and only if $s^j_{k_1} = 0$ and $k_2 \in A$ if and only if $s^j_{k_2} = 0$.

At least one of $k_1$, $k_2$ must be a variable dimension (let it be $k_1$), the other one can be a clause dimension $j$ or a variable dimension. Consequently, a variable $x_i$ for $i = k_1 - m$ is a witness of satisfiability of a clause $c_j$ because

$$\begin{array}{lllllll} c_j \text{ contains } x_i & \Rightarrow & s^j_{k_1=m+i} = 0 & \Rightarrow & k_1 = i+m \in A & \Rightarrow & \sigma(x_i) = 1, \\ c_j \text{ contains } \neg x_i & \Rightarrow & s^j_{k_1=m+i} = 1 & \Rightarrow & k_1 = i+m \notin A & \Rightarrow & \sigma(x_i) = 0. \end{array}$$

Therefore, every clause has a witness of satisfiability and the assignment $\sigma$ is a solution of the given 3-3-SAT problem.

To see that FibErrSubQ is NP, choose nondeterministically a subset $A$ and a permutation $\pi$ and check that the automorphism $g = f_\pi \circ h_A$ maps all faulty subcubes onto subcubes containing two consecutive ones. □

If we restrict FibErrSubQ only on direct embeddings generated by negation of dimensions (denote Neg-FibErrSubQ) it remains NP-complete problem.

**Theorem 3.5.2.** *Neg-FibErrSubQ is NP-complete.*

*Proof.* By a transformation from SAT ([22]). This time it is even more straightforward than in the proof of Theorem 3.5.1. Let $\Phi$ be a formula in CNF with clauses $c_1, \ldots, c_m$ and variables $x_1, \ldots, x_n$. We can assume that no clause contains both $x$ and $\neg x$, otherwise it is always true.

For each clause $c_j$ we construct a $\{0,1,*\}$-string $s^j = t^j_1 \ldots t^j_n$ of length $3n$ according to a following scheme and we put $F = \{s^j \mid j = 1, \ldots, m\}$.

$$t^j_i = \begin{cases} 100 & \text{if } c_j \text{ contains } x_i, \\ 110 & \text{if } c_j \text{ contains } \neg x_i, \qquad \text{for } i = 1, \ldots, n. \\ 1*0 & \text{else,} \end{cases}$$

If we have a satisfying assignment of variables $\sigma : X \to \{0,1\}$, define $A = \{i \mid \sigma(x_i) = 1\}$ and observe that $h_A$ maps all faulty subcubes onto subcubes with two consecutive ones so $h_A : FC_{3n} \to Q_{3n} \setminus F$ is the direct embedding for Neg-FibErrSubQ problem.

On the other hand, assume there is a required direct embedding $h_A$ given by a set $A$. Define an assignment of variables $\sigma : X \to \{0,1\}$ such that $\sigma(x_i) = 1$ if and only if $i \in A$ and observe that $\sigma$ satisfies $\Phi$.

To see that Neg-FibErrSubQ is NP, choose nondeterministically a subset $A$ and check that the automorphism $h_A$ maps all faulty subcubes onto subcubes containing two consecutive ones. $\square$

## Conclusions

We have presented an algorithm that constructs a direct embedding of $FC_n$ into faulty $Q_n$ with $\Omega(2^{0.25n})$ faults. This enables us to run efficiently various parallel algorithms with recursive design on hypercube architectures with many faulty or busy nodes ([6], [13], [12]). We have also given an upper and a lower bound on the number of tolerable faults $\Omega(2^{0.31n}) = \phi(n) = O(2^{0.82n})$.

We have proved that if we allow faulty subcubes on input, the problem of finding a direct embedding becomes NP-complete. Even we restrict on direct embeddings generated by negation of dimensions, it remains NP-complete. These results contrast with a polynomial algorithm (or nearly polynomial for a general case) that finds a direct embedding of a Fibonacci Cube into a hypercube of the same dimension if faults are given on input by enumeration.

We have investigated only faulty nodes. It might be interesting to admit also faulty links and consider embeddings of binomial trees. We think that our results can be extended to various modifications of Fibonacci Cubes such as Extended Fibonacci Cubes [66]. Remark that Algorithm 1 uses permutations only for small groups of dimensions. This indicates that some improvements are still possible.

# 4 Recursive Fault-Tolerance of Fibonacci Cube in Hypercubes

Let $Q_n$ denote $n$-dimensional binary hypercube, its set of vertices is the set of all binary strings of length $n$ and two vertices are adjacent if and only if they differ in exactly one bit. We say that an edge $\{u, v\} \in E(Q_n)$ has dimension $i$ if $u$, $v$ differ in $i$-th bit, counted from left.

Fibonacci cube $FC_n$ of dimension $n$ is defined recursively [35] as a subgraph of $Q_n$ induced on vertices

$$V(FC_n) = \{0u; u \in V(FC_{n-1})\} \cup \{10v; v \in V(FC_{n-2})\} \quad \text{for } n \geq 2, \tag{4.1}$$

and $V(FC_n) = V(Q_n)$ for $n < 2$. In hypercube $Q_n$ for $n \geq 2$, $FC_n$ is given recursively by connecting $FC_{n-2}$ in $10Q_{n-2}$ subcube with the corresponding vertices of $FC_{n-1}$ in $0Q_{n-1}$ subcube. For our purposes, let us use the notation $FC_n = 0FC_{n-1} \cup 10FC_{n-2}$. The number of vertices of $FC_n$ is $f_n$, the $(n+2)$-th Fibonacci number, i.e. $f_0 = 1$, $f_1 = 2$, $f_{n+2} = f_{n+1} + f_n$.

Observe from (4.1) that Fibonacci cube $FC_n$ can be characterized as

$$\text{a subgraph of } Q_n \text{ induced on vertices without two consecutive 1's.} \tag{4.2}$$

In this chapter, we study the problem of determining the sets of vertices whose removal from $n$-dimensional hypercube leaves no subgraph isomorphic to $m$-dimensional Fibonacci cube. Let $\mathcal{S}(n, m)$ be the collection of all such sets and denote $\psi(n, m)$ the minimum size of set in $\mathcal{S}(n, m)$. Note that $\mathcal{S}(n, m) \subseteq \mathcal{S}(n, m + 1)$ so $\psi(n, m) \geq \psi(n, m + 1)$. Define $\psi(n, m) = 0$ for $m > n$ and $\psi(n, m) = \psi(n, 0) = 2^n$ for $m \leq 0$. The informal term *faulty set* will always mean a set in $\mathcal{S}(n, m)$.

This question arises in a design of fault-tolerant applications for hypercube parallel architectures. If we consider busy vertices as faulty, how many faults in the worst case can

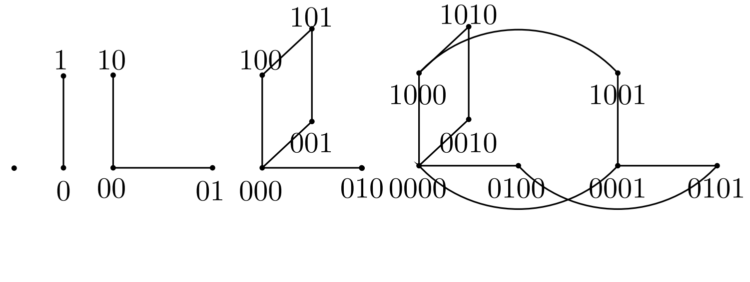



Figure 4.1: $FC_0$, $FC_1$, $FC_2$, $FC_3$, $FC_4$.

appear in $n$-dimensional hypercube while it is still possible to run applications designed for $m$-dimensional Fibonacci cube? The maximum number of tolerable faults is $\psi(n,m)-1$.

Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ via an isomorphism $h : FC_n \to G$. We say that $h$ is a direct embedding of $FC_n$ into $Q_n$. It is known [4] that $h$ can be uniquely extended to a hypercube automorphism $h : Q_n \to Q_n$. Thus, for a set $S$ of vertices of hypercube $Q_n$,

$$S \in \mathcal{S}(n,n) \Longleftrightarrow h^{-1}(S) \cap FC_n \neq \emptyset \text{ for every hyp. automorphism } h. \tag{4.3}$$

Let $h : Q_n \to Q_n$ be a hypercube automorphism. It is a well known fact that there exists exactly one permutation $\pi$ on $\{1,\ldots,n\}$ and exactly one binary string $w = w_1 \ldots w_n$ such that for every vertex $u = u_1 \ldots u_n$ we have

$$h(u_1 \ldots u_n) = v_1 \ldots v_n, \text{ where } v_i \equiv u_{\pi(i)} + w_{\pi(i)} \pmod 2, \tag{4.4}$$

for $1 \le i \le n$. In other words, hypercube automorphisms are composed of a permutation and a negation of bits in binary strings representing vertices.

There are several results on Fibonacci cube fault-tolerance in hypercube. A direct embedding of the $\lfloor \frac{n}{2} \rfloor$-th order Generalized Fibonacci cube of dimension $n + \lfloor \frac{n}{2} \rfloor$ into hypercube of dimension $n$ with no more than three faulty vertices was given [33].

In [4] authors present a construction of a direct embedding of $n$-dimensional Fibonacci cube into faulty $n$-dimensional hypercube with less or equal $2^{\lceil n/4 \rceil - 1}$ faulty vertices. The idea is to find a permutation $\pi$ and a binary string $w$ such that the automorphism $h$ from (4.4) maps the given set of faulty vertices $F$ to vertices with two consecutive 1's. By (4.2) it means that $h(F) \subseteq Q_n \setminus FC_n$ and by (4.3) it follows that $F \notin \mathcal{S}(n,n)$. Since $F$ is arbitrary we conclude that $\psi(n,n) > 2^{\lceil n/4 \rceil - 1} = \Omega(2^{0.25n})$. Also the exact values of $\psi(n,n)$ for small $n$ and the best known lower and upper bounds on the number of tolerable faulty vertices are given, in terms of $\psi(n,n)$ :

$$\Omega(2^{cn}) = \frac{2^n}{4f_{n-2}} < \psi(n,n) \le \kappa(n, \lceil n/2 \rceil) = O(2^{dn}), \tag{4.5}$$

where $c = 2 - log_2(1+\sqrt{5}) \doteq 0.31$, $d = 2 - \frac{3}{4} log_2 3 \doteq 0.82$ and $\kappa(n,m)$ is defined below.

The problem of a Fibonacci cube fault-tolerance in hypercube is closely related to a subcube fault-tolerance in hypercubes. Let $\kappa(n,m)$ denote the minimum number of vertices whose removal from $n$-dimensional hypercube leaves no $m$-dimensional subcube. There are many results on $\kappa(n,m)$, see [23] for survey. In this chapter, we derive similar results for $\psi(n,m)$.

In section 4.1 we give the exact values for $0 \le m \le 3$. For $\psi(n,3)$, we modify a proof of Johnson and Entringer [43] who determined the exact value of $\kappa(n,2)$. In section 4.2 we introduce Lucas cube, a special subgraph of Fibonacci cube with a symmetry property that leads to a recursive bound $\psi(n,n) \le \psi(n-1,n-3)$. Next, the main scope of the chapter, the technique of labeling from [23], is introduced. We use it in section 4.3 to establish recursive upper bounds on $\psi(n,m)$ for the case $m = n$, for example $\psi(n,n) \le 2\psi(n-3,n-5)$ for $n \ge 3$. Several labelings are constructed in section 4.4. In section 4.5 we extend this technique also for $m < n$. In section 4.6 we show that if we multiply by 3 any labeling for $\kappa(n, m)$-problem, we obtain a labeling for $\psi(n, m)$-problem. In section 4.7 we establish also recursive lower bounds $\psi(n, m) \geq 2\psi(n-4, m-4)$ for $n \geq m \geq 4$ and $\psi(n, m) \geq \frac{2^r}{f_r}\psi(n-r, m-r)$ for $n \geq m \geq r$. In section 4.7 we combine the results in previous sections to give the exact values or tight bounds on $\psi(n, m)$ for $0 \leq m \leq n \leq 10$ and we conclude with a discussion of related open problems.

## 4.1 Exact values of $\psi(n, m)$ for small $m$

In this section we focus on what we know about the concrete values of $\psi(n, m)$ for $0 \leq m \leq 3$. Let us start with some notations. Given $S \subseteq V(Q_n)$ and a binary string $a_1 \ldots a_r$ of length $r$, $0 \leq r \leq n$, let us denote $S(a_1 \ldots a_r) = \{v_1 \ldots v_n \in S \mid v_i = a_i \text{ for all } 1 \leq i \leq r\}$. For a subgraph $G = (S, E)$ of $Q_n$ let $G(a_1 \ldots a_r)$ denote its subgraph induced on $S(a_1 \ldots a_r)$.

**Lemma 4.1.1.** *$\psi(n, m) \leq \kappa(n, \lceil m/2 \rceil)$, for $n \geq m \geq 0$.*

*Proof.* Fibonacci cube of dimension $m$ contains a subcube of dimension $\lceil m/2 \rceil$ as a subgraph. To see this, consider the subcube on vertices with 0 in even bits, i.e. $*0 \cdots *0*$ for $m$ odd or $*0 \cdots *0$ for $m$ even, where $*$ is 0 or 1. These vertices do not contain two consecutive 1's so by (2) this subcube of dimension $\lceil m/2 \rceil$ is a subgraph of Fibonacci Cube of dimension $m$. Thus any faulty set for a direct embedding of $Q_{\lceil m/2 \rceil}$ into $Q_n$ is also a faulty set for a direct embedding of $FC_m$ into $Q_n$. $\square$

**Theorem 4.1.2.** (1) *$\psi(n, 0) = 2^n$ for $n \geq 0$,*

(2) *$\psi(n, 1) = 2^{n-1}$ for $n \geq 1$,*

(3) *$\psi(n, 2) = 2^{n-1}$ for $n \geq 2$.*

*Proof.* Part (1) and (2) follow immediately from $FC_0 = Q_0$, $FC_1 = Q_1$ and $\kappa(n, 0) = 2^n$, $\kappa(n, 1) = 2^{n-1}$, [23]. For (3), by Lemma 4.1.1 we have that $\psi(n, 2) \leq \kappa(n, 1) = 2^{n-1}$. But also $\psi(n, 2) \geq 2^{n-1}$ since we can divide $Q_n$ into $2^{n-2}$ disjoint $Q_2$ and every $Q_2$ must contain at least 2 vertices from any faulty set for $FC_2$. $\square$

**Theorem 4.1.3.** *$\psi(n, 3) = \lfloor 2^n/3 \rfloor$, for $n \geq 3$.*

*Proof.* Johnson and Entringer [43] determined, in terms of $\kappa$, that $\kappa(n, 2) = \lfloor 2^n/3 \rfloor$, so we obtain from Lemma 4.1.1 that $\psi(n, 3) \leq \lfloor 2^n/3 \rfloor$.

Further, we modify their proof [43] to show that, for $n \geq 3$,

$$\text{if } S \subseteq V(Q_n) \text{ and } |S| < \lfloor 2^n/3 \rfloor \text{ then } Q_n \setminus S \text{ contains } FC_3 \text{ as a subgraph.} \tag{4.6}$$

We prove (4.6) by induction but first of all we need the following observation:

$$\lfloor 2^{n+1}/3 \rfloor = \begin{cases} \frac{2^{n+1}-2}{3} = 2\frac{2^n-1}{3} = 2\lfloor 2^n/3 \rfloor, & n \text{ even}, \\ \frac{2^{n+1}-1}{3} = 2\frac{2^n-2}{3} + 1 = 2\lfloor 2^n/3 \rfloor + 1, & n \text{ odd}. \end{cases} \tag{4.7}$$

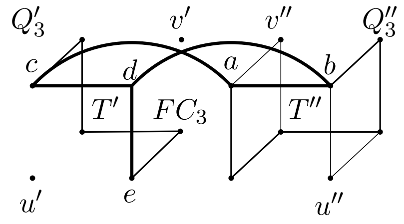



Figure 4.2: $\psi(4,3) \geq 5$ for otherwise we are led to a contradiction.

For $n = 3$ we have $\lfloor 2^n/3 \rfloor = 2$. If we remove any vertex from $Q_3$ we still have a subgraph isomorphic to $FC_3$, so (4.6) holds. Observe that if $\{u, v\} \in \mathcal{S}(3,3)$ then $u$, $v$ are antipodal vertices.

For $n = 4$ suppose that $|S| < \lfloor 2^n/3 \rfloor = 5$. If $Q_4 \setminus S$ does not contain $FC_3$ as a subgraph we are led to a contradiction as follows.

First, it must be that $|S(0)| = |S(1)| = 2$ and $u'$, $v'$ as well as $u''$, $v''$ are antipodes, where $S(0) = \{0u', 0v'\}$, $S(1) = \{1u'', 1v''\}$, for otherwise we can find a subgraph isomorphic to $FC_3$ in either $0Q_3' \setminus S(0)$ or $1Q_3'' \setminus S(1)$, where $Q_4 = 0Q_3' \cup 1Q_3''$. See Figure 4.2 for an illustration. Then there exists 6-cycle $T'$ in $0Q_3' \setminus S(0)$ and let $T''$ denote its corresponding 6-cycle in $1Q_3''$. There must exist two adjacent vertices $a$, $b \in T''$ disjoint from $u''$ and $v''$. Denote $c$, $d$ their corresponding vertices in $T'$ and $e$ the other vertex in $T'$ adjacent to $d$. These vertices form $FC_3$ in $Q_4 \setminus S$, a contradiction, so (4.6) holds.

For $n = 5$ if we have that $|S| = |S(0)| + |S(1)| < \lfloor 2^n/3 \rfloor = 10$ then either $|S(0)| < 5$ or $|S(1)| < 5$, so by the previous case for $n = 4$ we have that (4.6) holds.

Suppose (4.6) is true for $3, \ldots, n$ and let $S \subseteq V(Q_{n+1})$ with $|S| < \lfloor 2^{n+1}/3 \rfloor$. If $Q_{n+1} \setminus S$ does not contain $FC_3$ as a subgraph we are led to a contradiction as follows.

First, $n$ must be odd for otherwise, by (4.7) and the induction hypothesis, we would have

$$|S| = |S(0)| + |S(1)| \geq 2\lfloor 2^n/3 \rfloor = \lfloor 2^{n+1}/3 \rfloor.$$

Then, since $n$ is odd, we have, again from (4.7) and the induction hypothesis,

$$\begin{aligned} \lfloor 2^{n+1}/3 \rfloor - 1 \geq |S| = |S(0)| &+ |S(1)| \geq 2\lfloor 2^n/3 \rfloor = \lfloor 2^{n+1}/3 \rfloor - 1, \\ \text{so } |S(0)| &= |S(1)| = \lfloor 2^n/3 \rfloor. \end{aligned}$$

Similarly,

$$\begin{aligned} \lfloor 2^n/3 \rfloor = |S(0)| = |S(00)| &+ |S(01)| \geq 2\lfloor 2^{n-1}/3 \rfloor = \lfloor 2^n/3 \rfloor, \\ \lfloor 2^n/3 \rfloor = |S(1)| = |S(10)| &+ |S(11)| \geq 2\lfloor 2^{n-1}/3 \rfloor = \lfloor 2^n/3 \rfloor, \\ \text{so } |S(00)| = |S(01)| &= |S(10)| = |S(11)| = \lfloor 2^{n-1}/3 \rfloor. \end{aligned}$$

Next, for $n$ odd we have from (4.7) that $\lfloor 2^{n-1}/3 \rfloor$ is odd so that, for some $i \in \{0, 1\}$ we have $|S(00i)| = \lfloor 2^{n-2}/3 \rfloor + 1$ and $|S(001-i)| = \lfloor 2^{n-2}/3 \rfloor$. Since

$$|S(10i)| + |S(00i)| = \lfloor 2^{n-1}/3 \rfloor = |S(001-i)| + |S(101-i)|,$$

we obtain that $|S(10i)| = \lfloor 2^{n-2}/3 \rfloor$ and $|S(101-i)| = \lfloor 2^{n-2}/3 \rfloor + 1$. Similarly

$$\begin{aligned} |S(00i)| = |S(101-i)| = |S(011-i)| &= |S(11i)| = \lfloor 2^{n-2}/3 \rfloor + 1, \\ |S(001-i)| = |S(10i)| = |S(01i)| &= |S(111-i)| = \lfloor 2^{n-2}/3 \rfloor. \end{aligned}$$

From (4.7) we have that $\lfloor 2^{n-2}/3 \rfloor$ is even, so that

$$|S(001-i1)| = |S(10i1)| = |S(01i1)| = |S(111-i1)| = \lfloor 2^{n-3}/3 \rfloor.$$

Since $|S(11)| = \lfloor 2^{n-1}/3 \rfloor$, observe that by the same argument $|S(1**1)| = \lfloor 2^{n-1}/3 \rfloor$ and

$$|S(10i1)| + |S(11i1)| + |S(111-i1)| + |S(101-i1)| = \lfloor 2^{n-1}/3 \rfloor,$$

and this, combined with the previous result, gives

$$|S(11i1)| + |S(101-i1)| = \lfloor 2^{n-1}/3 \rfloor - 2\lfloor 2^{n-3}/3 \rfloor = 2\lfloor 2^{n-3}/3 \rfloor + 1.$$

Similar argument yields

$$\begin{aligned} |S(011-i1)| &+ |S(11i1)| = 2\lfloor 2^{n-3}/3 \rfloor + 1, \\ |S(101-i1)| &+ |S(011-i1)| = 2\lfloor 2^{n-3}/3 \rfloor + 1, \end{aligned}$$

so that, finally,

$$2(|S(101-i1)| + |S(011-i1)| + |S(11i1)|) = 6\lfloor 2^{n-3}/3 \rfloor + 3.$$

Since the left side is even but the right side is odd, this is a contradiction and (4.6) holds. Hence, $\psi(n,3) \geq \lfloor 2^n/3 \rfloor$. □

## 4.2 Lucas cube

In this section we introduce Lucas cube, a special subgraph of Fibonacci cube with a symmetry property that leads to recursive bounds on $\psi(n,n)$. It has been studied in [52].

Lucas cube $LC_n$ of dimension $n$ is a subgraph of $Q_n$ induced on vertices

$$V(LC_n) = \{0u; u \in V(FC_{n-1})\} \cup \{10v0; v \in V(FC_{n-3})\} \quad \text{for } n \geq 3, \tag{4.8}$$

$V(LC_2) = V(FC_2)$ and $V(LC_1) = \{0\}$. In our notation, $LC_n = 0FC_{n-1} \cup 10FC_{n-3}0$. For $n \geq 1$, the number of vertices is $l_n$, the $n$-th Lucas number, where $l_0 = 2$, $l_1 = 1$, $l_{n+2} = l_{n+1} + l_n$. Observe from (4.1) and (4.8) that Lucas cube is a subgraph of Fibonacci cube: if we remove from $FC_n$ all vertices with both the first and the last bit set to 1, we

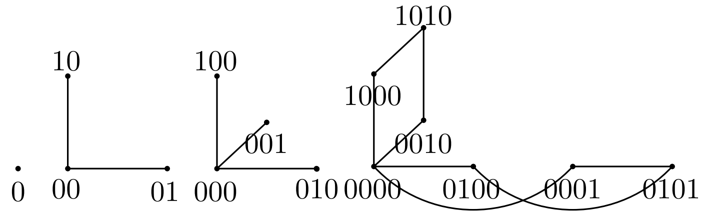



Figure 4.3: $LC_1$, $LC_2$, $LC_3$, $LC_4$.

obtain $LC_n$. We have

$$FC_n = \left\{ \begin{array}{l} LC_n = \left\{ \begin{array}{l} 0FC_{n-1} \\ 10FC_{n-3}0 \end{array} \right. \\ \left. \begin{array}{l} \\ 10FC_{n-4}01 \end{array} \right\} = 10FC_{n-2} \end{array} \right\} = FC_n \quad \text{for } n \geq 4,$$

and for $n = 3$ we have 101 instead of $10FC_{n-4}01$.

From (4.8) we obtain that the Lucas cube $LC_n$ can be characterized as

$$\begin{array}{c} \text{a subgraph of } Q_n \text{ induced on vertices without two consecutive 1's,} \\ \text{where the first and last bits are considered to be consecutive.} \end{array} \tag{4.9}$$

For $0 \leq i \leq n$ define an hypercube automorphism $\rho_i(v_1 \dots v_n) = v_{n-i+1} \dots v_n v_1 \dots v_{n-i}$ for $v \in V(Q_n)$, i.e. $\rho_i$ rotates the binary strings representing vertices $i$-times to the right. We have from (4.2) that this maps $LC_n$ to $LC_n$, so restricted on $LC_n$ it is an automorphism of Lucas cube. This leads to a symmetry property expressed in the following lemma.

**Lemma 4.2.1.** *For any $1 \leq i \leq n$, $n \geq 3$, if we remove all edges of dimension $i$ from $LC_n$ then it splits into two subgraphs isomorphic to $FC_{n-1}$ and $FC_{n-3}$.*

*Proof.* Split $LC_n$ along the first dimension into two subgraphs isomorphic to $FC_{n-1}$ and $FC_{n-3}$ by definition (4.8). Rotate it $(i-1)$-times to the right, so the removed edges of the first dimension are mapped to the edges of $i$-th dimension. Since the rotation is an automorphism of $LC_n$, we are done. □

**Theorem 4.2.2.** $\psi(n, n) \leq \psi(n-1, n-3)$, *for $n \geq 3$.*

*Proof.* Split $Q_n$ along an arbitrary dimension $i$ into two disjoint $Q'_{n-1}$ and $Q''_{n-1}$. Let $S \subseteq V(Q'_{n-1})$ be a set of size $\psi(n-1, n-3)$ in $\mathcal{S}(n-1, n-3)$. We will show that $S$ is in $\mathcal{S}(n, n)$. For a contradiction, suppose that $G$ is a subgraph of $Q_n \setminus S$ isomorphic to $FC_n$ via hypercube automorphism $h$. $G$ contains a subgraph isomorphic to $LC_n$ since $FC_n = LC_n \cup 10FC_{n-4}01$. Let $j$ be the dimension which automorphism $h$ maps to dimension $i$: $j = \pi^{-1}(i)$ where $\pi$ is given by (4.4). By Lemma 4.2.1, if we split $LC_n$ along the $j$-th dimension, it must be $h(LC_n) \cap Q'_{n-1} = h(FC_{n-1})$ or $h(LC_n) \cap Q'_{n-1} = h(FC_{n-3})$. In any case, this contradicts the choice of $S$. See Figure 4.4 for an illustration. □

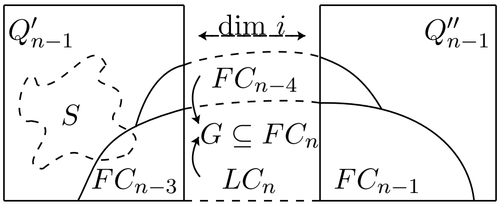



Figure 4.4: Intersection of $G$ with $S$.

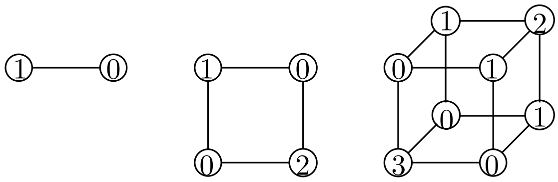


Figure 4.5: Examples of a labeling for the $\kappa(n, m)$-problem, $r = 1, 2, 3$.

As we can see in the proof of Theorem 4.2.2, any subgraph in hypercube of dimension $n$ isomorphic to Fibonacci cube of dimension $n$ intersects any subcube of dimension $n - 1$ with a graph containing a subgraph isomorphic to Fibonacci cube of dimension $n - 3$.

## 4.3 Labeling technique

In this section we introduce the concept of labeling [23], an useful tool for studying the recursive fault-tolerance in hypercubes and other self-similar networks.

Consider $(n - r)$-dimensional subcubes in $Q_n$ induced on vertices with first $r$ bits fixed. These $2^r$ subcubes form $r$-dimensional hypercube, say $C_r$, where each vertex $u$ represents one subcube $Q_n(u)$ and each edge represents the collection of all edges between two adjacent subcubes.

The following theorem [23] shows the use of labelings for the problem of subcube fault-tolerance in hypercube.

**Theorem 4.3.1.** *Label the vertices of $C_r$ with non-negative integers such that for every $0 \leq i \leq r$,*

$$\text{every } i\text{-dimensional subcube in } C_r \text{ has a vertex with label at least } i. \tag{4.10}$$

*Then for every $n \geq r$ and $m \leq n$ we have that*

$$\kappa(n, m) \leq \sum_{u \in V(C_r)} \kappa(n - r, m - l(u)),$$

*where $l(u)$ is the label of $u$ and for $m < l(u)$ we set $\kappa(n - r, m - l(u)) = \kappa(n - r, 0) = 2^{n-r}$.*

To avoid distraction, let us say that the labeling (4.10) is a labeling for the $\kappa(n, m)$-problem. See Figure 4.5 for examples. We modify this concept for the $\psi(n, m)$-problem, in

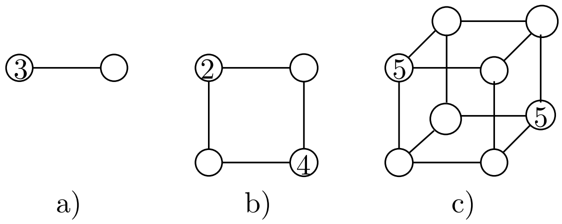



Figure 4.6: Examples of a labeling for the $\psi(n,n)$-problem, $r = 1, 2, 3$.

this section for $m = n$.

For a subgraph $G$ of $Q_n$ let us define its index of an inclusion of Fibonacci cube as

$$in(G) = \min\{i \mid G \text{ contains a copy of } FC_{n-i} \text{ as a subgraph}\}, \tag{4.11}$$

and leave $in(G)$ undefined if $G$ is an empty graph. Recall that $FC_0$ is a single vertex.

Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $r \leq n$. For every vertex $u \in V(C_r)$, consider subgraph $G(u)$ in $Q_n(u)$. We say that $in(G(u))$ is an indexing of $C_r$ with respect to $G$. We are considering partial indexings since $in(G(u))$ may be undefined. For two subgraphs $G_1$, $G_2$ of $Q_n$ isomorphic to $FC_n$, we say that their respective indexings are equivalent if there exists an automorphism $g : C_r \to C_r$ such that $in(G_1(u)) = in(G_2(g(u)))$ for all $u \in V(C_r)$.

Further, we define a (partial) labeling for the $\psi(n,n)$-problem. Label some vertices $u \in V(C_r)$ with non-negative integers $l(u)$ such that for $n \geq r$

$$\begin{aligned}&\text{for every subgraph } G \text{ of } Q_n \text{ isomorphic to } FC_n,\\ &\qquad\qquad \text{there exists } v \in V(C_r) \text{ with } in(G(v)) \leq l(v),\end{aligned} \tag{4.12}$$

i.e. $G(v)$ contains Fibonacci cube of dimension at least $n - l(v)$ as a subgraph.

Observe from the definition that a single vertex with label 0 is the only such labeling of $C_0$. For other examples see Figure 4.6.

**Theorem 4.3.2.** *Let $l$ be a (partial) labeling satisfying (4.12). Then for every $n \geq r$*

$$\psi(n,n) \leq \sum_{\substack{u \in V(C_r)\\ l(u) \text{ defined}}} \psi(n-r, n-l(u)).$$

*Proof.* For a vertex $u \in V(C_r)$ with $l(u)$ defined, let $S(u) \subseteq V(Q_n(u))$ be a faulty set of size $\psi(n-r, n-l(u))$ in $\mathcal{S}(n-r, n-l(u))$. We show that $S = \bigcup_{\substack{u \in V(C_r)\\ l(u) \text{ defined}}} S(u)$ is in $\mathcal{S}(n,n)$.

For a contradiction, let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and disjoint from $S$. By (4.12) we have a vertex $v$ with $in(G(v)) \leq l(v)$, hence $Q_n(v)$ contains $FC_{n-in(G(v))}$ as a subgraph which contains $FC_{n-l(v)}$ as a subgraph and disjoint from $S$. This contradicts the choice of $S$. $\square$

In the next section we show that labelings on Figure 4.6 satisfy (4.12) so we obtain

**Corollary 4.3.3.** (1) $\psi(n,n) \leq \psi(n-1, n-3)$ *for* $n \geq 1$,

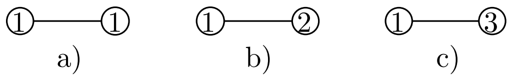



Figure 4.7: All indexings of $C_1$ up to an equivalence.

(2) $\psi(n,n) \leq \psi(n-2,n-2) + \psi(n-2,n-4)$ *for* $n \geq 2$,

(3) $\psi(n,n) \leq 2\psi(n-3,n-5)$ *for* $n \geq 3$.

Note that the part (1) is the same as in Theorem 4.2.2. We will see in the next section that labelings for the $\psi(n,n)$-problem can be constructed with a repeated use of a symmetry property of Lucas cubes from Lemma 4.2.1.

## 4.4 Construction of labelings

First of all, recall that for every subgraph $G$ of $Q_n$ isomorphic to $FC_n$ via $h : FC_n \to G$ there exists exactly one permutation $\pi$ on $\{1,\ldots,n\}$ and exactly one binary string $w$ of length $n$ such that $h$ is given by (4.4). The following lemma gives us a sufficient condition (4.13) for an equivalence of indexings with respect to two subgraphs isomorphic to $FC_n$.

**Lemma 4.4.1.** *For $r \leq n$, let $w_1$, $w_2$ be binary strings of length $n$ and $\pi_1$, $\pi_2$ be permutations on $\{1,\ldots,n\}$ such that*

$$\pi_1^{-1}(i) \leq r \text{ if and only if } \pi_2^{-1}(i) \leq r, \quad \text{for all } 1 \leq i \leq n. \tag{4.13}$$

*Denote $h_1$, $h_2$ automorphisms of $Q_n$, $h_1$ given by $\pi_1$, $w_1$ and $h_2$ given by $\pi_2$, $w_2$. Then the indexings of $C_r$ with respect to $G_1 = h_1(FC_n)$ and $G_2 = h_2(FC_n)$ are equivalent.*

*Proof.* Define hypercube automorphism $h = h_2 \circ h_1^{-1}$ and let a permutation $\pi$ and a binary string $w$ be given by (4.4). By (4.13) we have that

$$\pi(i) = \pi_1^{-1}(\pi_2(i)) \leq r \text{ if and only if } i \leq r, \quad \text{for all } 1 \leq i \leq n,$$

so we can define a permutation $\tau$ on $\{1,\ldots,r\}$ as a restriction of $\pi$ and a binary string $v$ of length $r$ as a prefix of $w$. Let $g$ denote the automorphism of $C_r$ from (4.4) given by $\tau$, $v$. Since $h(h_1(z)) = h_2(z)$ for all $z \in V(Q_n)$ it follows that

$$\text{if } z \in G_1(u) \text{ then } h(z) \in G_2(g(u)), \quad \text{for all } u \in V(C_r),$$

and we conclude $in(G_1(u)) = in(G_2(g(u)))$ for all $u \in V(C_r)$. □

**Lemma 4.4.2.** *Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $n \geq r = 1$. Then the indexing of $C_r$ with respect to $G$ is equivalent to one of indexings on Figure 4.7.*

*Proof.* By inspection of all cases, see Appendix. □

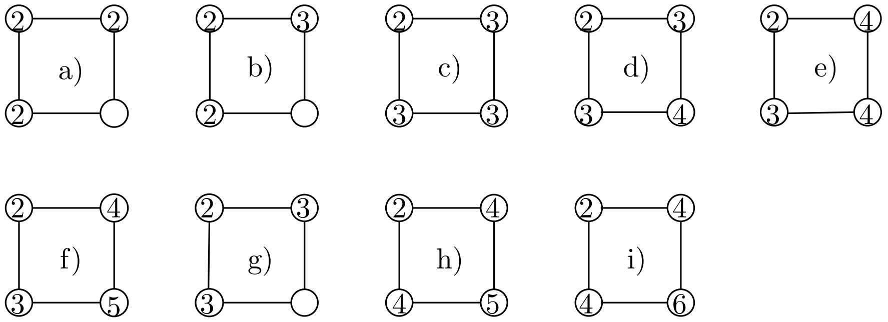



Figure 4.8: All indexings of $C_2$ up to an equivalence.

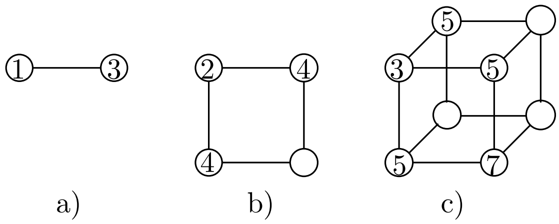



Figure 4.9: Majorant (4.14) of all indexings of $C_r$ with respect to some $G$.

**Lemma 4.4.3.** *Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $n \geq r = 2$. Then the indexing of $C_r$ with respect to $G$ is equivalent to one of indexings on Figure 4.8.*

*Proof.* By inspection of all cases, see Appendix. □

**Corollary 4.4.4.** *Labelings a) and b) on Figure 4.6 satisfy (4.12).*

*Proof.* Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $n \geq r$. Let $I$ denote the indexing on Figure 4.7 (resp. Figure 4.8) equivalent to the indexing with respect to $G$, i.e. $in(G(u)) = I(g(u))$ for all $u \in V(C_r)$ for some automorphism $g : C_r \to C_r$. On Figure 4.7 none index is higher than 3 and on Figure 4.8 if a vertex has index $> 4$ then its antipodal vertex has index 2. Thus there exists a vertex $v \in V(C_r)$ with $in(G(v)) = I(g(v)) \leq l(v)$ where $l$ is labeling on Figure 4.6a) (resp. Figure 4.6b). □

For $r > 2$ we can construct labelings for the $\psi(n, n)$-problem without the need to list indexings with respect to all subgraphs $G$ of $Q_n$ isomorphic to $FC_n$. Instead of this we find a (partial) function that majorizes them. Vertices of $C_r$ without two consecutive 1's form $FC_r$ in $C_r$. For $u \in V(C_r)$ define

$$m(u) = \begin{cases} 2w(u) + r & u \in V(FC_r), \\ \text{not def.} & u \notin V(FC_r), \end{cases} \tag{4.14}$$

where $w(u)$ denotes the weight of $u$, i.e. the number of 1's in $u$. See Figure 4.9 for examples.

**Lemma 4.4.5.** *Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$. Then there exists an automorphism $g : C_r \to C_r$ such that for all $u \in V(C_r)$,*

$$\text{if } m(g(u)) \text{ is defined then } G(u) \text{ is nonempty and } in(G(u)) \leq m(g(u)).$$

*Proof.* We can modify the mapping $h_1 : FC_n \to G$ to a mapping $h_2 : FC_n \to H$ such that the permutation $\pi_2$ and the binary string $w_2$ given by $h_2$ from (4.4) satisfy the condition that $\pi_2(i) \leq \pi_2(j)$ and $w_{2,\pi(i)} = 0$ for all $1 \leq i \leq j \leq r$. Moreover, we do it in a such way that also (4.13) is true, where $\pi_1$ and $w_1$ are given by $h_1$, so we can construct by Lemma 4.4.1 the automorphism $g : C_r \to C_r$ with $in(G(u)) = in(H(g(u)))$ for all $u \in V(C_r)$. We will show that for all $u \in V(C_r)$, if $m(u)$ is defined then $H(u)$ is nonempty and $in(H(u)) \leq m(u)$. So $g$ is the desired automorphism.

First of all, observe that $H(u)$ is nonempty for all $u = u_1 \dots u_r \in V(FC_r)$. Define $v = v_1 \dots v_n \in V(Q_n)$ with $v_{\pi_2(i)} = u_i$ for $1 \leq i \leq r$ and $v_{\pi_2(i)} = 0$ for $r < i \leq n$, so $h_2(v) \in Q_n(u)$. Since $u$ is without two consecutive 1's it follows that also $v$ is without two consecutive 1's, i.e. $v \in V(FC_n)$, and $h_2(v) \in H(u)$.

Secondly, we prove by induction on $r$ that $in(H(u)) \leq m(u)$ for all $u \in V(FC_r)$. By Lemma 4.4.2 and Lemma 4.4.3 it is true for $r \leq 2$. Suppose it is true for $r-1$ and let $v \in V(FC_{r-1})$ be the prefix of $u$, i.e. $u = v0$ or $u = v1$. Since $in(H(v))$ is defined we have $H(v)$ contains $h_2(FC_{n-in(H(v))})$ which contains $h_2(LC_{n-in(H(v))})$ as a subgraph. For $n - in(H(v)) \geq 3$, we have by a symmetry property in Lemma 4.2.1 that $h_2(LC_{n-in(H(v))})$ splits into $h_2(FC_{n-in(H(v))-1}) \subseteq H(v0)$ and $h_2(FC_{n-in(H(v))-3}) \subseteq H(v1)$. For $n - in(H(v)) < 3$, $FC_2$ splits into $FC_1$ and $FC_0$ and $FC_1$ splits into $FC_0$ and $FC_0$. Thus we conclude that

$$\begin{aligned} in(H(v0)) &\leq in(H(v)) + 1 \leq m(v) + 1 = m(v0), \\ in(H(v1)) &\leq in(H(v)) + 3 \leq m(v) + 3 = m(v1). \end{aligned}$$

□

**Corollary 4.4.6.** *Labeling c) on Figure 4.6 satisfies (4.12).*

*Proof.* Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $n \geq r = 3$. By Lemma 4.4.5 we have an automorphism $g : C_3 \to C_3$ with $in(G(u)) \leq m(g(u))$ for all $u \in V(C_3)$ with $m(g(u))$ defined. Denote $Z$ the set of vertices $u$ on Figure 4.9c) with $m(u) \leq 5$ and denote $v_1$, $v_2$ the vertices of Figure 4.6c) with label 5. Observe that for any automorphism $g' : C_3 \to C_3$ we have that $g'(Z)$ and $\{v_1, v_2\}$ intersect. Hence we conclude that there exist a vertex $v \in \{v_1, v_2\} \subset V(C_3)$ with $in(G(v)) \leq m(g(v)) \leq l(v) = 5$, where $l$ is the labeling on Figure 4.6c). □

To construct good labelings for a general $r$ is a challenging combinatorial problem in itself. However, we can generalize Corollary 4.4.6 to construct labelings of $C_r$ for a special $r$ in the form $r = 2^k - 1$, where $k$ is some integer. This result is based on well-known Hamming code, a perfect one-error correcting binary code, which can be interpreted as a partition of hypercube of dimension $r = 2^k - 1$ into $2^r/(r+1)$ disjoint 1-spheres. For an integer $z$,

$z$-sphere with center vertex $u$ is a subgraph of $Q_r$ induced on all vertices in a distance less or equal $z$ from $u$.

**Corollary 4.4.7.** *For every integer $k$ and $n \geq r = 2^k - 1$*

$$\psi(n, n) \leq \frac{2^r}{r+1}\psi(n-r, n-r-2).$$

*Proof.* Let $T$ denote the set of center vertices of disjoint 1-spheres given by Hamming codes. Label them with $r+2$ and other vertices leave unlabeled. Denote $Z$ the subgraph induced on vertices $u \in V(C_r)$ with $m(u) \leq r+2$. Since $Z$ is also 1-sphere, observe that for any automorphism $g' : C_r \to C_r$ we have that $g'(Z)$ intersects $T$. We can conclude by Lemma 4.4.5 that the constructed labeling satisfies (4.12). The rest follows from Theorem 4.3.2. $\square$

## 4.5 Recursive fault-tolerance for $m < n$

In this section we start with a generalization of Theorem 4.2.2 and we extend the labeling technique for $m < n$ in a way analogous to (4.10). First of all, note that a subgraph $G$ of $Q_n$ isomorphic to $FC_m$ is contained within a unique subcube given by $\{0, 1, *\}$-string $s = s_1 \dots s_n$ defined as follows. Let $h$ denote the isomorphism $h : FC_m \to G$, $u = h(0 \dots 0)$ and $I$ be the set of dimensions of edges on which edges of $FC_m$ from $0 \dots 0$ are mapped, i.e. $I = \{i; \exists v \in V(FC_m), \{0 \dots 0, v\} \in E(FC_m)$ and edge $\{h(0 \dots 0), h(v)\}$ has dimension $i\}$. Define $s_i = *$ for $i \in I$ and $s_i = u_i$ else.

Similarly like for $m = n$ the mapping $h$ can be uniquely extended to the automorphism of the whole subcube.

**Theorem 4.5.1.** *For $n \geq m \geq 0$*

(1) $\psi(n, m) \leq \min\{\psi(n-1, m) + \psi(n-1, m-3),\ 2\psi(n-1, m-1)\}$,

(2) $\psi(n, m) \geq \max\{2\psi(n-1, m),\ \psi(n-1, m-1)\}$.

*Proof.* For (1), split $Q_n$ along an arbitrary dimension $i$ into two disjoint $Q'_{n-1}$ and $Q''_{n-1}$. Let $S_1 \subseteq V(Q'_{n-1})$ and $S_2 \subseteq V(Q''_{n-1})$ be sets of size $\psi(n-1, m)$ and $\psi(n-1, m-3)$ (resp. both $\psi(n-1, m-1)$) in $\mathcal{S}(n-1, m)$ and $\mathcal{S}(n-1, m-3)$ (resp. both in $\mathcal{S}(n-1, m-1)$). We show that $S_1 \cup S_2$ is in $\mathcal{S}(n, m)$. Clearly, any $FC_m$ in $Q_n$ disjoint from $S_1 \cup S_2$ cannot be all in either $Q'_{n-1}$ or $Q''_{n-1}$. But it also cannot be in both subcubes, because, in that case, we have by Lemma 4.2.1 that $Q'_{n-1}$ and $Q''_{n-1}$ contain $FC_{m-1}$ and $FC_{m-3}$ as a subgraphs which contradicts the choice of $S_1 \cup S_2$.

For (2), note that at least $\psi(n-1, m)$ vertices must be removed from each $Q'_{n-1}$ and $Q''_{n-1}$ so that no $FC_m$ remains in either $Q'_{n-1}$ or $Q''_{n-1}$. Thus $\psi(n, m) \geq 2\psi(n-1, m)$. To prove the second inequality, let $S_1 \cup S_2$ be set of size $\psi(n, m)$ in $\mathcal{S}(n, m)$ and $S_1 \subseteq V(Q'_{n-1})$, $S_2 \subseteq V(Q''_{n-1})$. Denote by $T$ the set of vertices of $Q'_{n-1}$ that are adjacent to $S_2$. If $Q'_{n-1}$ contains $FC_{m-1}$ disjoint from $(S_1 \cup T)$ then we connect this $FC_{m-1}$ with the corresponding $FC_{m-2}$ in $Q''_{n-1}$ and we obtain $FC_n$ in $Q_n$ disjoint from $S_1 \cup S_2$. This contradicts the choice of $S$ so $S_1 \cup T$ must contain at least $\psi(n-1, m-1)$ vertices and, therefore, $\psi(n, m) \geq \psi(n-1, m-1)$. $\square$

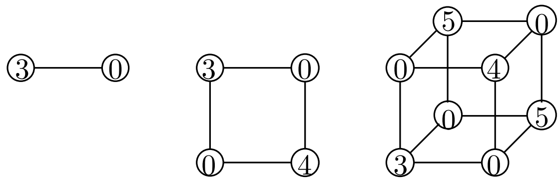


Figure 4.10: Examples of a labeling for the $\psi(n, m)$-problem.

Now we can make a step from a labeling for the $\psi(n, n)$-problem towards a labeling for the $\psi(n, m)$-problem for general $m \leq n$. Label vertices $u \in V(C_r)$ with non-negative integers $l(u)$ such that for $m \leq n \geq r$

$$\begin{aligned}&\text{for every subgraph } G \text{ of } Q_n \text{ isomorphic to } FC_m,\\ &\qquad \text{there exists } v \in V(C_r) \text{ with } in(G(v)) - n + m \leq l(v),\end{aligned} \tag{4.15}$$

i.e. $G(v)$ contains Fibonacci cube of dimension at least $m - l(v)$ as a subgraph.

Note that condition (4.15) can be rewritten more informally and analogously to (4.10) as follows. Label vertices $u \in V(C_r)$ with non-negative integers $l(u)$ such that for every $0 \leq i \leq r$,

$$l \text{ restricted on every } i\text{-dimensional subcube } D_i \text{ in } C_r \text{ satisfies (4.12).}$$

For example, observe on Figure 4.10 that every vertex, i.e. subcube $D_0$, has label at least 0, every edge, i.e. subcube $D_1$, has a vertex with label at least 3, every 4-cycle, i.e subcube $D_2$, has two antipodal vertices with labels at least 2 and 4, and labeling c) of $C_3$ satisfies (4.12) by an argument similar to Corollary 4.4.6. Hence all labelings on Figure 4.10 satisfy (4.15).

**Theorem 4.5.2.** *Let $l$ be a labeling of $C_r$ satisfying (4.15). For every $n \geq r$, $m \leq n$ we have*

$$\psi(n, m) \leq \sum_{u \in V(C_r)} \psi(n - r, m - l(u)).$$

*Proof.* For a vertex $u \in V(C_r)$ let $S(u) \subseteq V(Q_n(u))$ be a faulty set of size $\psi(n-r, m-l(u))$ in $\mathcal{S}(n - r, m - l(u))$. We show that $S = \bigcup_{u \in V(C_r)} S(u)$ is in $\mathcal{S}(n, m)$.

For a contradiction, let $G$ be a subgraph of $Q_n$ isomorphic to $FC_m$ and disjoint from $S$. By (4.15) we have a vertex $v \in V(C_r)$ with $in(G(v)) - n + m \leq l(v)$, hence $Q_n(v)$ contains $FC_{n-in(G(v))} = FC_{m-(in(G(v))-n+m)}$ as a subgraph which contains $FC_{m-l(v)}$ as a subgraph and disjoint from $S$. This contradicts the choice of $S$. $\square$

From Figure 4.10 we immediately obtain

**Corollary 4.5.3.** *For $m \leq n$*

(1) $\psi(n, m) \leq \psi(n - 1, m) + \psi(n - 1, m - 3)$ *for $n \geq 1$,*

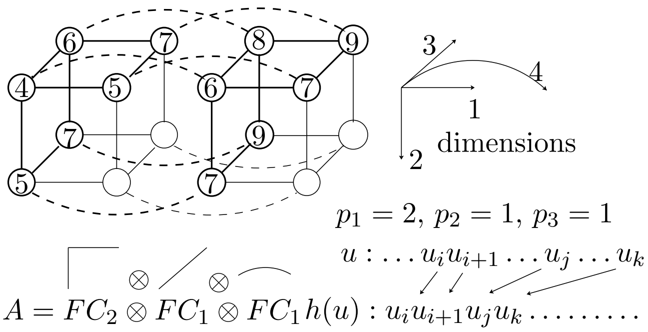



Figure 4.11: $A$ is a product of Fibonacci cubes.

(2) $\psi(n, m) \leq 2\psi(n-2, m) + \psi(n-2, m-3) + \psi(n-2, m-4)$ *for* $n \geq 2$,

(3) $\psi(n, m) \leq 4\psi(n-3, m) + \psi(n-3, m-3) + \psi(n-3, m-4) + 2\psi(n-3, m-5)$ *for* $n \geq 3$.

Note that the part (1) is included also in Theorem 4.5.1.

## 4.6 Relation to subcube fault-tolerance labeling

In this section we show that labelings for Fibonacci cube fault-tolerance can easily be derived from labelings for subcube fault-tolerance. Although it is an interesting relation, it should be noted that labelings obtained in this way are less effective than labelings constructed directly in previous sections. For example, the labeling for $r = 2$ in Figure 4.5 is optimal for $\kappa(n, m)$-problem. From this, Theorem 4.6.1 and Theorem 4.5.2 we obtain that $\psi(n, m) \leq 2\psi(n-2, m) + \psi(n-2, m-3) + \psi(n-2, m-6)$ for $n \geq 2$, $m \leq n$. But if we compare this to Corollary 4.5.3 (2), we can see that the labeling for $r = 2$ in Figure 4.10 is better, since $\psi(n-2, m-4) \leq \psi(n-2, m-6)$.

**Theorem 4.6.1.** *Let $l$ be a labeling of $C_r$ satisfying (4.10). Then the labeling $3l$ satisfies (4.15).*

*Proof.* Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_m$ via $h : FC_m \to G$. Denote $A$ the subgraph of $C_r$ induced on vertices $u$ with $G \cap Q_n(u) \neq \emptyset$, i.e. $in(G(u))$ is defined. We will show that for some $s$, $0 \leq s \leq m$, $A$ contains a subcube $C_s$ of dimension $s$ with $in(G(u)) - n + m \leq 3s$ for all $u \in V(C_s)$. From (4.10) it follows that there is a vertex $v \in V(C_s)$ with $s \leq l(v)$, thus $in(G(v)) - n + m \leq 3l(v)$ and (4.15) holds.

First of all, observe that $A$ is a product of several Fibonacci cubes. Product $FC_i \otimes FC_j$ is a subgraph of $Q_{i+j}$ induced on vertices $\{uv \mid u \in V(FC_i), v \in V(FC_j)\}$ where $uv$ means concatenation. Namely $A = \bigotimes_{i=1}^{z} FC_{p_i}$ where $p_i$ is the length of $i$-th block of consecutive bits mapped to $\{1, \ldots, r\}$ by isomorphism $h$ and $z$ is the number of such blocks. See Figure 4.11 for example.

As we have seen in Lemma 4.1.1, every $FC_{p_i}$ contains a subcube of dimension $\lceil p_i/2 \rceil$ so the product $A$ contains a subcube $C_s$ of dimension $s = \sum_{i=1}^{z} \lceil p_i/2 \rceil$. Denote $a$ the number of bits mapped to $\{1,\ldots,r\}$ by isomorphism $h$, i.e. $a = \sum_{i=1}^{z} p_i$. Clearly, $in(G(u)) - n + m \geq a$ for all $u \in V(C_s)$, and the equality holds for vertex $u$ with $h(0\ldots0) \in Q_n(u)$.

For every edge $\{u,v\}$ of $C_s$, observe that $in(G(u))$ and $in(G(v))$ differ by at most 1 if it is an edge of $FC_{p_i>1}$, and by at most 2 if it is an edge of $FC_{p_i=1}$. The first statement follows from the fact that if we denote $j$ the dimension mapped by $h$ to dimension of $\{u,v\}$, then $j+1$ (or similarly for $j-1$) is also mapped to some dimension in $\{1,\ldots,r\}$, otherwise $p_i = 1$. Thus for some $w \in V(FC_m)$ if $h(w_1\ldots w_{j-1}00w_{j+2}\ldots w_m) \in Q_n(u)$ then $h(w_1\ldots w_{j-2}010w_{j+2}\ldots w_m) \in Q_n(v)$. The second statement follows from the similar argument and the symmetry property of Lucas cubes.

From above we obtain $in(G(u)) - n + m \leq a + b + 2c$ for all $u \in V(C_s)$, where $b = \sum_{p_i>1} \lceil p_i/2 \rceil$ and $c = \sum_{p_i=1} \lceil p_i/2 \rceil$, i.e. $b + c = s$. Since

$$2\sum_{i=1}^{z} \lceil p_i/2 \rceil \geq \sum_{i=1}^{z} p_i + \sum_{\substack{i=1 \\ p_i=1}}^{z} p_i, \quad \text{for every } p_1,\ldots,p_z,$$

we have $2s \geq a + c$ by definition so $3s \geq a + b + 2c \geq in(G(u)) - n + m$ for all vertices $u \in V(C_s)$ and we are done. □

## 4.7 Recursive lower bounds

The methods for lower bounds in this section are modifications for a general $m$ of the methods that were essentially used in [4].

**Lemma 4.7.1.** *Let $G$ be a subgraph of $C_r$ isomorphic to $FC_r$ and $S \in \mathcal{S}(n,m)$, $n \geq m \geq r$. Then*

$$\sum_{u \in V(G)} |S(u)| \geq \psi(n-r, m-r).$$

*Proof.* Set $T = \{v \in V(Q_{n-r}); \exists u \in V(G) \text{ and } uv \in S\}$, i.e. $T$ is a subset of $V(Q_{n-r})$ obtained by a projection of $S \cap G$ over last $n-r$ dimensions, so $\sum_{u \in V(G)} |S(u)| \geq |T|$. Suppose that $|T| < \psi(n-r, m-r)$. Then there is a copy of $FC_{m-r}$ in $Q_{n-r}$ disjoint from $T$. By connecting such copies in every $Q_n(u)$ for $u \in V(G)$ we obtain a copy of $FC_r \otimes FC_{m-r}$ in $Q_n$ disjoint from $S$. Since $FC_r \otimes FC_{m-r}$ contains $FC_m$ as a subgraph (more precisely, in our notation $FC_r FC_{m-r} = FC_m \cup FC_{r-2}0110FC_{m-r-2}$ for $n \geq 4$, $r \geq 2$ and similarly for $r < 2$ or $n < 4$), this contradicts the choice of $S$ and we are done. □

Selecting the minimal $S$ yields $\psi(n,m) = |S| \geq \sum_{u \in V(G)} |S(u)| \geq \psi(n-r, m-r)$. Note that this generalizes the second statement in Theorem 4.5.1, the argument there is the same. The following results are based on better estimates than $|S| \geq \sum_{u \in V(G)} |S(u)|$.

**Theorem 4.7.2.** $\psi(n,m) \geq 2\psi(n-4, m-4)$, *for $n \geq m \geq 4$.*

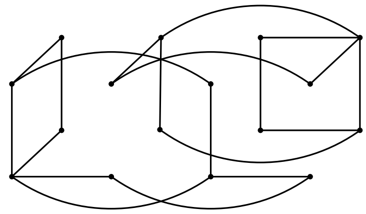


Figure 4.12: Partition of $C_4$ into two disjoint copies of $FC_4$.

*Proof.* Observe from Figure 4.12 that $C_4$ can be partitioned into two disjoint copies $G$, $G'$ of $FC_4$. So for the minimal set $S \in \mathcal{S}(n,m)$ we have by Lemma 4.7.1 that $\psi(n,m) = |S| = \sum\limits_{u \in V(G)} |S(u)| + \sum\limits_{u \in V(G')} |S(u)| \geq 2\psi(n-4, m-4)$. □

**Theorem 4.7.3.** *$\psi(n,m) \geq \frac{2^r}{f_r}\psi(n-r, m-r)$, for $n \geq m \geq r$.*

*Proof.* For $u$, $v \in V(Q_n)$ the number of automorphisms $g : Q_n \to Q_n$ with $g(u) = v$ is $n!$. Thus for $A$, $B \subseteq V(Q_n)$ we obtain

$$\sum_{g: Q_n \to Q_n} |g(A) \cap B| = \sum_{u \in A, v \in B} n! = |A|.|B|.n!.$$

Since there is exactly $n!2^n$ automorphisms of $Q_n$ we deduce that there exists an automorphism $g : Q_n \to Q_n$ with $|g(A) \cap B| \leq \frac{|A|.|B|}{2^n}$.

Let $B$ be the minimal set in $\mathcal{S}(n,m)$ and $A$ be the product $FC_r \otimes Q_{n-r}$, i.e. $A$ consists of vertices without two consecutive ones on first $r$ bits. By the argument above we have an automorphism $g : Q_n \to Q_n$ with $|g(A) \cap B| \leq \frac{f_r \psi(n,m)}{2^r}$. We can suppose without lost of generality that $g$ maps dimensions $\{1, \ldots, r\}$ to themselves, so $g$ restricted on $C_r$ is an automorphism $g'$ of $C_r$. Let $G$ denote the subgraph of $C_r$ isomorphic to $FC_r$ via $g'$, i.e. $g(A) = \bigcup\limits_{u \in V(G)} Q_n(u)$. By Lemma 4.7.1 we obtain that

$$\frac{f_r \psi(n,m)}{2^r} \geq \sum_{u \in V(G)} |B(u)| \geq \psi(n-r, m-r),$$

and we are done. □

## Conclusions

The combined results from this chapter and a few previous results for small values of $n$ (namely, $\psi(n,n)$ for $n \leq 6$, [4]) are summarized in Figure 4.13. There is still some improvement possible.

All recursive lower bounds were constructive, in a sense that given faulty sets for smaller $n$ and $m$ we recursively construct the faulty set in $\mathcal{S}(n,m)$ by adding a prefix corresponding to a respective subcube. The faulty sets for $m \leq 3$ in section 4.1 are taken as faulty sets for

| $n/m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | |
| 1 | 2 | 1 | | | | | | | | | |
| 2 | 4 | 2 | 2 | | | | | | | | |
| 3 | 8 | 4 | 4 | 2 | | | | | | | |
| 4 | 16 | 8 | 8 | 5 | 4 | | | | | | |
| 5 | 32 | 16 | 16 | 10 | 8-10 | 6 | | | | | |
| 6 | 64 | 32 | 32 | 21 | 16-20 | 12 | 8 | | | | |
| 7 | 128 | 64 | 64 | 42 | 32-42 | 24 | 16-24 | 8-12 | | | |
| 8 | 256 | 128 | 128 | 85 | 64-84 | 48-56 | 32-48 | 16-24 | 8-20 | | |
| 9 | 512 | 256 | 256 | 170 | 128-170 | 96-120 | 64-112 | 32-64 | 22-48 | 12-24 | |
| 10 | 1024 | 512 | 512 | 341 | 256-340 | 192-240 | 128-240 | 64-165 | 44-128 | 26-68 | 16-48 |

Figure 4.13: Table of known values or bounds on $\psi(n,m)$ for $0 \le m \le n \le 10$.

$\kappa(n, \lceil m/2 \rceil)$-problem. They can be constructed as a union of several level sets, i.e. vertices $u$ with weight $w(u) \equiv k \bmod m$ for a properly chosen $k$.

The upper bound in Theorem 4.7.2 is also constructive, in a sense that given a set $S$ of vertices with $|S| < 2\psi(n-4, m-4)$ we recursively find a copy of $FC_m$ in $Q_n$ disjoint from $S$, and each such step can be done in time $O(|S|)$. Thus Theorem 4.7.2 can be used for construction of fault-tolerant embedding algorithm running in polynomial time $O(n.|S|)$. On the other hand, in Theorem 4.7.3 the situation is different, we have no better way than to search all $n!2^n$ automorphisms in every step.

The labeling method can be applied in the problem of fault-tolerance also for other networks with recursive structure. What is necessary is some kind of a symmetry property, like in Lemma 4.2.1 for Lucas cubes.

The construction of labelings satisfying (4.12) (resp. (4.15)) for higher dimension $r$ remains a challenging combinatorial problem. In Corollary 4.4.7 we have shown that some results from coding theory might be useful for such tasks.

For practical purposes, it might be interesting to consider faulty edges instead of vertices, or faulty subgraphs in general. We believe that similar results could be formulated for this problem as well.

We have shown the close relationship to the problem of subcube fault-tolerance in hypercubes. This problem is well known also as a problem of $(n,k)$-universal sets, and there are connections to $k$-independence problem, partitions and linear codes.

# Appendix

All graphs in this section are induced subgraphs of hypercube. For a simplicity, we work with sets of binary strings of the same length representing vertices. Let us write $FC_n$ instead of $V(FC_n)$, and $FC_0$ is a nonempty set containing just the empty string denoted $\lambda$. For $G_1$ subgraph of $Q_i$ and $G_2$ subgraph of $Q_j$ define a concatenation $G_1G_2 = \{uv \mid u \in G_1, v \in G_2\}$

subgraph of $Q_{i+j}$. For $i \geq -1$ define

$$FC_i{}^0 = \begin{cases} \emptyset & i = -1 \\ FC_i 0 & i \geq 0 \end{cases} \quad \text{and} \quad {}^0FC_i = \begin{cases} \emptyset & i = -1 \\ 0FC_i & i \geq 0 \end{cases}.$$

Observe from characterization of $FC_n$ in (4.2) that

i) $FC_iFC_j = FC_{i+j}$ if $i = 0$ or $j = 0$,

ii) $FC_iFC_j = FC_{i+j} \cup FC_{i-2}{}^0 11 {}^0FC_{j-2}$ for all $i$, $j \geq 1$,

iii) $FC_n = FC_{i-1}0FC_{n-i} \cup FC_{i-2}{}^0 1 {}^0FC_{n-i-1}$ for all $1 \leq i \leq n$.

**Lemma 4.7.4.** *Let $w$ be a binary string of length $i$, $G$ be a nonempty subgraph of $Q_{n-i}$, $1 \leq i \leq n$.*

*1) $in(FC_i) = 0$,*

*2) $in(w) = i$,*

*3) $in(Gw) = in(G) + i$ for $i < n$,*

*4) $in(GFC_i) = in(G)$ for $i < n$,*

*5) $in(w_{i_0}FC_{j_1}w_{i_1}\ldots FC_{j_k}w_{i_k}) = \sum_{a=0}^{k} i_a$, for any binary strings $w_{i_a}$ of length $i_a$, $0 \leq a \leq k$, and any integers $j_b$, $1 \leq b \leq k$, such that $\sum_{a=0}^{k} i_a + \sum_{b=1}^{k} j_b = n$.*

*Proof.* Part 1) and 2) follow directly from the definition. By (4.11) $G$ contains $FC_{(n-i)-in(G)}$. Subgraph $Gw$ of $Q_n$ is isomorphic to $G$ so the highest dimension of Fibonacci cube in $Gw$ is the same: $FC_{n-(in(G)+i)}$ and 3) holds. Subgraph $GFC_i$ of $Q_n$ contains $FC_{(n-i)-in(G)}FC_i$ which by ii) contains $FC_{n-in(G)}$ and it is Fibonacci cube of the highest dimension in $GFC_i$ so 4) holds. Part 5) follows from 3) and 4). □

Now we are ready to prove two remaining lemmas from Section 4.4.

**Lemma 4.4.2.** *Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $n \geq r = 1$. Then the indexing of $C_r$ with respect to $G$ is equivalent to one of indexings on Figure 4.7.*

*Proof.* Let $\pi$ be the permutation and $w$ be the binary string such that $h : FC_n \to G$ where $h$ is given by (4), $i = \pi(1)$. By Lemma 4.4.1 we can suppose without a lost of the generality that $w_{\pi(i)} = 0$.

"a)" For $n = 1$ we have $FC_n = 0 \cup 1$ so

$$G(0) = h(0) = h(0FC_{n-1}) \Rightarrow in(G(0)) = 1 \text{ by Lemma 4.7.4}$$
$$G(1) = h(1) = h(0FC_{n-1}) \Rightarrow in(G(1)) = 1 \text{ by Lemma 4.7.4.}$$

"b1)" For $i = 1$ and $n > 1$ we have $FC_n = 0FC_{n-1} \cup 10FC_{n-2}$ so

$$G(0) = h(0FC_{n-1}) \quad \Rightarrow in(G(0)) = 1 \text{ by Lemma 4.7.4,}$$
$$G(1) = h(10FC_{n-2}) \quad \Rightarrow in(G(1)) = 2 \text{ by Lemma 4.7.4.}$$

"b2)" For $i = n$ and $n > 1$ we have $FC_n = FC_{n-1}0 \cup FC_{n-2}01$ so

$$\begin{aligned} G(0) &= h(FC_{n-1}0) && \Rightarrow in(G(0)) = 1 \text{ by Lemma 4.7.4,} \\ G(1) &= h(FC_{n-2}01) && \Rightarrow in(G(1)) = 2 \text{ by Lemma 4.7.4.} \end{aligned}$$

"c)" Else from iii) we have $FC_n = FC_{i-1}0FC_{n-i} \cup FC_{i-2}010FC_{n-i-1}$ so

$$\begin{aligned} G(0) &= h(FC_{i-1}0FC_{n-i}) && \Rightarrow in(G(0)) = 1 \text{ by Lemma 4.7.4,} \\ G(1) &= h(FC_{i-2}010FC_{n-i-1}) && \Rightarrow in(G(1)) = 3 \text{ by Lemma 4.7.4.} \end{aligned}$$

□

**Lemma 4.4.3.** *Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $n \geq r = 2$. Then the indexing of $C_r$ with respect to $G$ is equivalent to one of indexings on Figure 4.8.*

*Proof.* Let $\pi$ be the permutation and $w$ be the binary string such that $h : FC_n \to G$ where $h$ is given by (4), $i = \pi(1)$, $j = \pi(2)$. By Lemma 4.4.1 we can suppose without a lost of the generality that $i < j$ and $w_{\pi(i)} = w_{\pi(j)} = 0$.

a) For $n = 2$ we have

$$h(FC_n) = \begin{cases} h(00) = h(00FC_{n-2}) = G(00) \Rightarrow in(G(00)) = 2, \\ h(01) = h(01FC_{n-2}) = G(01) \Rightarrow in(G(01)) = 2, \\ h(10) = h(10FC_{n-2}) = G(10) \Rightarrow in(G(10)) = 2, \end{cases}$$
$$\emptyset = G(11) \Rightarrow in(G(11)) \text{ not def.}$$

b) For $n > 2$, $i = 1$ and $j = 2$ we have $h(FC_n) = \{$

$$h(0FC_{n-1}) = \begin{cases} h(00FC_{n-2}) = G(00) \Rightarrow in(G(00)) = 2, \\ h(010FC_{n-3}) = G(01) \Rightarrow in(G(01)) = 3, \end{cases}$$
$$h(10FC_{n-2}) = G(10) \Rightarrow in(G(10)) = 2,$$
$$\emptyset = G(11) \Rightarrow in(G(11)) \text{ not def.}$$

c) For $n = 3$, $i = 1$ and $j = 3$ we have $h(FC_n) = \{$

$$h(0FC_{n-1}) = \begin{cases} h(0FC_{n-2}\ 0) = G(00) \Rightarrow in(G(00)) = 2, \\ h(0FC_{n-3}01) = G(01) \Rightarrow in(G(01)) = 3, \end{cases}$$
$$h(10FC_{n-2}) = \begin{cases} h(10FC_{n-3}0) = G(10) \Rightarrow in(G(10)) = 3, \\ h(10FC_{n-3}1) = G(11) \Rightarrow in(G(11)) = 3. \end{cases}$$

d) For $n > 3$, $i = 1$ and $j = n$ we have $h(FC_n) = \{$

$$\begin{aligned} h(0FC_{n-1}) \quad &= \begin{cases} h(0\ \ FC_{n-2}\ \ 0) = G(00) \Rightarrow in(G(00)) = 2, \\ h(0\ \ FC_{n-3}01) = G(01) \Rightarrow in(G(01)) = 3, \end{cases} \\ h(10FC_{n-2}) &= \begin{cases} h(10FC_{n-3}\ \ 0) = G(10) \Rightarrow in(G(10)) = 3, \\ h(10FC_{n-4}01) = G(11) \Rightarrow in(G(11)) = 4. \end{cases} \end{aligned}$$

e) For $n > 3$, $i = 1$ and $j = 3$ we have $h(FC_n) = \{$

$$\begin{aligned} h(0FC_{n-1}) \quad &= \begin{cases} h(0FC_1 0FC_{n-3}) = G(00) \Rightarrow in(G(00)) = 2, \\ h(0010FC_{n-4}) = G(01) \Rightarrow in(G(01)) = 4, \end{cases} \\ h(10FC_{n-2}) &= \begin{cases} h(100\ \ FC_{n-3}) = G(10) \Rightarrow in(G(10)) = 3, \\ h(1010FC_{n-4}) = G(11) \Rightarrow in(G(11)) = 4. \end{cases} \end{aligned}$$

f) For $n > 4$, $i = 1$ and $3 < j < n$ we have $h(FC_n) = \{$

$$\begin{aligned} h(0FC_{n-1}) \quad &= \begin{cases} h(0\ \ FC_{j-2}\ \ 0\ \ FC_{n-j}\quad) = G(00) \Rightarrow in(G(00)) = 2, \\ h(0\ \ FC_{j-3}010FC_{n-j-1}) = G(01) \Rightarrow in(G(01)) = 4, \end{cases} \\ h(10FC_{n-2}) &= \begin{cases} h(10FC_{j-3}\ \ 0\ \ FC_{n-j}\quad) = G(10) \Rightarrow in(G(10)) = 3, \\ h(10FC_{j-4}010FC_{n-j-1}) = G(11) \Rightarrow in(G(11)) = 5. \end{cases} \end{aligned}$$

g) For $n > 3$, $i > 1$ and $j = i + 1 < n$ we have $h(FC_n) = \{$

$$\begin{aligned} h(FC_{i-1}0FC_{n-i}) &= \begin{cases} h(FC_{i-1}\ \ 00\ \ FC_{n-i-1}) \Rightarrow in(G(00)) = 2, \\ h(FC_{i-1}\ \ 010FC_{n-i-2}) \Rightarrow in(G(01)) = 3, \end{cases} \\ &\quad\ \ h(FC_{i-2}010\ \ FC_{n-i-1}) \Rightarrow in(G(10)) = 3, \\ &\qquad\qquad\qquad \emptyset = G(11) \Rightarrow in(G(11)) \text{ not def.} \end{aligned}$$

h) For $n > 4$, $i > 1$ and $j = i + 2 < n$ we have $h(FC_n) = \{$

$$\begin{aligned} h(FC_{i-1}\ \ 0\ \ FC_{n-i}\quad) &= \begin{cases} h(FC_{i-1}0FC_1 0FC_{n-i-2}) \Rightarrow in(G(00)) = 2, \\ h(FC_{i-1}\ \ 0010FC_{n-i-3}) \Rightarrow in(G(01)) = 4, \end{cases} \\ h(FC_{i-2}010FC_{n-i-1}) &= \begin{cases} h(FC_{i-2}0100\ \ FC_{n-i-2}) \Rightarrow in(G(10)) = 4, \\ h(FC_{i-2}01010FC_{n-i-3}) \Rightarrow in(G(11)) = 5. \end{cases} \end{aligned}$$

i) For $n > 5$, $i > 1$ and $i + 2 < j < n$ we have $h(FC_n) = \{$

$$h(FC_{i-1}\ 0\ FC_{n-i}\quad) = \begin{cases} h(FC_{i-1}\ 0\ FC_{j-i-1}\ 0\ FC_{n-j}\quad) = G(00), \\ h(FC_{i-1}\ 0\ FC_{j-i-2}010FC_{n-j-1}) = G(01), \end{cases}$$

$$h(FC_{i-2}010FC_{n-i-1}) = \begin{cases} h(FC_{i-2}010FC_{j-i-2}\ 0\ FC_{n-j}\quad) = G(10), \\ h(FC_{i-2}010FC_{j-i-3}010FC_{n-j-1}) = G(11). \end{cases}$$

By Lemma 4.7.4 it follows $in(G(00)) = 2$, $in(G(01)) = in(G(10)) = 4$, $in(G(11)) = 6$.

j) For $n > 2$, $i > 1$ and $j = n$ observe from 4.2) that the automorphism $h'$ of $Q_n$ that reverses vertices, i.e. $h'(u_1 \dots u_n) = u_n \dots u_1$, restricted on $FC_n$ is an automorphism of $FC_n$. Thus the indexing with respect to $G = h(FC_n) = h(h'(FC_n))$ is equivalent to the case $i = 1$ and $j < n$.

□

# 5 Hamiltonian Paths with Prescribed Edges in Hypercubes

## Introduction

The graph of the $n$-dimensional hypercube $Q_n$ is known to be hamiltonian for any $n \geq 2$ and the investigation of properties of hamiltonian cycles in $Q_n$ has received a considerable amount of attention ([61]). Moreover, a classical result of Havel [32] says that *if the distance $d(u, v)$ of vertices $u$ and $v$ of $Q_n$ ($n \geq 1$) is odd, there exists a hamiltonian path of $Q_n$ between $u$ and $v$.* Note that the condition on the odd distance between endvertices is trivially necessary since every hypercube is a bipartite graph with even number of vertices.

The advent of massively parallel computers ([48]) inspired the study of hypercubes with faulty links, which lead to the investigation of hamiltonian cycles and paths of $Q_n$ avoiding certain set of forbidden edges ([7, 9, 34, 62]). A related question has been recently proposed by R. Caha and V. Koubek: Given two vertices and a set of edges of $Q_n$, does there exist a hamiltonian path between given vertices, passing through every edge of this set? In [5] they observed that any proper subset $\mathcal{P}$ of edges of a hamiltonian path between $u$ and $v$ necessarily *induces a subgraph $\langle \mathcal{P} \rangle$ consisting of pairwise vertex-disjoint paths such that $\langle \mathcal{P} \rangle$ contains no path between $u$ and $v$, and neither $u$ nor $v$ is incident with more than one edge of $\mathcal{P}$.* Moreover, they showed that in case $|\mathcal{P}| \leq n - 2$, $n \geq 2$, this natural necessary condition (NNC) is also sufficient for the existence of a hamiltonian path of $Q_n$ between vertices $u$ and $v$ with $d(u, v)$ odd, passing through every edge of $\mathcal{P}$. On the other hand, for any $n \geq 3$ and vertices $u, v$ of $Q_n$ with $d(u, v)$ odd, there is a set of $2n - 3$ edges satisfying NNC, but not contained in any hamiltonian path between $u$ and $v$. Indeed, let $u, v$ be vertices of $Q_n$, $x \neq v$ a neighbor of $u$ and $\mathcal{P}$ a set of edges incident with neighbors of $x$ so that $u$ is incident with one edge of $\mathcal{P}$, each but one of the remaining neighbors is incident with two edges of $\mathcal{P}$, but no edge of $\mathcal{P}$ is incident with $x$. It is not difficult to see that this can always be done in such a way that NNC is preserved. Since $Q_n$ is a regular graph of degree $n$, it follows that $|\mathcal{P}| = 2n - 3$ and obviously, any path between $u$ and $v$ passing through all edges of $\mathcal{P}$ avoids $x$.

The main result of this chapter closes the gap between the lower bound $n - 2$, for which NNC becomes sufficient, and the upper bound $2n - 3$, for which it fails: We show that given vertices $u$ and $v$ with $d(u, v)$ odd and a set of at most $2n - 4$ prescribed edges of $Q_n$ satisfying NNC, there exists a hamiltonian path between $u$ and $v$ passing through every edge of the prescribed set. This holds for any $n \geq 2$ with two exceptions, namely two forbidden configurations for $n \in \{3, 4\}$. A similar problem for hamiltonian cycles formulated in [5] has been resolved in [15]. Our methods are similar to those of [15], however, the construction needed for the proof of this result is substantially more complex.

The rest of the chapter is divided into four sections. We start with a brief summary of concepts, notation and well-known properties of hypercubes. The next part is devoted to auxiliary results, preparing the necessary technique for an inductive construction, which forms the core of the proof of the main theorem in the following section. We conclude the chapter by showing how our result relates to the classical problem of hamiltonicity of hypercubes with faulty edges.

## 5.1 Preliminaries

We deal with finite undirected graphs, without loops or multiple edges. Our terminology and notation generally follows [28]. The vertex and edge sets of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. The distance of vertices $x$ and $y$ in $G$ is denoted by $d_G(x,y)$ with the subscript omitted if no ambiguity may arise. For $e, e' \in E(G)$, $d_G(e,e') = \min\{d_G(x,y) \mid x \in e, y \in e'\}$. Given a set $\mathcal{E} \subseteq E(G)$ of edges of a graph $G$, $\langle \mathcal{E} \rangle$ denotes the subgraph of $G$, induced by $\mathcal{E}$, i. e. $V(\langle \mathcal{E} \rangle) = \bigcup_{e \in \mathcal{E}} e$, $E(\langle \mathcal{E} \rangle) = \mathcal{E}$. Given a set $V' \subseteq V(G)$ of vertices of a graph $G$, $G - V'$ denotes the graph obtained from $G$ by removing all vertices of $V'$ and edges incident with them.

A *path* $P = x_0, x_1, \ldots, x_n$ of length $n$ between $x_0$ and $x_n$ is a graph $P$ with $V(P) = \{x_0, \ldots, x_n\}$ and $E(P) = \{\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}\}$. Vertices $x_0$ and $x_n$ are called the *endvertices* of $P$. A *cycle* of length $n$ is a connected 2-regular graph on $n$ vertices. Given a set $\mathcal{E}$ of edges, we say that a path $P$ (cycle $C$)

- *passes through* $\mathcal{E}$ if $\mathcal{E} \subseteq E(P)$ ($\mathcal{E} \subseteq E(C)$),
- *avoids* $\mathcal{E}$ if $\mathcal{E} \cap E(P) = \emptyset$ ($\mathcal{E} \cap E(C) = \emptyset$).

The $n$-dimensional hypercube $Q_n$ is a graph whose vertex set consists of binary vectors of length $n$, two vertices being adjacent whenever the corresponding vectors differ in exactly one coordinate. Following [48], the *dimension* of an edge $e = \{x, y\} \in E(Q_n)$, denoted by $\dim(e)$, is defined as the integer $i$ such that $x$ and $y$ differ in the $i$-th coordinate.

Let us summarize several fundamental properties of hypercubes (cf. [29] for details):

1. The removal of all edges of the same dimension splits $Q_{n+1}$ into two disjoint copies of $Q_n$, denoted by $Q_n^L$ and $Q_n^R$ (the "left" and the "right" subcube). Note that any vertex $x \in V(Q_n^L)$ has in $Q_n^R$ a unique neighbor, which shall be denoted by $x^R$. Similarly, any vertex $y \in V(Q_n^R)$ has in $Q_n^L$ a unique neighbor, denoted by $y^L$. For an edge $e = \{x, y\} \in E(Q_n^L)$, $e^R$ denotes the edge $\{x^R, y^R\} \in E(Q_n^R)$. Given a set $\mathcal{P} \subseteq E(Q_{n+1})$, we denote its subsets $\mathcal{P} \cap E(Q_n^L)$ and $\mathcal{P} \cap E(Q_n^R)$ by $\mathcal{P}^L$ and $\mathcal{P}^R$, respectively.

2. $Q_n$ is a bipartite graph for any $n \geq 1$, which means that for any $u, v \in V(Q_n)$, the length of each path between $u$ and $v$ has the same parity as $d(u,v)$. In particular, if there is a hamiltonian path between $u$ and $v$, then $u$ and $v$ belong to different partite sets and therefore $d(u,v)$ is odd.

3. The (0,2)–property: Any two distinct vertices of $Q_n$ ($n \geq 1$) have either exactly two neighbors in common or none at all.

For the rest of this section assume that the set of prescribed edges $\mathcal{P} \subseteq E(Q_n)$ is fixed. We say that $\mathcal{P}$ *contains a path* $P$ (between $u$ and $v$) if $P$ is a path (with endvertices $u$ and

$v$) forming a connected component of $\langle \mathcal{P} \rangle$. Note that any path contained in $\mathcal{P}$ is of length at least one.

Given a pair of vertices $u, v \in V(Q_n)$, we say that $(Q_n, \mathcal{P}, u, v)$ satisfies a *path condition* and write $PC(Q_n, \mathcal{P}, u, v)$ if

(i) $d(u, v)$ is odd,

(ii) $\langle \mathcal{P} \rangle$ consists of pairwise vertex-disjoint paths such that none of them has both $x$ and $y$ as its endvertices,

(iii) neither $x$ nor $y$ is incident with more than one edge of $\mathcal{P}$.

Given a subcube $S$ of $Q_n$, an edge $e \in E(S)$ is called *free in* $S$ if $e \notin \mathcal{P}$ and $\langle (\mathcal{P} \cap E(S)) \cup \{e\} \rangle$ consists of pairwise vertex-disjoint paths.

Let $Q_{n+1}$ be split into subcubes $Q_n^L$ and $Q_n^R$ such that $\mathcal{P} \setminus (\mathcal{P}^L \cup \mathcal{P}^R) = \{\{x, x^R\}\}$. For an edge $\{x, y\} \in E(Q_n^L)$, we say that the dimension $\dim(\{x, y\})$ is *blocked* for $\{x, x^R\}$ if

(i) $\mathcal{P}^L$ contains a path between $x$ and $y$, while $y^R$ is not incident with any edge of $\mathcal{P}^R$, or

(ii) $\mathcal{P}^R$ contains a path between $x^R$ and $y^R$, while $y$ is not incident with any edge of $\mathcal{P}^L$.

Note that if $\langle \mathcal{P} \rangle$ consists of pairwise disjoint paths, then each of conditions (i)-(ii) may hold for at most one edge $\{x, y\} \in E(Q_n^L)$ and hence at most two dimensions may be blocked for $\{x, x^R\}$.

## 5.2 Lemmas on paths and splits

The purpose of this section is to derive auxiliary results that shall be useful for the proof of the main theorem.

**Lemma 5.2.1.** *Let $n \geq 3$, $\mathcal{P} \subseteq E(Q_{n+1})$ and $u, v \in V(Q_{n+1})$ be such that $|\mathcal{P}| \leq 2(n+1) - 4$ and $PC(Q_{n+1}, \mathcal{P}, u, v)$. Then there exists a split of $Q_{n+1}$ into subcubes $Q_n^L$ and $Q_n^R$ such that $|\mathcal{P} \setminus (\mathcal{P}^L \cup \mathcal{P}^R)| \leq 1$. Moreover, if $\mathcal{P} \setminus (\mathcal{P}^L \cup \mathcal{P}^R) = \{\{x, x^R\}\}$, then the following conditions hold:*

(i) $\{x, x^R\} \cap \{u, v\} = \emptyset$,

(ii) *if there exists $\{\hat{u}, \hat{v}\} \in E(Q_{n+1})$ such that $\mathcal{P}$ contains paths between $u$ and $\hat{u}$ and between $v$ and $\hat{v}$, then* $\dim(\{\hat{u}, \hat{v}\}) \neq \dim(\{x, x^R\})$,

(iii) *if $\mathcal{P}$ contains a path of even length with one endvertex in $\{u, v\}$ and the other in $\{x, x^R\}$, then $\mathcal{P}$ contains another path with an endvertex in $\{u, v\}$ of length at least two.*

*Proof.* Put $\mathcal{P}_i = \{e \in \mathcal{P} \mid \dim(e) = i\}$ for $i \in \{1, 2, \ldots, n+1\}$. If there exists $i_0$ such that $\mathcal{P}_{i_0} = \emptyset$, then the removal of all edges of dimension $i_0$ splits $Q_{n+1}$ into subcubes such that $\mathcal{P} \setminus (\mathcal{P}^L \cup \mathcal{P}^R) = \emptyset$.

Otherwise let $I \subseteq \{1, 2, \ldots, n+1\}$ be such that $|\mathcal{P}_i| = 1$ for each $i \in I$; our assumption on the cardinality of $\mathcal{P}$ then implies that $|I| \geq 4$. Since neither $u$ nor $v$ may be incident with more than one edge of $\mathcal{P}$, it follows that there are at least two distinct edges $e, e' \in \bigcup_{i\in I} \mathcal{P}_i$ such that $(e \cup e') \cap \{u, v\} = \emptyset$. Consequently, the removal of all edges of dimension $\dim(e)$ or $\dim(e')$ splits $Q_{n+1}$ into subcubes such that (i) holds.

It remains to show that one of these two splits also satisfies both conditions (ii) and (iii). Note that if $\mathcal{P}$ contains at most one path with an endvertex in $\{u, v\}$, then at most one edge of $e, e'$ may be incident with the other endvertex of that path. Then split $Q_{n+1}$ by the dimension of the other edge and we are done.

Hence we can assume that $\mathcal{P}$ contains paths $P_1$ between $u$ and $\hat{u}$ and $P_2$ between $v$ and $\hat{v}$ for some $\hat{u}, \hat{v} \in V(Q_{n+1})$. First, suppose that $\{\hat{u}, \hat{v}\} \notin E(Q_{n+1})$, i. e. (ii) holds, and split $Q_{n+1}$ by $\dim(e)$. If (iii) does not hold, then one of the paths, say $P_1$, has even length, $e$ is incident with $\hat{u}$ and $P_2$ has length one. But then $e'$ cannot be incident with $\hat{u}$ and therefore splitting $Q_{n+1}$ by $\dim(e')$ satisfies (iii).

Now assume that $\{\hat{u}, \hat{v}\} \in E(Q_{n+1})$. Since $\dim(e) \neq \dim(e')$, splitting $Q_{n+1}$ by one of these dimensions must satisfy (ii). We claim that this split must also satisfy (iii). Indeed, otherwise it must be the case that, say, $P_1$ has even length and $P_2$ length one. But then $d(\hat{u}, \hat{v}) = 1$ implies that $d(u, v)$ must be even, contrary to our assumption that $PC(Q_{n+1}, \mathcal{P}, u, v)$ holds. $\square$

**Lemma 5.2.2.** *Let $n \geq 3$, $\mathcal{P} \subseteq E(Q_{n+1})$ and $u, v \in V(Q_{n+1})$ be such that $|\mathcal{P}| \leq 2(n+1)-4$ and $PC(Q_{n+1}, \mathcal{P}, u, v)$ and let $Q_{n+1}$ be split into subcubes $Q_n^L$ and $Q_n^R$. Then the following holds:*

- (i) *if $u, v \in V(Q_n^L)$, then there exists an edge $\{x, y\} \in E(Q_n^L) \setminus \mathcal{P}^L$ such that $PC(Q_n^L, \mathcal{P}^L \cup \{\{x, y\}\}, u, v)$ and $PC(Q_n^R, \mathcal{P}^R, x^R, y^R)$,*
- (ii) *if $u \in V(Q_n^L)$ and $v \in V(Q_n^R)$, then there exists a vertex $x \in V(Q_n^L)$ such that $PC(Q_n^L, \mathcal{P}^L, u, x)$ and $PC(Q_n^R, \mathcal{P}^R, x^R, v)$.*

*Proof.* First observe that for any $e \in E(Q_n^L)$, $e$ is not free in $Q_n^L$ or $e^R$ is not free in $Q_n^R$ if and only if $e$ or $e^R$

- belongs to $\mathcal{P}$, or
- is incident with a vertex of $\langle \mathcal{P}^L \rangle$ of degree two, or
- connects endvertices of a path of $\mathcal{P}$.

Let $p$ and $t$ denote the number of paths contained in $\mathcal{P}$ and the number of vertices of $\langle \mathcal{P} \rangle$ of degree two, respectively. Observe that $t \leq |\mathcal{P}| - 1$ and $t + p = |\mathcal{P}|$. Consequently, the number of edges $e \in E(Q_n^L)$ such that $e$ is not free in $Q_n^L$ or $e^R$ is not free in $Q_n^R$ does not exceed

$$|\mathcal{P}| + t(n-2) + p = 2|\mathcal{P}| + t(n-3) \leq 2n^2 - 5n + 5 < n2^{n-1} - 2 = |E(Q_n)| - 2$$

for $n \geq 3$. Hence there must exists edges $e_1, e_2 \in E(Q_n^L)$ such that $e_i$ is free in $Q_n^L$ and $e_i^R$ is free in $Q_n^R$ for $i \in \{1, 2\}$. Moreover, since $\mathcal{P} \cup \{e_i\}$ may contain a path between $u$ and $v$ for at most one $i \in \{1, 2\}$, at least one of $e_1, e_2$ must satisfy (i).

To show the validity of part (ii), suppose that $u \in V(Q_n^L)$ and $v \in V(Q_n^R)$. Let $x \in V(Q_n^L)$ be such that $d(u,x)$ is odd. Note that then $d(x^R, v)$ must have the same parity as $d(u,v)$, which is odd by our assumption. Consequently, $PC(Q_n^L, \mathcal{P}^L, u, x)$ or $PC(Q_n^R, \mathcal{P}^R, x^R, v)$ fail to hold if and only if

- $x$ or $x^R$ is incident with two edges of $\mathcal{P}$, or
- $\mathcal{P}^L$ contains a path between $u$ and $x$, or
- $\mathcal{P}^R$ contains a path between $x^R$ and $v$.

Put $A = \{x \in V(Q_n^L) \mid d(u,x) \text{ is odd}\}$. Let $p$ denote the number of paths of $\mathcal{P}$ starting at $u$ or $v$ and $t$ denote the number of vertices of $A$ which are incident with two edges of $\mathcal{P}$. Since $p \leq 2$ and $t \leq \frac{|\mathcal{P}|-p}{2}$, it follows that the number of vertices of $A$ for which (ii) fails does not exceed

$$t + p \leq \frac{|\mathcal{P}| - p}{2} + p = \frac{|\mathcal{P}| + p}{2} \leq n < 2^{n-1} = |A|,$$

for $n \geq 3$. Hence there must be a vertex for which (ii) holds. $\square$

**Lemma 5.2.3.** *Let $n \geq 3$, $\mathcal{P} \subseteq E(Q_{n+1})$ and $u, v \in V(Q_{n+1})$ be such that $|\mathcal{P}| \leq 2(n+1)-4$ and $PC(Q_{n+1}, \mathcal{P}, u, v)$. Further suppose that $Q_{n+1}$ is split into subcubes $Q_n^L$ and $Q_n^R$ such that $\mathcal{P} \setminus (\mathcal{P}^L \cup \mathcal{P}^R) = \{\{x, x^R\}\}$. Then*

(i) *there exists a free edge $\{x, y\}$ in $Q_n^L$ such that $\{x^R, y^R\}$ is free in $Q_n^R$, moreover, at least one of $y, y^R$ is not incident with any edge of $\mathcal{P}$,*

(ii) *if there is exactly one free edge $\{x, y\}$ in $Q_n^L$ such that $\{x^R, y^R\}$ is free in $Q_n^R$ and one vertex of $y, y^R$ is incident with an edge of $\mathcal{P}$, then each edge of $\mathcal{P}$ is incident with a neighbor of $x$ or $x^R$,*

(iii) *if at most one dimension is blocked for $\{x, x^R\}$, then there exists an edge $\{x, y\} \in E(Q_n^L)$ satisfying* (i) *such that neither $y$ nor $y^R$ is incident with any edge of $\mathcal{P}$, or there exist two distinct edges $\{x, y'\}, \{x, y\} \in E(Q_n^L)$ satisfying* (i),

(iv) *if no dimension is blocked for $\{x, x^R\}$ then there exist two distinct edges $\{x, y'\}, \{x, y\} \in E(Q_n^L)$ satisfying* (i).

*Proof.* Assume that $\mathcal{P} \setminus (\mathcal{P}^L \cup \mathcal{P}^R) = \{\{x, x^R\}\}$ for some $x \in V(Q_n^L)$. If condition (i) does not hold, then for any edge $\{x, y\} \in E(Q_n^L)$, at least one of the following must be true:

(a) $\mathcal{P}^L$ contains a path between $x$ and $y$, while $y^R$ is not incident with any edge of $\mathcal{P}^R$,

(b) $\mathcal{P}^R$ contains a path between $x^R$ and $y^R$, while $y$ is not incident with any edge of $\mathcal{P}^L$,

(c) there exist two edges in $\mathcal{P}$ such that each of them is incident with $y$ or $y^R$.

Note that as any hypercube is a triangle-free graph, no edge of $\mathcal{P}^L$ ($\mathcal{P}^R$) can be incident with two distinct neighbors of $x$ ($x^R$). Moreover, since $x$ has exactly $n$ distinct neighbors in $Q_n^L$ and each of conditions (a) and (b) may hold for at most one of them, it follows that there are at least $2 + 2(n-2) = 2n - 2$ edges of $\mathcal{P}^L \cup \mathcal{P}^R$, incident either with a neighbor of

$x$ in $Q_n^L$ or with a neighbor of $x^R$ in $Q_n^R$. Taking into account that $\{x, x^R\} \in \mathcal{P}$, it follows that $|\mathcal{P}| \geq 2n-1$, contrary to our assumption that $\mathcal{P} \leq 2(n+1)-4 = 2n-2$. This proves part (i). Moreover, if the assumptions of (ii) are satisfied, then the above argument shows that there are at least $2+2(n-3)+1 = 2n-3$ edges of $\mathcal{P}^L \cup \mathcal{P}^R$, incident with a neighbor of $x$ in $Q_n^L$ or with a neighbor of $x^R$ in $Q_n^R$, and the conclusion of (ii) follows.

To see that (iii) holds as well, observe that if at most one dimension is blocked for $\{x, x^R\}$, one of conditions (a) and (b) does not hold for any neighbor of $x$ in $Q_n^L$. If $y$ is the only neighbor of $x$ satisfying (i), there must be at least $1+2(n-2) = 2n-3$ edges of $\mathcal{P} \setminus \{\{x, x^R\}\}$, incident with neighbors of $x$ in $Q_n^L$ or with neighbors of $x^R$ in $Q_n^R$, distinct from $y$ and $y^R$. It follows that neither $y$ nor $y^R$ may be incident with any edge of $\mathcal{P}$.

In case (iv), when no dimension is blocked for $\{x, x^R\}$, neither (a) nor (b) may hold for any neighbor of $x$ in $Q_n^L$, and the above counting argument can be used to show that at least two distinct edges of $E(Q_n^L)$, incident with $x$, must satisfy condition (i). □

**Lemma 5.2.4.** *Let $n \geq 3$, $\mathcal{P} \subseteq E(Q_{n+1})$ and $u, v \in V(Q_{n+1})$ be such that $|\mathcal{P}| \leq 2(n+1)-4$ and $PC(Q_{n+1}, \mathcal{P}, u, v)$. Further suppose that $Q_{n+1}$ is split into subcubes $Q_n^L$ and $Q_n^R$ such that $\mathcal{P} \setminus (\mathcal{P}^L \cup \mathcal{P}^R) = \{\{x, x^r\}\}$, $u \in V(Q_n^L)$, $v \in V(Q_n^R)$, both $d(u, x)$ and $d(v, x^R)$ are odd and $\mathcal{P}^R$ contains a path between $x^R$ and $v$. Then there is a path $x, y, z$ in $Q_n^L$ such that $PC(Q_n^L, \mathcal{P}^L \cup \{\{y, z\}\}, u, x)$ and $PC(Q_n^R, \mathcal{P}^R \cup \{\{x^R, y^R\}\}, z^R, v)$.*

*Proof.* By Lemma 5.2.3 $x$ has a neighbor $\hat{y}$ in $Q_n^L$ such that edge $\{x, \hat{y}\}$ is free in $Q_n^L$ and edge $\{x^R, \hat{y}^R\}$ is free in $Q_n^R$.

1. First assume that $\hat{y} \neq u$ or $\hat{y}$ is not incident with any edge of $\mathcal{P}$. Then put $y = \hat{y}$. If $y$ has a neighbor $z \neq x$ in $Q_n^L$ such that neither $z$ nor $z^R$ is incident with any edge of $\mathcal{P}$, then clearly $PC(Q_n^L, \mathcal{P}^L \cup \{\{y, z\}\}, u, x)$ and $PC(Q_n^R, \mathcal{P}^R \cup \{\{x^R, y^R\}\}, z^R, v)$ hold as required. Otherwise it must be the case that

(∗) for any neighbor $z \neq x$ of $y$ in $Q_n^L$, there is edge of $\mathcal{P}$ incident with $z$ or $z^R$.

Since at least $n-1$ distinct edges of $\mathcal{P}$ satisfy (∗) and $\{x, x^R\} \in \mathcal{P}$ as well, it follows that there are at most $2(n+1)-4-n = n-2$ edges of $\mathcal{P}$ left that could be incident with neighbors of $x$ in $Q_n^L$ or neighbors of $x^R$ in $Q_n^R$, distinct from $y$ and $y^R$. But $x$ has exactly $n$ neighbors in $Q_n^L$ and therefore one of them, say $\hat{y} \neq y$, must be such that neither $\hat{y}$ nor $\hat{y}^R$ is incident with any edge of $\mathcal{P}$. Now suppose, by way of contradiction, that (∗) holds for $\hat{y}$. Since $y$ and $\hat{y}$ possess, by the (0,2)-property, exactly one common neighbor distinct from $x$, it follows that at least $2(n-1)-1 = 2n-3$ edges of $\mathcal{P}$ are incident with neighbors of $y$, $\hat{y}$, $y^R$ and $\hat{y}^R$, distinct from $x$ and $x^R$. Regarding that $x^R$ is incident with two edges of $\mathcal{P}$, it follows that $|\mathcal{P}| \geq 2n-1 = 2(n+1)-3$, contrary to our assumption. Hence there must be a neighbor $z$ of $\hat{y}$ in $Q_n^L$ such that neither $z$ nor $z^R$ is incident with any edge of $\mathcal{P}$. It follows that $PC(Q_n^L, \mathcal{P}^L \cup \{\{\hat{y}, z\}\}, u, x)$ and $PC(Q_n^R, \mathcal{P}^R \cup \{\{x^R, \hat{y}^R\}\}, z^R, v)$ hold as required.

2. Now we shall deal with the case when the only edge $e$, incident with $x$, having the property that $e$ is free in $Q_n^L$ and $e^R$ is free in $Q_n^R$, is the edge $e = \{x, u\}$, and moreover, $u$ is incident with an edge of $\mathcal{P}$. But then part (iii) of Lemma 5.2.3 implies that two dimensions are blocked for $\{x, x^R\}$. In particular, there must be a neighbor $y$ of $x$ in $Q_n^L$ such that $\mathcal{P}^L$ contains a path between $y$ and $x$ and $y^R$ is not incident with any edge of $\mathcal{P}^R$. Note that as

$\{x, y\}$ is not free in $Q_n^L$, $y \neq u$. Similarly, the path of $\mathcal{P}^R$ between $x^R$ and $v$, which exists by our assumptions, must have the property that $v$ is a neighbor of $x^R$, while $v^L$ is not incident with any edge of $\mathcal{P}^L$. Moreover, the assumption of case 2 implies that for any neighbor $w$ of $x$ in $Q_n^L$, $w \notin \{u, y, v^L\}$, there exist two distinct edges of $\mathcal{P}$ such that each of them is incident with one of $w, w^R$. This sums up to at least $2(n-3) = 2n-6$ distinct edges of $\mathcal{P}$. Regarding that there are another three edges of $\mathcal{P}$, incident with $u, y$ and $v$, we can conclude that there are at least $2n-3$ distinct edges of $\mathcal{P}$, incident with neighbors of $x$ in $Q_n^L$ or with neighbors of $x^R$ in $Q_n^R$. Since $\{x, x^R\} \in \mathcal{P}$ and we assume that $|\mathcal{P}| \leq 2(n+1) - 4 = 2n-2$, it follows that

(**) each edge of $\mathcal{P}$ must be incident with a neighbor of $x$ or $x^R$.

It remains to observe that $y$ has $n-1$ neighbors in $Q_n^L$ distinct from $x$, and at most one of them may be an endvertex of a path of $\mathcal{P}^L$, starting at $u$. Hence there are at least $n-2 \geq 1$ ways to choose a neighbor $z \neq x$ of $y$ in $Q_n^L$ such that $\mathcal{P}^L \cup \{\{y, z\}\}$ contains no path between $u$ and $x$.

Moreover, since $z$ and $x$ possess by the (0,2)-property exactly two common neighbors, one of them being $y$ and $\{y, z\} \notin \mathcal{P}$, (**) implies that at most one edge of $\mathcal{P}^L$ may be incident with $z$. It follows that $PC(Q_n^L, \mathcal{P}^L \cup \{\{y, z\}\}, u, x)$ holds. Applying the same argument to $z^R$ we can conclude that at most one edge of $\mathcal{P}^R$ may be incident with $z^R$ as well. Regarding that $y^R$ is not incident with any edge of $\mathcal{P}^R$, it follows that $PC(Q_n^R, \mathcal{P}^R \cup \{\{x^R, y^R\}\}, z^R, v)$ also holds as required. □

**Lemma 5.2.5.** *Let $n \geq 3$, $\mathcal{P} \subseteq E(Q_{n+1})$ and $u, v \in V(Q_{n+1})$ be such that $|\mathcal{P}| \leq 2(n+1)-4$ and $PC(Q_{n+1}, \mathcal{P}, u, v)$. Further suppose that $Q_{n+1}$ is split into subcubes $Q_n^L$ and $Q_n^R$ such that $\mathcal{P} \setminus (\mathcal{P}^L \cup \mathcal{P}^R) = \{x, x^R\}$, $u \in V(Q_n^L)$, $v \in V(Q_n^R)$, both $d(u, x)$ and $d(v, x^R)$ are even and no dimension is blocked for $\{x, x^R\}$. Then*

(i) *there is a 4-cycle $x, y, z, w$ in $Q_n^L$ such that edges $\{x, y\}, \{x, w\}$ are free in $Q_n^L$, edges $\{x^R, y^R\}, \{w^R, z^R\}$ are free in $Q_n^R$, neither $y^R$ nor $z^R$ is incident with any edge of $\mathcal{P}^R$ and $\mathcal{P}^L$ contains no path between $y$ and $w$ or between $u$ and $w$, or*

(ii) *there is a 4-cycle $x^R, y, z, w$ in $Q_n^R$ such that edges $\{x^R, y\}, \{x^R, w\}$ are free in $Q_n^R$, edges $\{x, y^L\}, \{w^L, z^L\}$ are free in $Q_n^L$, neither $y^L$ nor $z^L$ is incident with any edge of $\mathcal{P}^L$ and $\mathcal{P}^R$ contains no path between $y$ and $w$ or between $v$ and $w$.*

*Proof.* For any neighbor $y$ of $x$ in $Q_n^L$, $\dim(\{x, y\})$ is called a *free dimension* if $\{x, y\}$ is free in $Q_n^L$, $\{x^R, y^R\}$ is free in $Q_n^R$ and at least one of $y, y^R$ is not incident with any edge of $\mathcal{P}$. Since no dimension is blocked for $\{x, x^R\}$, it follows that $\dim(\{x, y\})$ is free if and only if there is at most one edge of $\mathcal{P}$, incident with $y$ or $y^R$.

Now assume that exactly $k$ dimensions are free; note that part (iv) of Lemma 5.2.3 implies that $k \geq 2$. Let $y_1, y_2, \ldots, y_k$ denote the neighbors of $x$ in $Q_n^L$ such that $\dim(\{x, y_i\})$ is free for each $i = 1, \ldots, k$. For any $I \subseteq \{1, 2, \ldots, k\}$ put $N(I) = |\{e \in \mathcal{P} \mid e \cap \{y_i, y_i^R, z_{ij}, z_{ij}^R\} \neq \emptyset, i, j \in I, i \neq j\}|$, where $z_{ij}$ denotes the common neighbor of $y_i$ and $y_j$, distinct from $x$. It follows that $N(\{1, 2, \ldots, k\}) \leq 2(n+1) - 4 - (2(n-k)+1) = 2k-3$. In particular in case

$k = 2$, $N(\{1,2\}) \leq 1$ and hence either $\{y,w\} = \{y_1, y_2\}$ satisfies (i), or $\{y,w\} = \{y_1^R, y_2^R\}$ satisfies (ii).

It remains to deal with the case when $k \geq 3$. We claim that then for any $I \subseteq \{1,2,\ldots,k\}$

$$(*)\ N(I) \leq 2|I| - 3 \Rightarrow \exists I_0 \subseteq I, |I_0| = 3 \text{ such that } N(I_0) \leq 3.$$

To verify the claim, argue by induction on $s = |I|$. As the claim is trivially true for $s = 3$, let $4 \leq s \leq k$ and suppose that $(*)$ holds for any subset of $\{1,2,\ldots,k\}$ of cardinality $s-1$. Let $I \subseteq \{1,2,\ldots,k\}$, $|I| = s$ such that $N(I) \leq 2s-3$ and assume that there exists $i_0 \in I$ such that $|\{e \in \mathcal{P} \mid e \cap \{y_{i_0}, y_{i_0}^R, z_{i_0 j}, z_{i_0 j}^R \mid j \in I, j \neq i_0\}\}| \geq 2$, for otherwise the conclusion of $(*)$ holds for any three-element subset of $I$. By the (0,2)-property $z_{i_0 j} \neq z_{lm}$ for any $j,l,m \in I \setminus \{i_0\}$, $l \neq m$, and since hypercubes are triangle-free, it follows that $N(I \setminus \{i_0\}) \leq N(I) - 2 \leq 2s - 5 = 2|I \setminus \{i_0\}| - 3$. Hence by the induction hypothesis there exist $I_0 \subseteq I \setminus \{i_0\} \subset I, |I_0| = 3$ satisfying $(*)$ as claimed.

We have just showed that $x$ has three distinct neighbors $y_{i_1}, y_{i_2}, y_{i_3}$ in $Q_n^L$ such that

- $|\{e \in \mathcal{P} \mid e \cap \{y_i, y_i^R, z_{ij}, z_{ij}^R \mid i,j \in \{i_1,i_2,i_3\}, i \neq j\} \neq \emptyset\}| \leq 3$, and
- $|\{e \in \mathcal{P} \mid e \cap \{y_i, y_i^R \mid i \in \{i_1,i_2,i_3\}\} \neq \emptyset\}| \leq 1$.

It is straightforward to verify by inspection that there exist $\{y,w\} \subseteq \{y_{i_1}, y_{i_2}, y_{i_3}\}$ satisfying (i) or $\{y,w\} \subseteq \{y_{i_1}^R, y_{i_2}^R, y_{i_3}^R\}$ satisfying (ii). $\square$

The following result is proved in [15]:

**Lemma 5.2.6.** *Let $u,v,x,y$ be pairwise distinct vertices of $Q_n$, $n \geq 2$, such that both $d(u,v)$ and $d(x,y)$ is odd. Then*

(i) *there exist paths between $u$ and $v$ and between $x$ and $y$ whose vertex sets partition $V(Q_n)$,*

(ii) *if $d(x,y) = 1$, there exists a hamiltonian path of $Q_n - \{x,y\}$ between $u$ and $v$ unless $n = 3$, $d(u,v) = 1$ and $d(\{u,v\},\{x,y\}) = 2$,*

(iii) *for any $e \in E(Q_n)$ such that $e \neq \{u,v\}$ there exists a hamiltonian path of $Q_n$ between $u$ and $v$ passing through edge $e$.*

The next lemma is a weaker form of a theorem proved in [34]:

**Lemma 5.2.7.** *For $n \geq 3$ and pairwise distinct vertices $x,y,u \in Q_n$ such that both $d(u,x)$ and $d(v,x)$ is odd, there exists a hamiltonian path of $Q_n - \{u\}$ between $x$ and $y$.*

We have a similar result on cycles avoiding two forbidden vertices:

**Lemma 5.2.8.** *Let $u,v,x,y$ be pairwise distinct vertices of $Q_n$, $n \geq 2$, such that $d(u,v)$ is odd and $\{x,y\} \in E(Q_n)$. Then there exists a hamiltonian cycle of $Q_n - \{u,v\}$ passing through edge $\{x,y\}$ unless $n = 3$, $d(u,v) = 1$ and $d(\{u,v\},\{x,y\}) = 2$.*

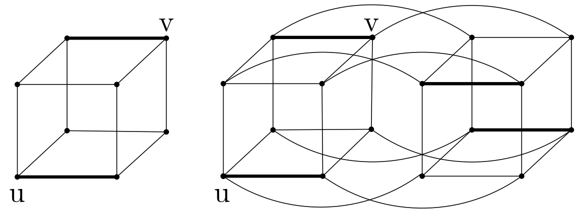



Figure 5.1: The only special configurations for $Q_3$ and $Q_4$.

*Proof.* Since in case $d(u, v) = 1$ the statement follows from Lemma 5.2.6, assume that $d(u, v) \geq 3$. Then shortest paths between $u$ and $v$ in $Q_{n+1}$ ($n \geq 2$) must contain an edge $e$ such that $\dim(e) \neq \dim(\{x, y\})$ and hence removing all edges of dimension $\dim(e)$ splits $Q_{n+1}$ into subcubes such that $u$ and $v$ belong to different subcubes, while $x$ and $y$ to the same subcube. Without a loss of generality assume that $u, x, y \in V(Q_n^L)$ and $v \in V(Q_n^R)$. Choose an arbitrary neighbor $w \in V(Q_n^R)$ of $v$ and observe that $d(w^L, u)$ must have the same parity as $d(u, v)$, which is odd by our assumptions. Hence we can apply Lemma 5.2.6 to obtain a hamiltonian path $P_1$ of $Q_n^L$ between $u$ and $w^L$, passing through edge $\{x, y\}$. Denote the neighbor of $u$ on $P_1$ by $z$ and apply Lemma 5.2.6 again, this time to obtain a hamiltonian path $P_2$ of $Q_n^R$ between $z^R$ and $v$, passing through edge $\{w, v\}$. It remains to observe that the desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$, passing through edge $\{x, y\}$ is induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{w^L, w\}\}) \setminus (\{u, z\}, \{w, v\})$. □

We complete this section with a lemma that resolves our problem for hypercubes of small dimensions. Note that we decided to rely on computers to verify the statement of the lemma by an exhaustive search.

**Lemma 5.2.9.** *Let $n \in \{2, 3, 4, 5\}$, $\mathcal{P} \subseteq E(Q_n)$ and $u, v \in V(Q_n)$ be such that $|\mathcal{P}| \leq 2n - 4$ and $PC(Q_n, \mathcal{P}, u, v)$ holds. Then there exists a hamiltonian path of $Q_n$ between $u$ and $v$ passing through $\mathcal{P}$ except the case when $n \in \{3, 4\}$, $d(u, v) = 3$ and $\mathcal{P}$ consists of $2n - 4$ edges of the same dimension such that any pair of them has mutual distance two and each of $u, v$ is incident with one edge of $\mathcal{P}$.*

*Proof.* The case $n \in \{2, 3\}$ may be verified by inspection (which is done in [5] in detail). The cases $n \in \{4, 5\}$ were verified by a computer search. □

The only "forbidden" cases in dimensions $n \in \{3, 4\}$ are depicted on Fig. 5.1.

## 5.3 Main theorem

**Theorem 5.3.1.** *Let $n \geq 2$, $\mathcal{P} \subseteq E(Q_n)$ and $u, v \in V(Q_n)$ be such that $|\mathcal{P}| \leq 2n - 4$ and $d(u, v)$ is odd. Then there exists a hamiltonian path of $Q_n$ between $u$ and $v$ passing through $\mathcal{P}$ if and only if $\langle \mathcal{P} \rangle$ consists of pairwise vertex-disjoint paths, $\langle \mathcal{P} \rangle$ contains no path between $u$ and $v$ and neither $u$ nor $v$ is incident with more than one edge of $\mathcal{P}$, except the case when $n \in \{3, 4\}$, $d(u, v) = 3$ and $\mathcal{P}$ consists of $2n - 4$ edges of the same dimension such that any pair of them has mutual distance two and each of $u, v$ is incident with one edge of $\mathcal{P}$.*

*Proof.* Since the necessity of the condition is obvious, the purpose of this proof is to show that it is also sufficient. We argue by induction on $n$. Recalling that the case $n \leq 5$ was settled by Lemma 5.2.9, let $n \geq 5$ and assume that the statement of the theorem holds for the hypercube of dimension $n$. In the following we shall prove that it also holds for the hypercube of dimension $n+1$. The desired hamiltonian path $H$ of $Q_{n+1}$ between $u$ and $v$ passing through $\mathcal{P}$ shall be described in terms of its edge set, i. e. $H = \langle \mathcal{E} \rangle$ for a suitable $\mathcal{E} \subseteq E(Q_{n+1})$. Using Lemma 5.2.1, split $Q_{n+1}$ into subcubes $Q_n^L$ and $Q_n^R$ such that $|\mathcal{P} \setminus (\mathcal{P}^L \cup \mathcal{P}^R)| \leq 1$. Assuming without a loss of generality that $|\mathcal{P}| = 2(n+1) - 4$ and $u \in V(Q_n^L)$, consider the following cases:

Case (1): There exists $x \in V(Q_n^L)$ such that $\mathcal{P} \setminus (\mathcal{P}^L \cup \mathcal{P}^R) = \{\{x, x^R\}\}$. Note that we can assume that conditions (i)–(iii) of Lemma 5.2.1 hold.

Subcase (1.1): $v \in V(Q_n^L)$.

(1.1.1) $|\mathcal{P}^L| < 2n-4$ and $|\mathcal{P}^R| \leq 2n-4$: By Lemma 5.2.3 there is a free edge $\{x, y\}$ in $Q_n^L$ such that $\{x^R, y^R\}$ is free in $Q_n^R$.

(1.1.1.1) $y$ is not incident with any edge of $\mathcal{P}^L$ or $y \notin \{u, v\}$: Then both $PC(Q_n^L, \mathcal{P}^L \cup \{\{x, y\}\}, u, v)$ and $PC(Q_n^R, \mathcal{P}^R, x^R, y^R)$ hold and hence by the induction hypothesis there exist hamiltonian paths $P_1$ of $Q_n^L$ between $u$ and $v$ and $P_2$ of $Q_n^R$ between $x^R$ and $y^R$, passing through $\mathcal{P}^L \cup \{x, y\}$ and $\mathcal{P}^R$, respectively. Then $H = \langle (E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}\}) \setminus \{\{x, y\}\} \rangle$.

(1.1.1.2) $\{x, u\}$ is the only free edge in $Q_n^L$ such that $\{x^R, u^R\}$ is free in $Q_n^R$; moreover, $u$ is incident with an edge of $\mathcal{P}^L$: Then condition (iii) of Lemma 5.2.3 says that two dimensions are blocked for $\{x, x^R\}$. This in particular means that $x$ has a neighbor $z \in V(Q_n^L)$ which is not incident with any edge of $\mathcal{P}^L$, while $\mathcal{P}^R$ contains a path between $x^R$ and $z^R$. Note that as $u$ is incident with an edge of $\mathcal{P}^L$, it follows that $z \neq u$, and as $d(u, z)$ is even, while $d(u, v)$ is odd, $z \neq v$ as well. Hence we can apply the induction to obtain a hamiltonian path $P_1$ of $Q_n^L$ between $u$ and $v$ passing through $\mathcal{P}^L \cup \{\{x, z\}\}$.

Let $w$ denote the neighbor of $z$ on $P_1$, distinct from $x$. Observe that since $z^R$ is incident with exactly one edge of $\mathcal{P}^R$, we can choose a neighbor $r \in V(Q_n^L)$ of $z$ such that $r \notin \{x, v, w\}$ and $\{z^R, r^R\} \notin \mathcal{P}^R$. Next observe that by the (0,2)-property, $r$ and $x$ have exactly two neighbors in $Q_n^L$ in common, one of them being $z$. But since $\{x, z\}, \{z, w\} \in E(P_1)$ and $z \notin \{u, v\}$ is not an endvertex of $P_1$, there must exist an edge $\{r, s\} \in E(P_1)$ such that $d(x, s) = 3$.

At this point recall that Lemma 5.2.3 also says that in this case every edge of $\mathcal{P}$ is incident with a neighbor of $x$ or $x^R$, which in particular means that $\{r, s\} \notin \mathcal{P}^L$, while $r^R$ may be incident with at most one and $s^R$ with no edge of $\mathcal{P}^R$. Hence $PC(Q_n^R, \mathcal{P}^R \cup \{\{z^R, r^R\}\}, x^R, s^R)$ holds, but before using the induction hypothesis we need to prove that $|\mathcal{P}^R| < 2n-4$. Indeed, if $|\mathcal{P}^R| = 2n-4$, then $|\mathcal{P}^L| = 1$, and we assume that one edge of $\mathcal{P}^L$ is already incident with $u$, but not with $x$. Consequently, $\mathcal{P}^L$ contains no path connecting $x$ to one of its neighbors, but then at most one dimension may be blocked for $\{x, x^R\}$, contrary to our assumption.

Therefore it is safe to apply the induction to obtain a hamiltonian path $P_2$ of $Q_n^R$ between $x^R$ and $s^R$, passing through $\mathcal{P}^R \cup \{\{z^R, r^R\}\}$. Then $H = \langle (E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{z, z^R\}, \{r, r^R\}, \{s, s^R\}\}) \setminus \{\{x, z\}, \{r, s\}, \{r^R, z^R\}\} \rangle$.

(1.1.2) $|\mathcal{P}^L| = 2n-4$ and $|\mathcal{P}^R| = 1$: Using the induction, find a hamiltonian path $P_1$ between $u$ and $v$ in $Q_n^L$ passing through $\mathcal{P}^L$. Since $\{x, x^R\} \in \mathcal{P}$, there has to be an edge $\{x, y\} \in E(P_1) \setminus \mathcal{P}^L$. If $\{x^R, y^R\} \notin \mathcal{P}^R$, then simply use the induction hypothesis to find in $Q_n^R$ a hamiltonian path $P_2$ between $x^R$ and $y^R$ passing through $\mathcal{P}^R$ and observe that $H = \langle (E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}\}) \setminus \{\{x, y\}\}\rangle$.

If, however, $\mathcal{P}^R = \{\{x^R, y^R\}\}$, choose on $P_1$ an edge $\{r, s\} \notin \mathcal{P}^L$ such that $\{r, s\} \cap \{x, y\} = \emptyset$. Then apply Lemma 5.2.6 to find in $Q_n^R - \{x^R, y^R\}$ a hamiltonian path $P_2$ between $r^R$ and $s^R$ and observe that $H = \langle (E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{x^R, y^R\}, \{y, y^R\}, \{r, r^R\}, \{s, s^R\}\}) \setminus \{\{x, y\}, \{r, s\}\}\rangle$.

(1.1.3) $|\mathcal{P}^L| = 2n-3$ and $|\mathcal{P}^R| = 0$: Recall that the edge $\{x, x^R\}$ was chosen in such a way that $x \notin \{u, v\}$. First choose an edge $\{r, s\} \in \mathcal{P}^L$ such that if $\mathcal{P}^L$ contains a path between $u$ or $v$ and $x$, then $r = x$. Otherwise make the choice in such a way that $s$ is an endvertex of a path of $\mathcal{P}^L$. Moreover, if $\mathcal{P}^L$ contains a path starting at $x$ then $s$ should be the other endvertex of that path. If there is not a path in $\mathcal{P}^L$ starting at $x$ but a path starting at $u$ or $v$ then $s$ should be also the other endvertex of that path.

Next, apply the induction hypothesis to find a hamiltonian path $P$ of $Q_n^L$ between $u$ and $v$ passing through $\mathcal{P}^L \setminus \{\{r, s\}\}$. If $\{r, s\} \in E(P)$, then simply choose on $P$ a neighbor $y$ of $x$ such that $\{x, y\} \notin \mathcal{P}^L$, apply the induction hypothesis to find a hamiltonian path $P_1$ of $Q_n^R$ between $x^R$ and $y^R$ and observe that $H = \langle (E(P) \cup E(P_1) \cup \{\{x, x^R\}, \{y, y^R\}\}) \setminus \{\{x, y\}\}\rangle$. If, however, $\{r, s\} \notin E(P)$, choose on $P$ neighbors $p$ and $q$ of $x$ and $s$, respectively, such that neither $\{x, p\}$ nor $\{s, q\}$ belongs to $\mathcal{P}^L$ and an extra condition specified below is satisfied. Let $P'$ denote the subpath of $P$ between $r$ and $s$.

(1.1.3.1) $r = x$: We can assume that the choice of $p$ and $q$ has been made in such a way that $P'$ contains exactly one of $p, q$. Let $y$ be the neighbor of $x$ on $P$, distinct from $p$. Apply Lemma 5.2.6 to find in $Q_n^R$ paths $P_1$ between $x^R$ and $y^R$ and $P_2$ between $p^R$ and $q^R$ whose vertex sets partition $V(Q_n^R)$ and observe that $H = \langle (E(P) \cup E(P_1) \cup (P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{p, p^R\}, \{q, q^R\}\}) \setminus \{\{x, p\}, \{s, q\}, \{x, y\}\}\rangle$.

(1.1.3.2) $r \neq x$: In this case we can assume $P'$ contains exactly one of $q$ and $x$. In case $p = x$ simply apply the induction hypothesis to find a hamiltonian path $P_1$ of $Q_n^R$ between $x^R$ and $q^R$ and observe that $H = \langle (E(P) \cup E(P_1) \cup \{\{x, x^R\}, \{q, q^R\}, \{r, s\}\}) \setminus \{\{r, x\}, \{s, q\}\}\rangle$. If, however, $p \neq x$, consider two subcases:

(1.1.3.2.1) $P'$ contains exactly one of $p$ and $x$: Choose on $P$ a neighbor $y$ of $x$ such that $\{x, y\} \notin P^L$. We may assume that $d(x, p)$ is odd, interchanging $x$ and $y$ if necessary. Then $d(y, q)$ is odd as well and we can apply Lemma 5.2.6 to find in $Q_n^R$ paths $P_1$ between $x^R$ and $p^R$ and $P_2$ between $y^R$ and $q^R$ whose vertex sets partition $V(Q_n^R)$. It remains to observe that $H = \langle (E(P) \cup E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{p, p^R\}, \{q, q^R\}, \{r, s\}\}) \setminus \{\{r, p\}, \{s, q\}, \{x, y\}\}\rangle$.

(1.1.3.2.2) $P'$ contains either both $p$ and $x$ or none of them: Choose on $P$ a neighbor $y$ of $x$ such that $\{x, y\} \notin P^L$ and $y \neq p$. Then apply Lemma 5.2.6 to find in $Q_n^R$ paths $P_1$ between $x^R$ and $y^R$ and $P_2$ between $p^R$ and $q^R$ whose vertex sets partition $V(Q_n^R)$ and observe that $H = \langle (E(P) \cup E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{p, p^R\}, \{q, q^R\}, \{r, s\}\}) \setminus \{\{r, p\}, \{s, q\}, \{x, y\}\}\rangle$.

(1.1.4) $|\mathcal{P}^L| = 0$ and $|\mathcal{P}^R| = 2n-3$: By the induction hypothesis $Q_n^R$ contains a hamiltonian cycle $C$ passing through $\mathcal{P}^R$. Since $\{x, x^R\} \in \mathcal{P}$, at least one of the two edges of $C$, incident with $x^R$, does not belong to $\mathcal{P}^R$. Hence there must exist an edge $\{x^R, y^R\} \in E(C) \setminus \mathcal{P}^R$. It remains to apply the induction again to obtain a hamiltonian path $P$ between $u$ and $v$ in $Q_n^L$ passing through edge $\{x, y\}$ and observe that $H = \langle (E(P) \cup E(C) \cup \{\{x, x^R\}, \{y, y^R\}\}) \setminus \{\{x, y\}, \{x^R, y^R\}\} \rangle$.

Subcase (1.2): $v \in V(Q_n^R)$.

(1.2.1) $\max\{|\mathcal{P}^L|, |\mathcal{P}^R|\} \leq 2n - 4$:

(1.2.1.1) $d(u, x)$ is odd: Then our assumption that $d(u, v)$ is odd implies that $d(x^R, v)$ must be odd as well.

(1.2.1.1.1) $\mathcal{P}^L$ contains no path between $u$ and $x$ and $\mathcal{P}^R$ contains no path between $v$ and $x^R$: In this case it suffices to apply the induction hypothesis to obtain a hamiltonian path $P_1$ of $Q_n^L$ between $u$ and $x$, passing through $\mathcal{P}^L$, and a hamiltonian path $P_2$ of $Q_n^R$ between $x^R$ and $v$, passing through $\mathcal{P}^R$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is obtained as a concatenation of $P_1$ and $P_2$.

(1.2.1.1.2) There is either a path between $u$ and $x$ in $\mathcal{P}^L$ or between $v$ and $x^R$ in $\mathcal{P}^R$, but not both, as then there would be a path between $u$ and $v$ in $\langle \mathcal{P} \rangle$, contrary to our assumption. Assuming without a loss of generality that the latter case applies, consider the following cases:

(1.2.1.1.2.1) $\max\{|\mathcal{P}^L|, |\mathcal{P}^R|\} < 2n - 4$: By Lemma 5.2.4 a path $x, y, z$ exists in $Q_n^L$ such that $PC(Q_n^L, \mathcal{P}^L \cup \{\{y, z\}\}, u, x)$ and $PC(Q_n^R, \mathcal{P}^R \cup \{\{x^R, y^R\}\}, z^R, v)$. It remains to use the induction hypothesis to obtain a hamiltonian path $P_1$ of $Q_n^L$ between $u$ and $x$, passing through $\mathcal{P}^L \cup \{\{y, z\}\}$, and a hamiltonian path $P_2$ of $Q_n^R$ between $z^R$ and $v$, passing through $\mathcal{P}^R \cup \{\{x^R, y^R\}\}$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{z, z^R\}\}) \setminus \{\{y, z\}, \{x^R, y^R\}\}$.

(1.2.1.1.2.2) $|\mathcal{P}^L| = 1$ and $|\mathcal{P}^R| = 2n - 4$: First note that part (iii) of Lemma 5.2.1 implies that $\mathcal{P}^L = \{\{x, y\}\}$, while the assumption of case (1.2.1.1.2) implies that $y \neq u$. Next observe that $y^R$ may have in $Q_n^R$ at most $|\mathcal{P}^R|/2 = n - 2$ neighbors different from $x^R$ such that each of them is either incident with two edges of $\mathcal{P}^R$ or is an endvertex of a path of $\mathcal{P}^R$ starting at $x^R$. Hence we can choose a neighbor $z \in V(Q_n^R)$ of $x^R$ such that $PC(Q_n^R, \mathcal{P}^R, v, z)$ holds. Now use the induction hypothesis to obtain a hamiltonian path $P_1$ of $Q_n^R$ between $v$ and $z$ passing through $\mathcal{P}^R$. Note that there must exist an edge $\{x^R, w\} \in E(P_1) \setminus \mathcal{P}^R$.

(1.2.1.1.2.2.1) $w^L = y$: Apply Lemma 5.2.6 to obtain a hamiltonian path $P_2$ of $Q_n^L - \{x, y\}$ between $u$ and $z^L$ and observe that the desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{x, y\}, \{x, x^R\}, \{y, y^R\}, \{z^L, z\}\}) \setminus \{\{x^R, y^R\}\}$.

(1.2.1.1.2.2.2) $w^L \neq y$ and $\{w, z\} \in E(Q_{n+1})$: Apply the induction hypothesis to obtain a hamiltonian path $P_2$ of $Q_n^L$ between $u$ and $x$ passing through $\mathcal{P}^L \cup \{\{w^L, z^L\}\}$ and observe that the desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{w^L, w\}, \{z^L, z\}\}) \setminus \{\{w^L, z^L\}, \{x^R, w\}\}$.

(1.2.1.1.2.2.3) $w^L \neq y$ and $\{w, z\} \notin E(Q_{n+1})$: Let $q$ be the common neighbor of $y$ and $w^L$, distinct from $x$. Note that $\{w, z\} \notin E(Q_{n+1})$ implies that $q \neq z^L$. Apply the induction hypothesis to obtain a hamiltonian path $P_2$ of $Q_n^L$ between $u$ and $x$ passing through $\mathcal{P}^L \setminus \{\{x, y\}\} \cup \{\{x, w^L\}, \{w^L, q\}, \{q, y\}, \{y, z^L\}\}$ and observe that the desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(P_1) \cup E(P_2) \cup$

$\{\{x,y\},\{x,x^R\},\{w^L,w\},\{z^L,z\}\}) \setminus \{\{y,z^L\},\{x,w^L\},\{x^R,w\}\}$.

(1.2.1.1.2.3) $|\mathcal{P}^L| = 2n-4$ and $|\mathcal{P}^R| = 1$: Note that then $\mathcal{P}^R = \{\{x^R,v\}\}$. Use the induction hypothesis to obtain a hamiltonian path $P_1$ of $Q_n^L$ between $u$ and $x$, passing through $\mathcal{P}^L$. Since $|\mathcal{P}^L| = 2n-4 < 2^n-4 = |E(P_1)|-3$, there must exist an edge $\{y,z\} \in E(P_1)\setminus\mathcal{P}^L$, not incident with $x$ or $v^L$. It remains to apply Lemma 5.2.6 to obtain a hamiltonian path $P_2$ of $Q_n^R - \{x^R,v\}$ and observe that the desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(P_1)\cup E(P_2)\cup\{\{x,x^R\},\{x^R,v\},\{y,y^R\},\{z,z^R\}\})\setminus\{\{y,z\}\}$.

(1.2.1.2) $d(u,x)$ is even: Then $d(x^R,v)$ must be even as well.

(1.2.1.2.1) $\max\{|\mathcal{P}^L|,|\mathcal{P}^R|\} < 2n-4$:

(1.2.1.2.1.1) $\mathcal{P}^L$ contains a path between $x$ and its neighbor $z$, while $z^R$ is not incident with any edge of $\mathcal{P}$: By Lemma 5.2.3 $x$ has a neighbor $y$ in $Q_n^L$ such that edge $\{x,y\}$ is free in $Q_n^L$, edge $\{x^R,y^R\}$ is free in $Q_n^R$ and at most one vertex of $y,y^R$ is incident with an edge of $\mathcal{P}$. Note that as $\{x,z\}$ is not free in $Q_n^L$, it follows that $y \neq z$.

(1.2.1.2.1.1.1) $\langle\mathcal{P}^L\rangle$ contains no path between $u$ and $y$: Then $PC(Q_n^L,\mathcal{P}^L\cup\{\{x,y\}\},u,z)$ and $PC(Q_n^R,\mathcal{P}^R\cup\{\{x^R,z^R\}\},y^R,v)$ hold and hence we can apply the induction hypothesis to obtain a hamiltonian path $P_1$ of $Q_n^L$ between $u$ and $z$, passing through $\mathcal{P}^L\cup\{\{x,y\}\}$, and a hamiltonian path $P_2$ of $Q_n^R$ between $y^R$ and $v$, passing through $\mathcal{P}^R\cup\{\{x^R,z^R\}\}$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(P_1)\cup E(P_2)\cup\{\{x,x^R\},\{y,y^R\},\{z,z^R\}\})\setminus\{\{x,y\},\{x^R,z^R\}\}$.

(1.2.1.2.1.1.2) $\mathcal{P}^L$ contains a path between $u$ and $y$: Then $y$ must be incident with an edge of $\mathcal{P}^L$ and as noted above, it means that $y^R$ is not incident with any edge of $\mathcal{P}^R$. Consequently, $\mathcal{P}^R$ contains no path between $v$ and $y^R$.

(1.2.1.2.1.1.2.1) $\mathcal{P}^R$ contains a path from $x^R$ to its neighbor $w$, while $w^L$ is not incident with any edge of $\mathcal{P}$: Then $PC(Q_n^R,\mathcal{P}^R\cup\{\{x^R,z^R\}\},v,w)$ and $PC(Q_n^L,\mathcal{P}^L\cup\{\{x,w^L\}\},u,z)$ and we can use a construction similar to (1.2.1.2.1.1.1); applying the induction hypothesis to obtain a hamiltonian path $P_1$ of $Q_n^R$ between $v$ and $w$, passing through $\mathcal{P}^L\cup\{\{x^R,z^R\}\}$ and a hamiltonian path $P_2$ of $Q_n^L$ between $u$ and $z$, passing through $\mathcal{P}^L\cup\{\{x,w^L\}\}$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(P_1)\cup E(P_2)\cup\{\{x,x^R\},\{z,z^R\},\{w^L,w\}\})\setminus\{\{x^R,z^R\},\{x,w^L\}\}$.

(1.2.1.2.1.1.2.2) The condition of the previous case does not hold: Then only one dimension is blocked for $\{x,x^R\}$, and hence by Lemma 5.2.3 either $y$ is not incident with any edge of $\mathcal{P}^L$ or there are at least two ways to choose it. In any case, the choice of $y$ can be made in such a way that $\langle\mathcal{P}^L\rangle$ contains no path between $u$ and $y$ and case (1.2.1.2.1.1.1) applies.

(1.2.1.2.1.2) $\langle\mathcal{P}^R\rangle$ contains a path from $x^R$ to its neighbor $z$, while $z^L$ is not incident with any edge of $\mathcal{P}$: This is a symmetric version of case (1.2.1.2.1.1).

(1.2.1.2.1.3) No dimension is blocked for $\{x,x^R\}$: By Lemma 5.2.5 we can assume that there is a 4-cycle $x,y,z,w$ in $V(Q_n^L)$ such that $PC(Q_n^L,\mathcal{P}^L\cup\{\{x,y\}\},u,w)$. Apply the induction hypothesis to obtain a hamiltonian path $P_1$ of $Q_n^L$ between $u$ and $w$ passing through $\mathcal{P}^L\cup\{\{x,y\}\}$ and consider two subcases:

(1.2.1.2.1.3.1) The subpath of $P_1$ between $u$ and $x$ contains $y$: Lemma 5.2.5 implies that $PC(Q_n^R,\mathcal{P}^R\cup\{\{x^R,w^R\}\},y^R,v)$. Hence we can apply the induction hypothesis to obtain a hamiltonian path $P_2$ of $Q_n^R$ between $y^R$ and $v$, passing through $\mathcal{P}^R\cup\{\{x^R,w^R\}\}$ and observe that the desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is induced by

$(E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{w, w^R\}\}) \setminus \{\{x, y\}, \{x^R, w^R\}\}$.

(1.2.1.2.1.3.2) The subpath of $P_1$ between $u$ and $x$ does not contain $y$:

(1.2.1.2.1.3.2.1) $|\mathcal{P}^R| < 2n - 5$: Lemma 5.2.5 implies that $PC(Q_n^R, \mathcal{P}^R \cup \{\{x^R, y^R\}, \{z^R, w^R\}\}, y^R, v)$. Hence we can apply the induction hypothesis to obtain a hamiltonian path $P_2$ of $Q_n^R$ between $y^R$ and $v$, passing through $\mathcal{P}^R \cup \{\{x^R, y^R\}, \{w^R, z^R\}\}$ and observe that the hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is induced by $(E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{w, w^R\}, \{y^R, z^R\}\}) \setminus \{\{x, y\}, \{x^R, y^R\}, \{z^R, w^R\}\}$.

(1.2.1.2.1.3.2.2) $|\mathcal{P}^L| = 2$ and $|\mathcal{P}^R| = 2n - 3$: If $\mathcal{P}^L$ contains a path of length two between $u$ and $x$, then no edge of $\mathcal{P}^L$ may be incident with $y$,$z$ or $w$. It follows that part (ii) of Lemma 5.2.5 holds for $x^R, y^R, z^R, w^R$ and hence we can use the construction of case (1.2.1.2.1.3.1) or (1.2.1.2.1.3.2.1) with the roles of $Q_n^L$ and $Q_n^R$ interchanged.

Otherwise we shall use the following strategy: Add additional prescribed edges to $\mathcal{P}^L$ to make sure that any hamiltonian path of $Q_n^L$ between $u$ and $w$ passing through them satisfies the condition of case (1.2.1.2.1.3.1). In order to do that, first verify by inspection that there exists a set $\mathcal{E} \subseteq E(Q_n^L) \setminus \mathcal{P}^L$, $|\mathcal{E}| \leq 3$ such that $\mathcal{P}^L \cup \mathcal{E}$ contains a path between $x$ and $w$ and $PC(Q_n^L, \mathcal{P}^L \cup \mathcal{E} \cup \{\{x, y\}\}, u, w)$. Since $|\mathcal{P}^L \cup \mathcal{E} \cup \{\{x, y\}\}| \leq 6 \leq 2n - 4$ for $n \geq 5$, we can apply the induction hypothesis to obtain a hamiltonian path $P_1$ of $Q_n^L$ between $u$ and $w$, passing through $\mathcal{P}^L \cup \mathcal{E} \cup \{\{x, y\}\}$. Then the subpath of $P_1$ between $u$ and $x$ contains $y$ and hence we can use the same construction as in case (1.2.1.2.1.3.1).

(1.2.1.2.2) $|\mathcal{P}^L| = 2n - 4$ and $|\mathcal{P}^R| = 1$:

First note that the number of neighbors of $x$ in $Q_n^L$ which are incident with two edges of $\mathcal{P}^L$ or form an endvertex of a path of $\mathcal{P}^L$ starting at $u$ does not exceed $n - 2$. Moreover, there is only one edge in $\mathcal{P}^R$ which may possibly be incident with $x^R$. Hence we can choose a neighbor $z$ of $x$ in $Q_n^L$ such that

(i) $PC(Q_n^L, \mathcal{P}^L, u, z)$ and $\{x^R, z^R\} \notin \mathcal{P}^R$.

Moreover, the choice can be made in such a way that

(ii) if $x^R$ has a neighbor $\hat{z}$ in $Q_n^R$ such that $\{\hat{z}, v\} \in \mathcal{P}^R$ and $\hat{z}^L$ is incident with at most one edge of $\mathcal{P}^L$, then $z = \hat{z}^L$.

Note that if (ii) holds, then by Lemma 5.2.1 $\mathcal{P}^L$ really contains no path between $u$ and $z$ and hence (ii) does not contradict (i). In any case, we can use the induction hypothesis to obtain a hamiltonian path $P_1$ between $u$ and $z$ in $Q_n^L$, passing through $\mathcal{P}^L$. Observe that there has to be an edge $\{x, y\} \in E(P_1) \setminus \mathcal{P}^L$. Assume that $y$ is chosen in such a way that, if possible, $y \neq z$ and consider the following subcases:

(1.2.1.2.2.1) $y \neq z$:

(1.2.1.2.2.1.1) $\mathcal{P}^R = \{\{x^R, y^R\}\}$: By Lemma 5.2.6 there is a hamiltonian path $P_2$ of $Q_n^R - \{x^R, y^R\}$ between $z^R$ and $v$ and the desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{x^R, y^R\}, \{y, y^R\}, \{z, z^R\}\}) \setminus \{\{x, y\}\}$.

(1.2.1.2.2.1.2) $\mathcal{P}^R = \{\{z^R, v\}\}$: By Lemma 5.2.6 there is a hamiltonian path $P_2$ of $Q_n^R - \{z^R, v\}$ between $x^R$ and $y$ and the desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{z, z^R\}, \{z^R, v\}\}) \setminus \{\{x, y\}\}$.

(1.2.1.2.2.1.3) None of the above two cases applies:

(1.2.1.2.2.1.3.1) $y$ belongs to the subpath of $P_1$ between $u$ and $x$: Note that the way $z$ was chosen (condition (ii)) implies that $\{y^R, v\} \notin \mathcal{P}^R$. It follows that $PC(Q_n^R, \mathcal{P}^R \cup \{\{x^R, z^R\}\}, y^R, v)$ holds and hence we can apply the induction hypothesis to obtain a hamiltonian path $P_2$ of $Q_n^R$ between $y^R$ and $v$, passing through $\mathcal{P}^R \cup \{\{x^R, z^R\}\}$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{z, z^R\}\}) \setminus \{\{x, y\}, \{x^R, z^R\}\}$.

(1.2.1.2.2.1.3.2) $y$ belongs to the subpath of $P_1$ between $x$ and $z$: Then the construction depends on the position of the only edge of $\mathcal{P}^R$.

(1.2.1.2.2.1.3.2.1) $\mathcal{P}^R = \{\{x^R, w\}\}$ for some $w \notin \{y^R, z^R\}$: Let $\hat{y}$ (resp. $\hat{z}$) be the common neighbor of $w$ and $y^R$ (of $w$ and $z^R$), distinct from $x^R$. Note that since $\hat{y} \neq \hat{z}$ by the (0,2)-property, at most one of them may be equal to $v$. If $\hat{y} \neq v$, observe that $PC(Q_n^R, \mathcal{P}^R \cup \{\{y^R, \hat{y}\}, \{\hat{y}, w\}, \{x^R, z^R\}\}, y^R, v)$ and hence we can apply the induction hypothesis to obtain a hamiltonian path $P_2$ of $Q_n^R$ between $y^R$ and $v$, passing through $\mathcal{P}^R \cup \{\{y^R, \hat{y}\}, \{\hat{y}, w\}, \{x^R, z^R\}\}$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{z, z^R\}\}) \setminus \{\{x, y\}, \{x^R, z^R\}\}$. If $\hat{y} = v$, then $\hat{z} \neq v$ and similarly as above, apply the induction to obtain a hamiltonian path $P_2$ of $Q_n^R$ between $z^R$ and $v$, passing through $\mathcal{P}^R \cup \{\{z^R, \hat{z}\}, \{\hat{z}, w\}, \{x^R, y^R\}\}$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{z, z^R\}\}) \setminus \{\{x, y\}, \{x^R, y^R\}\}$.

(1.2.1.2.2.1.3.2.2) $\mathcal{P}^R = \{\{z^R, w\}\}$ for some $w \notin \{x^R, v\}$: Apply the induction to obtain a hamiltonian path $P_2$ of $Q_n^R$ between $y^R$ and $v$, passing through $\mathcal{P}^R \cup \{\{x^R, y^R\}, \{x^R, z^R\}\}$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{z, z^R\}\}) \setminus \{\{x, y\}, \{x^R, z^R\}\}$.

(1.2.1.2.2.1.3.2.3) None of the above two cases applies: Observe that then $PC(Q_n^R, \mathcal{P}^R \cup \{\{x^R, y^R\}\ \{x^R, z^R\}\}, z^R, v)$ holds and hence we can use the induction to obtain a hamiltonian path $P_2$ of $Q_n^R$ between $z^R$ and $v$, passing through $\mathcal{P}^R \cup \{\{x^R, y^R\}, \{x^R, z^R\}\}$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{z, z^R\}\}) \setminus \{\{x, y\}, \{x^R, y^R\}\}$.

(1.2.1.2.2.2) $y = z$:

(1.2.1.2.2.2.1) $\mathcal{P}^R = \{\{z^R, v\}\}$: Recall that $v \neq x^R$.

(1.2.1.2.2.2.1.1) There exists an edge $\{r, s\} \in E(P_1) \setminus \mathcal{P}^L$ such that $s$ is a neighbor of $z$ and $r$ belongs to the subpath of $P_1$ between $u$ and $s$: Note that then $d(r, x) = d(r^R, x^R)$ is odd and hence we can apply Lemma 5.2.6 to find a hamiltonian path $P_2$ of $Q_n^R - \{z^R, v\}$ between $r^R$ and $x^R$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{r, r^R\}, \{x, x^R\}, \{z, z^R\}, \{z^R, v\}\}) \setminus \{\{r, s\}, \{x, z\}\}$.

(1.2.1.2.2.2.1.2) The condition of the previous case does not hold: We claim that then we can choose two distinct edges $\{p, q\}, \{r, s\} \in E(P_1) \setminus \mathcal{P}^L$ such that $p$ (resp. $r$) belongs to the subpath of $P_1$ between $u$ and $q$ ($u$ and $s$). To prove the claim, note that one edge of $\mathcal{P}^L$ must be incident with $x$, otherwise case (1.2.1.2.2.1) applies. Hence there are at most $2n-5$ edges of $\mathcal{P}^L$ than may be incident with $n-1$ neighbors of $z$, distinct from $x$. It follows that there exist neighbors $w_1, w_2, \ldots, w_k$ of $z$, distinct from $x$, such that

(i) $k = 2$ and each $w_i$ ($i \in \{1, 2\}$) is incident with at most $i-1$ edges of $\mathcal{P}^L$, or

(ii) $k \geq 3$ and each $w_i$ ($1 \leq i \leq k\}$) is incident with at most one edge of $\mathcal{P}^L$.

If (i) holds then $w_1 = u$, for otherwise the previous case (1.2.1.2.2.2.1.1) applies. But then we can choose $\{p,q\}$ and $\{r,s\}$ as the edges of $E(P_1)\setminus\mathcal{P}^L$, incident with $w_1$ and $w_2$, respectively. Note that vertex $p = u$ obviously belongs to the subpath of $P_1$ between $u$ and $q$. Vertex $r = w_2$ must belong to the subpath of $P_1$ between $u$ and $s$, for otherwise the previous case (1.2.1.2.2.2.1.1) applies. In case (ii) there must be at least two neighbors different from $u$ and we can use the same argument as above to show that they are incident with edges $\{p,q\}$ and $\{r,s\}$ of the desired properties. This completes the proof of the claim.

(1.2.1.2.2.2.1.2.1) $p^R = v$: The existence of a path $x,z,r,s$ guarantees that $d(x,s) = d(x^R,s^R)$ is odd. Hence we can apply the induction hypothesis to find a hamiltonian path of $Q_n^R$ between $x^R$ and $s^R$, passing through $\mathcal{P}^R \cup \{\{x^R,z^R\},\{v,q^R\}\}$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{p,z\},\{r,z\},\{q,q^R\},\{s,s^R\},\{x,x^R\}\}) \setminus \{\{p,q\},\{r,s\},\{x,z\},\{q,v\}\}$.

(1.2.1.2.2.2.1.2.2) $p^R \neq v$: Since the existence of a path $s^R,s,r,z,z^R,v$ guarantees that $d(s^R,v) = d(x^R,s^R)$ is odd and $\{p^R,q^R,s^R\} \cap \{x^R,z^R,w\} = \emptyset$, we can apply the induction hypothesis to obtain a hamiltonian path $P_2$ of $Q_n^R$ between $s^R$ and $v$, passing through $\mathcal{P}^R \cup \{\{p^R,q^R\},\{x^R,z^R\}\}$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is induced by $(E(P_1) \cup E(P_2) \cup \{\{p,p^R\},\{q,q^R\},\{s,s^R\},\{x,x^R\},\{z,z^R\}\}) \setminus \{\{p,q\},\{r,s\},\{x,z\},\{x^R,z^R\},\{p^R,q^R\}\}$.

(1.2.1.2.2.2.2) $\mathcal{P}^R \neq \{\{z^R,v\}\}$: We claim that then $z$ has a neighbor $w \neq x$ in $Q_n^L$ such that

(i) $w$ has a neighbor $q$ on $P_1$ such that $\{w,q\} \notin \mathcal{P}^L$,

(ii) $q^R$ is not incident with the only edge of $\mathcal{P}^R$.

To prove the claim, we use similar arguments as in case (1.2.1.2.2.2.1). At most $2n-5$ edges of $\mathcal{P}^L$ may be incident with $n-1$ neighbors of $z$, distinct from $x$, and hence at least two such neighbors are incident with at most one edge of $\mathcal{P}^L$. It follows that at least two neighbors of $z$ satisfy (i), and (ii) must hold for at least one of them.

If $q$ belongs to the subpath of $P_1$ between $u$ and $w$, apply the induction hypothesis to obtain a hamiltonian path $P_2$ between $q^R$ and $v$ in $Q_n^R$, passing through $\mathcal{P}^R \cup \{\{x^R,z^R\}\}$. If this is not the case, it is not difficult to see that there exists a set $\mathcal{E} \subseteq (E(Q_n^R) \setminus \mathcal{P}^R)$ of at most three edges not incident with $v$ such that $\mathcal{P}^R \cup \mathcal{E}$ contains a path between $z^R$ and $q^R$. Hence we can apply the induction hypothesis to obtain a hamiltonian path $P_2$ between $q$ and $v$ in $Q_n^R$, passing through $\mathcal{P}^R \cup \mathcal{E} \cup \{\{x^R,z^R\}\}$. It remains to observe that the desired hamiltonian path between $u$ and $v$ in $Q_{n+1}$ is induced by edges of $(E(P_1) \cup E(P_2) \cup \{\{q,q^R\},\{x,x^R\},\{z,z^R\}\}) \setminus \{\{x,z\},\{w,q\},\{x^R,z^R\}\}$.

(1.2.1.2.3) $|\mathcal{P}^L| = 1$ and $|\mathcal{P}^R| = 2n-4$: This is a symmetric version of case (1.2.1.2.2).

(1.2.2) $|\mathcal{P}^L| = 2n-3$ and $|\mathcal{P}^R| = 0$:

Recall that edge $\{x,x^R\}$ was chosen such that $u \neq x$ and $v \neq x^R$. Apply the induction to find a hamiltonian cycle $C$ of $Q_n^L$ passing through $\mathcal{P}^L$. Choose on $C$ neighbors $w$ and $y$ of $u$ and $x$, respectively, such that $\{u,w\} \notin \mathcal{P}^L$, $\{x,y\} \notin \mathcal{P}^L$. Moreover, choose $y \neq w$, if it is possible.

(1.2.2.1) $w = x$ or $y = u$: Apply the induction hypothesis to find a hamiltonian path $P$ of $Q_n^R$ between $x^R$ and $v$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(C) \cup E(P) \cup \{\{x, x^R\}\}) \setminus \{\{u, x\}\}$.

(1.2.2.2) $w = y$: We claim that there exists an edge $\{a, b\} \notin \mathcal{P}^L$ on $C$ such that $\{y, a\} \in E(Q^L) \setminus \mathcal{P}^L$. Indeed, the way $y$ has been chosen implies that both $u$ and $x$ are incident with edges of $\mathcal{P}^L$. Hence there remain at most $2n-5$ edges of $\mathcal{P}^L$ that can be incident with $n-2$ neighbors of $y$ in $Q_n^L$ different from $u$, $x$, each can be incident with at most one neighbor. So there is a neighbor $a$ incident with at most one edge.

(1.2.2.2.1) Each of two paths on $C$ between $y$ and $a$ contains exactly one of $u, b$: Apply Lemma 5.2.6 to find in $Q_n^R$ paths $P_1$ and $P_2$ between $x^R$ and $y^R$ and between $b^R$ and $v$ whose vertex sets partition $V(Q_n^R)$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(C) \cup E(P_1) \cup (P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{b, b^R\}, \{y, a\}\}) \setminus \{\{u, w\}, \{x, y\}, \{a, b\}\}$.

(1.2.2.2.2) One of two paths on $C$ between $y$ and $a$ contains both $u$ and $b$: Apply Lemma 5.2.6 to find in $Q_n^R$ paths $P_1$ and $P_2$ between $x^R$ and $b^R$ and between $y^R$ and $v$ whose vertex sets partition $V(Q_n^R)$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(C) \cup E(P_1) \cup (P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{b, b^R\}, \{y, a\}\}) \setminus \{\{u, w\}, \{x, y\}, \{a, b\}\}$.

(1.2.2.3) $w \neq x$, $w \neq y$, $y \neq u$, $v \neq y^R$: Apply Lemma 5.2.6 to find in $Q_n^R$ paths $P_1$ and $P_2$ between $x^R$ and $y^R$ and between $w^R$ and $v$ whose vertex sets partition $V(Q_n^R)$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(C) \cup E(P_1) \cup (P_2) \cup \{\{x, x^R\}, \{y, y^R\}, \{w, w^R\}\}) \setminus \{\{u, w\}, \{x, y\}\}$.

(1.2.2.4) $w \neq x$, $w \neq y$, $y \neq u$, $v = y^R$:

(1.2.2.4.1) One of the two paths on $C$ between $u$ and $x$ contains both $w$ and $y$: Apply the induction hypothesis to find a hamiltonian path $P$ of $Q_n^R$ between $w^R$ and $v$ passing through edge $\{x^R, y^R\}$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(C) \cup E(P) \cup \{\{x, x^R\}, \{y, y^R\}, \{w, w^R\}\}) \setminus \{\{u, w\}, \{x, y\}, \{x^R, y^R\}\}$.

(1.2.2.4.2) Each of the two paths on $C$ between $u$ and $x$ contains exactly one of $w, y$:

(1.2.2.4.2.1) $\mathcal{P}^L$ contains no path between $u$ and $y$: Then there exists an edge $\{a, b\} \in E(C) \setminus \mathcal{P}^L$ on the subpath of $C$ between $u$ and $y$ not containing $x$. It remains to apply the induction hypothesis to find a hamiltonian path $P$ between $w^R$ and $v$ in $Q_n^R$ passing through edges $\{x^R, y^R\}$ and $\{a^R, b^R\}$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(C) \cup E(P) \cup \{\{x, x^R\}, \{y, y^R\}, \{a, a^R\}, \{b, b^R\}, \{w, w^R\}\}) \setminus \{\{u, w\}, \{x, y\}, \{a, b\}, \{x^R, y^R\}, \{a^R, b^R\}\}$.

(1.2.2.4.2.2) $\mathcal{P}^L$ contains a path between $u$ and $y$: Since $y$ has $n-1$ neighbors in $Q_n^L$ different from $x$ and $|\mathcal{P}^L| = 2n-3$, there is a neighbor $a \neq x$ that is incident with at most one edge of $P^L$. Note that $a \neq u$ since $d(u, y)$ is even. If $a = w$, then simply apply the induction hypothesis to find a hamiltonian path $P$ between $x^R$ and $v$ in $Q_n^R$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(C) \cup E(P) \cup \{\{x, x^R\}\}) \setminus \{\{u, w\}, \{x, y\}\}$. If, however, $a \neq w$, let $b$ be a neighbor of $a$ such that $\{a, b\} \in E(C) \setminus \mathcal{P}^L$. Put $c = v$ and $d = b^R$, if one of two paths on $C$ between $a$ and $y$ contains both $b$ and $x$, otherwise put $c = b^R$ and $d = v$. Next apply Lemma 5.2.6 to find in $Q_n^R$ paths $P_1$ and $P_2$ between $x^R$ and $c$ and between $w^R$ and $d$ whose vertex sets partition

$V(Q_n^R)$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(C) \cup E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{w, w^R\}, \{b, b^R\}, \{y, a\}\} \setminus \{\{u, w\}, \{x, y\}, \{a, b\}\}$.

Case (2): $\mathcal{P} = \mathcal{P}^L \cup \mathcal{P}^R$.

Subcase (2.1): $v \in V(Q_n^L)$.

(2.1.1) $|\mathcal{P}^L| < 2n-4$ and $|\mathcal{P}^R| \leq 2n-4$: Lemma 5.2.2 guarantees the existence of an edge $\{x, y\} \in E(Q_n^L) \setminus \mathcal{P}^L$ such that $PC(Q_n^L, \mathcal{P}^L \cup \{\{x, y\}\}, u, v)$ and $PC(Q_n^R, \mathcal{P}^R, x^R, y^R)$ hold. Hence we can use the induction hypothesis to obtain hamiltonian paths $P_1$ of $Q_n^L$ between $u$ and $v$ and $P_2$ of $Q_n^R$ between $x^R$ and $y^R$, passing through $\mathcal{P}^L \cup \{\{x, y\}\}$ and $\mathcal{P}^R$, respectively. Then $H = \langle (E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}\}) \setminus \{\{x, y\}\} \rangle$.

(2.1.2) $|\mathcal{P}^L| = 2n-4$ and $|\mathcal{P}^R| = 2$: First apply the induction to find a hamiltonian path $P_1$ of $Q_n^L$ between $u$ and $v$ passing through $\mathcal{P}^L$. Since $|E(P_1)| - |\mathcal{P}| = 2^n - 1 - (2n-2) > 3$ for $n \geq 5$, there has to be an edge $\{x, y\} \in E(P_1) \setminus \mathcal{P}^L$ such that $\{x^R, y^R\} \notin \mathcal{P}^R$ and neither $x^R$ nor $y^R$ is incident with both edges of $\mathcal{P}^R$. It follows that $\{x^R, y^R\}$ is free in $Q_n^R$ and hence we can apply the induction again to obtain a hamiltonian path $P_2$ of $Q_n^R$ between $x^R$ and $y^R$ passing through $\mathcal{P}^R$. Then $H = \langle (E(P_1) \cup E(P_2) \cup \{\{x, x^R\}, \{y, y^R\}\}) \setminus \{\{x, y\}\} \rangle$.

(2.1.3) $|\mathcal{P}^L| = 2n-3$ and $|\mathcal{P}^R| = 1$: First choose an edge $\{x, y\} \in \mathcal{P}^L$ such that $y$ is an endvertex of a path of $\langle \mathcal{P}^L \rangle$. Moreover, if $\langle \mathcal{P}^L \rangle$ contains a path starting at $u$ or $v$, then $y$ should be the other endvertex of that path. Next apply the induction to find a hamiltonian path $P_1$ of $Q_n^L$ between $u$ and $v$ passing through $\mathcal{P}^L \setminus \{\{x, y\}\}$. If $\{x, y\} \in E(P_1)$, use the same construction as in the previous case (2.1.2) to obtain the desired hamiltonian path. If this is not the case, choose on $P_1$ neighbors $r$ and $s$ of $x$ and $y$, respectively, such that

(i) $\{x, r\} \notin \mathcal{P}^L$,

(ii) the subpath of $P_1$ between $x$ and $y$ contains exactly one of $r, s$.

The way $y$ has been chosen implies that $\{y, s\} \notin \mathcal{P}^L$. The existence of a path of length three between $r$ and $s$ means that $d(r, s)$ is odd and the same has to be true about $d(r^R, s^R)$. If $\{r^R, s^R\} \notin \mathcal{P}^R$, use the induction to find a hamiltonian path $P_2$ of $Q_n^R$ between $r^R$ and $s^R$, passing through $\mathcal{P}^R$ and observe that then $H = \langle (E(P_1) \cup E(P_2) \cup \{\{x, y\}, \{r, r^R\}, \{s, s^R\}\}) \setminus \{\{x, r\}, \{y, s\}\} \rangle$.

If, however, $\mathcal{P}^R = \{\{r^R, s^R\}\}$, which means that $d(r, s) = 1$, choose an edge $\{p, q\} \in E(P_1) \setminus (\mathcal{P}^L \cup \{\{x, r\}, \{y, s\}\})$. Note that as $|E(P_1)| - (|\mathcal{P}^L \setminus \{\{x, y\}\}| + 2) = 2^n - 1 - (2n-4) - 2 > 1$ for $n \geq 5$, this is always possible. Then use the induction hypothesis to obtain a hamiltonian path $P_2$ of $Q_n^R$ between $p^R$ and $q^R$, passing through $\mathcal{P}^R$ and observe that $H = \langle (E(P_1) \cup E(P_2) \cup \{\{x, y\}, \{r, s\}, \{p, p^R\}, \{q, q^R\}\}) \setminus \{\{x, r\}, \{y, s\}, \{p, q\}\} \rangle$.

(2.1.4) $|\mathcal{P}^L| = 2n-2$ and $|\mathcal{P}^R| = 0$:

First choose an edge $\{x, y\} \in \mathcal{P}^L$ such that $y$ is an endvertex of a path of $\langle \mathcal{P}^L \rangle$. Moreover, if $\langle \mathcal{P}^L \rangle$ contains a path starting at $u$ or $v$, then $y$ should be the other endvertex of that path. We can assume that this path starts at $u$, interchanging $u$ and $v$ if necessary. Note that this implies that $v \neq x$, $v \neq y$, $u \neq y$. Next apply the induction to find in $Q_n^L$ a hamiltonian cycle $C$ passing through $\mathcal{P}^L \setminus \{\{x, y\}\}$.

(2.1.4.1) $\{x, y\} \in E(C)$: Choose on $C$ neighbors $r$ and $s$ of $u$ and $v$, respectively, such that $\{u, r\} \notin P^L$ and $\{v, s\} \notin P^L$.

(2.1.4.1.1) $\{u, v\} \in E(C)$: Choose an edge $\{a, b\} \in E(C) \setminus P^L$, apply the induction hypothesis to find a hamiltonian path $P$ of $Q_n^R$ between $a^R$ and $b^R$ and observe that $H = \langle (E(C) \cup E(P) \cup \{\{a, a^R\}, \{b, b^R\}\}) \setminus \{\{u, v\}, \{a, b\}\} \rangle$.

(2.1.4.1.2) $\{u, v\} \notin E(C)$ and each of two paths on $C$ between $u$ and $v$ contains exactly one of $r, s$: Apply the induction hypothesis to find a hamiltonian path $P$ of $Q_n^R$ between $r^R$ and $s^R$. The desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$ is then induced by edges of $(E(C) \cup E(P) \cup \{\{r, r^R\}, \{s, s^R\}\}) \setminus \{\{u, r\}, \{v, s\}\}$.

(2.1.4.1.3) $\{u, v\} \notin E(C)$ and each of two paths on $C$ between $u$ and $v$ contains both or none of $r, s$: First note that $r \neq s$, for otherwise $d(u, v) = 2$, contrary to our assumption that this distance is odd. Moreover, the assumption that $\mathcal{P}^L$ contains no path between $u$ and $v$ implies that there exists an edge $\{a, b\} \in E(C) \setminus P^L$ on the path of $C$ between $u$ and $v$ containing none of $r, s$. We may assume that $d(r, a)$ is odd, interchanging $a$ and $b$ if necessary. Then $d(s, b)$ is odd as well. Hence we can apply Lemma 5.2.6 to find in $Q_n^R$ paths $P_1$ between $r^R$ and $a^R$ and $P_2$ between $s^R$ and $b^R$ such that $V(P_1)$ and $V(P_2)$ form a partition of $V(Q_n^R)$. Then $H = \langle (E(C) \cup E(P_1) \cup E(P_2) \cup \{\{r, r^R\}, \{s, s^R\}, \{a, a^R\}, \{b, b^R\}\}) \setminus \{\{u, r\}, \{v, s\}, \{a, b\}\} \rangle$.

(2.1.4.2) $\{x, y\} \notin E(C)$

(2.1.4.2.1) $\{u, v\} \in E(C)$: Choose on $C$ neighbors $r$ and $s$ of $x$ and $y$, respectively, such that $\{x, r\} \notin P^L$ and each of two paths on $C$ between $x$ and $y$ contains exactly one of $r, s$. Moreover, if $u = x$ then choose $r$ different from $v$. Note that the way $y$ has been chosen implies that $\{y, s\} \notin P^L$. It remains to apply the induction hypothesis to find a hamiltonian path $P$ of $Q_n^R$ between $r^R$ and $s^R$ and observe that then $H = \langle (E(C) \cup E(P) \cup \{\{x, y\}, \{r, r^R\}, \{s, s^R\}\}) \setminus \{\{x, r\}, \{y, s\}, \{u, v\}\} \rangle$.

(2.1.4.2.2) $\{u, v\} \notin E(C)$ and $u = x$: Choose on $C$ neighbors $b$ and $s$ of $v$ and $y$, respectively, such that neither $\{v, b\}$ nor $\{y, s\}$ belongs to $P^L$ and each of the two paths on $C$ between $v$ and $y$ contains exactly one of $b, s$. Further choose distinct neighbors $r$ and $a$ of $u$ such that each of the two paths on $C$ between $u$ and $y$ contains exactly one of $r, v$. Note that the way $y$ has been chosen implies that neither $\{u, r\}$ nor $\{u, a\}$ belongs to $P^L$. Next apply Lemma 5.2.6 to find in $Q_n^R$ paths $P_1$ between $r^R$ and $b^R$ and $P_2$ between $s^R$ and $a^R$ whose vertex sets partition $V(Q_n^R)$ and observe that then $H = \langle (E(C) \cup E(P_1) \cup E(P_2) \cup \{\{x, y\}, \{r, r^R\}, \{s, s^R\}, \{a, a^R\}, \{b, b^R\}\}) \setminus \{\{u, a\}, \{v, b\}, \{u, r\}, \{y, s\}\} \rangle$.

(2.1.4.2.3)$\{u, v\} \notin E(C)$ and each of two paths on $C$ between $u$ and $v$ contains exactly one of $x, y$: Choose on $C$ neighbors $r$ and $s$ of $x$ and $y$, respectively, such that $\{x, r\} \notin P^L$ and each of two paths on $C$ between $x$ and $y$ contains both or none of $r, s$. Note that the way $y$ has been chosen implies that $\{y, s\} \notin P^L$ and the fact that hypercubes are triangle-free implies $r \neq s$. Further choose on $C$ neighbors $a$ and $b$ of $u$ and $v$, respectively, such that $\{u, a\} \notin P^L$ and $\{v, b\} \notin P^L$.

First consider the special case when $\{\{r, x\}, \{s, y\}\} \cap \{\{u, a\}, \{v, b\}\} \neq \emptyset$. Note that we can assume that $\{r, x\} = \{u, a\}$, interchanging $\{r, x\}$ with $\{s, y\}$ or $\{u, a\}$ with $\{v, b\}$ if necessary. Then simply apply the induction hypothesis to find a hamiltonian path $P$ of $Q_n^R$ between $s^R$ and $b^R$ and observe that $H = \langle (E(C) \cup E(P) \cup \{\{x, y\}, \{s, s^R\}, \{b, b^R\}\}) \setminus \{\{x, r\}, \{y, s\}, \{v, b\}\} \rangle$.

If the special case does not apply, use Lemma 5.2.6 to find in $Q_n^R$ paths $P_1$ between $r^R$ and $s^R$ and $P_2$ between $a^R$ and $b^R$ such that their vertex sets partition $V(Q_n^R)$ and observe that $H = \langle (E(C) \cup E(P_1) \cup E(P_2) \cup \{\{x,y\},\{r,r^R\},\{s,s^R\}, \{a,a^R\},\{b,b^R\}\}) \setminus \{\{u,a\},\{v,b\},\{x,r\},\{y,s\}\}\rangle$.

(2.1.4.2.4) $\{u,v\} \notin E(C)$ and each of two paths on $C$ between $u$ and $v$ contains both or none of $x,y$: Choose on $C$ neighbors $r$ and $s$ of $x$ and $y$, respectively, such that $\{x,r\} \notin P^L$ and each two paths on $C$ between $x$ and $y$ contains exactly one of $r,s$. Note that the way $y$ has been chosen implies that $\{y,s\} \notin P^L$. Further choose on $C$ neighbors $a$ and $b$ of $u$ and $v$, respectively, such that $\{u,a\} \notin P^L$ and $\{v,b\} \notin P^L$. If the choice can be made in such a way that each of two paths on $C$ between $u$ and $v$ contains exactly one of $a,b$, then we can use the same construction as in the previous case (2.1.4.2.3).

It remains to settle the case that $a$ and $b$ had to be chosen on one subpath of $C$ between $u$ and $v$: The way $y$ was chosen implies that $x$, $y$ must be on the other subpath of $C$ between $u$ and $v$, for otherwise $\mathcal{P}^L$ would contain a path starting at $u$, but not ending at $y$. We may assume that $d(r,a)$ is odd, interchanging $r$ and $s$ if necessary. Then $d(s,b)$ is odd as well and we can apply Lemma 5.2.6 to find in $Q_n^R$ paths $P_1$ between $r^R$ and $a^R$ and $P_2$ between $s^R$ and $b^R$ whose vertex sets partition $V(Q_n^R)$. It remains to observe that $H = \langle (E(C) \cup E(P_1) \cup E(P_2) \cup \{\{x,y\},\{r,r^R\},\{s,s^R\},\{a,a^R\},\{b,b^R\}\}) \setminus \{\{u,a\},\{v,b\},\{x,r\},\{y,s\}\}\rangle$.

(2.1.5) $|\mathcal{P}^L| = 1$ and $|\mathcal{P}^R| = 2n-3$: By the induction hypothesis $Q_n^R$ contains a hamiltonian cycle $C$ passing through $\mathcal{P}^R$. Since $|E(C)| - |\mathcal{P}^R| = 2^n - (2n-3) > 3$ for $n \geq 5$, there has to be an edge $\{x,y\} \in E(C) \setminus \mathcal{P}^R$ such that $\{x^L,y^L\} \notin \mathcal{P}^L$ and $\{x^L,y^L\} \neq \{u,v\}$. It remains to use the induction again to obtain a hamiltonian path $P$ of $Q_n^L$ between $u$ and $v$ passing through $\mathcal{P}^L \cup \{\{x^L,y^L\}\}$ and observe that $H = \langle (E(P) \cup E(C) \cup \{\{x^L,x\},\{y^L,y\}\}) \setminus \{\{x^L,y^L\}\}\rangle$.

(2.1.6) $|\mathcal{P}^L| = 0$ and $|\mathcal{P}^R| = 2n-2$:

First choose an edge $\{x,y\} \in \mathcal{P}^R$ such that $y$ is an endvertex of a path of $\mathcal{P}^R$. Next apply the induction to find in $Q_n^R$ a hamiltonian cycle $C$ passing through $\mathcal{P}^R \setminus \{\{x,y\}\}$. If $\{x,y\} \in E(C)$, use the same construction as in the previous case (2.1.5) to obtain the desired hamiltonian path of $Q_{n+1}$ between $u$ and $v$. Otherwise choose on $C$ neighbors $r$ and $s$ of $x$ and $y$, respectively, such that $\{x,r\} \notin P^R$ and each of two paths on $C$ between $x$ and $y$ contains exactly one of $r,s$. Note that the way $y$ has been chosen implies that $\{y,s\} \notin \mathcal{P}^R$.

(2.1.6.1) $|\{u,v\} \cap \{r^L,s^L\}| = 0$: We may assume that $d(r^L,u)$ is odd, interchanging $r$ and $s$ if necessary. Then $d(s^L,v)$ must be odd as well and we can apply Lemma 5.2.6 to obtain paths $P_1$ between $u$ and $r^L$ and $P_2$ between $v$ and $s^L$, whose vertex sets partition $V(Q_n^L)$. It remains to observe that $H = \langle (E(P_1) \cup E(P_2) \cup E(C) \cup \{\{r^L,r\},\{s^L,s\},\{x,y\}\}) \setminus \{\{x,r\},\{y,s\}\}\rangle$.

(2.1.6.2) $|\{u,v\} \cap \{r^L,s^L\}| = 1$: Due to the symmetry we can assume that $u = r^L$. Since both $d(u,v)$ and $d(r^L,s^L)$ are odd, $d(s^L,v)$ must be even and hence we can apply Lemma 5.2.7 to find a hamiltonian path $P$ of $Q_n^L - \{u\}$ between $s^L$ and $v$. Then $H = \langle (E(C) \cup E(P) \cup \{\{s,s^L\},\{u,r\},\{x,y\}\}) \setminus \{\{x,r\},\{y,s\}\}\rangle$.

(2.1.6.3) $|\{u,v\} \cap \{r^L,s^L\}| = 2$: Choose an edge $\{a,b\} \in E(C) \setminus P^R$ such that $\{a,b\} \cap \{r,s\} = \emptyset$ and apply Lemma 5.2.8 to find a hamiltonian cycle $C'$ of $Q_n^L - \{u,v\}$ passing through edge $\{a^L,b^L\}$. Then $H = \langle (E(C) \cup E(C') \cup \{\{u,u^R\}, \{v,v^R\}, \{a,a^L\}, \{b,b^L\}\}) \setminus \{\{x,r\}, \{y,s\}, \{a,b\}, \{a^L,b^L\}\}\rangle$.

Subcase (2.2): $v \in V(Q_n^R)$.

(2.2.1) $|\mathcal{P}^L| \leq 2n-4$ and $|\mathcal{P}^R| \leq 2n-4$: Lemma 5.2.2 guarantees the existence of a vertex $x \in V(Q_n^L)$ such that $PC(Q_n^L, \mathcal{P}^L, u, x)$ and $PC(Q_n^R, \mathcal{P}^R, x^R, v)$ hold. It remains to apply the induction hypothesis to obtain hamiltonian paths $P_1$ of $Q_n^L$ between $u$ and $x$ and $P_2$ of $Q_n^R$ between $x^R$ and $v$, passing through $\mathcal{P}^L$ and $\mathcal{P}^R$, respectively. The desired hamiltonian path $H$ is a concatenation of $P_1$ and $P_2$.

(2.2.2) $|\mathcal{P}^L| = 2n-3$ and $|\mathcal{P}^R| = 1$: By the induction hypothesis $Q_n^L$ contains a hamiltonian cycle $C$ passing through $\mathcal{P}^L$. Since $u$ may be incident with at most one edge of $\mathcal{P}$, there has to be a neighbor $x$ of $u$ on $C$ such that $\{u, x\} \notin \mathcal{P}^L$. First assume that there is not a path in $\mathcal{P}^R$ between $x^R$ and $v$. Since the fact that $d(u, x^R) = 2$ implies that $d(x^R, v)$ has the same parity as $d(u, v)$, i. e. odd, we can safely apply the induction hypothesis to obtain a hamiltonian path $P$ of $Q_n^R$ between $x^R$ and $v$, passing through $\mathcal{P}^R$ and observe that $H = \langle (E(C) \cup E(P) \cup \{\{x, x^R\}\}) \setminus \{\{u, x\}\} \rangle$.

Otherwise it must be the case that $\mathcal{P}^R = \{\{x^R, v\}\}$. Then choose an arbitrary edge $\{r, s\} \in E(C) \setminus \mathcal{P}^L$ such that $\{r, s\} \cap \{x, v^L\} = \emptyset$ and apply Lemma 5.2.6 to obtain a hamiltonian path $P$ of $Q_n^R - \{x^R, v\}$ between $r^R$ and $s^R$. Now observe that $H = \langle (E(C) \cup E(P) \cup \{\{x, x^R\}, \{x^R, v\}, \{r, r^R\}, \{s, s^R\}\}) \setminus \{\{u, x\}, \{r, s\}\} \rangle$.

(2.2.3) $|\mathcal{P}^L| = 2n-2$ and $|\mathcal{P}^R| = 0$: First choose an edge $\{x, y\} \in \mathcal{P}^L$ such that $y$ is an endvertex of a path of $\mathcal{P}^L$. Moreover, if $\mathcal{P}^L$ contains a path starting at $u$, then $y$ should be the other endvertex of that path. Next apply the induction to find a hamiltonian cycle $C$ of $Q_n^L$ passing through $\mathcal{P}^L \setminus \{\{x, y\}\}$. If $\{x, y\} \in E(C)$, use the construction of the previous case (2.2.2) to obtain the desired hamiltonian path. Otherwise choose on $C$ neighbors $r, s$ and $a$ of $x, y$ and $u$, respectively, such that $a \neq r$, $\{u, a\} \notin P^L$, $\{x, r\} \notin P^L$ and each of the two paths on $C$ between $x$ and $y$ contains exactly one of $r, s$. Moreover, choose $r \neq v^L$ and $s \neq v^L$, if possible. Note that the way $y$ has been chosen implies that $\{y, s\} \notin P^L$.

(2.2.3.1) $u = r$ or $u = s$: Due to the symmetry we can assume that $u = r$. Note that then $d(u, s)$ is odd, which means that $d(s^R, v)$ must be odd as well. Hence we can simply apply the induction hypothesis to find a hamiltonian path $P$ of $Q_n^R$ between $s^R$ and $v$ and observe that $H = \langle (E(C) \cup E(P) \cup \{\{x, y\}, \{s, s^R\}\}) \setminus \{\{x, r\}, \{y, s\}\} \rangle$.

(2.2.3.2) $a = s$: Let $b$ denote the neighbor of $y$ on $C$, different from $s$. Note that then $d(u, b)$ is odd and therefore $d(b^R, v)$ has the same parity as $d(u, v)$, which is odd by our assumption. Hence we can apply Lemma 5.2.6 to find in $Q_n^R$ paths $P_1$ between $r^R$ and $s^R$ and $P_2$ between $b^R$ and $v$ whose vertex sets partition $V(Q_n^R)$. Then $H = \langle (E(C) \cup E(P_1) \cup E(P_2) \cup \{\{r, r^R\}, \{s, s^R\}, \{b, b^R\}, \{x, y\}\}) \setminus \{\{x, r\}, \{y, b\}, \{u, a\}\} \rangle$.

(2.2.3.3) $r = v^L$: The conditions used for the choice of $r$ imply that in this case there must be a path on $C$ between $u$ and $x$ containing only edges of $\mathcal{P}$. Let $b$ be the neighbor of $y$ on $C$, different from $s$. Note that the existence of a path of length three between $b$ and $r$ means that $d(b, r) = d(b^R, v)$ is odd. Moreover, the fact that $d(u, a^R) = 2$ implies that $d(a^R, v)$ must have the same parity as $d(u, v)$, i. e. odd. Hence we can apply Lemma 5.2.7 to find a hamiltonian path $P$ of $Q_n^R - \{v\}$ between $a^R$ and $b^R$ observe that $H = \langle (E(C) \cup E(P) \cup \{\{a, a^R\}, \{b, b^R\}, \{r, v\}, \{x, y\}\}) \setminus \{\{x, r\}, \{y, b\}, \{u, a\}\} \rangle$.

(2.2.3.4) $s = v^L$: Since case $u = s$ was settled previously in (2.2.3.1), we can also assume that $u \neq s$. First note that the conditions used for the choice of $s$ imply that in this case there must be a path on $C$ between $u$ and $x$ containing only edges of $\mathcal{P}$. Next observe that the existence of a path of length three between $r$ and $s$ means that $d(r,s) = d(r^R, v)$ is odd. Since $d(a^R, v)$ is odd as well for the same reasons as in the previous case, we can apply Lemma 5.2.7 to find a hamiltonian path $P$ of $Q_n^R - \{v\}$ between $a^R$ and $r^R$. It remains to observe that $H = \langle (E(C) \cup E(P) \cup \{\{a,a^R\}, \{r,r^R\}, \{s,v\}, \{x,y\}\}) \setminus \{\{x,r\}, \{y,s\}, \{u,a\}\}\rangle$.

(2.2.3.5) $\{r,s\} \cap \{u, v^L\} = \emptyset$ and $a \neq s$: Apply Lemma 5.2.6 to find in $Q_n^R$ paths $P_1$ between $r^R$ and $s^R$ and $P_2$ between $a^R$ and $v$ whose vertex sets partition $V(Q_n^R)$ and observe $H = \langle (E(C) \cup E(P_1) \cup E(P_2) \cup \{\{r,r^R\}, \{s,s^R\}, \{a,a^R\},\ \{x,y\}\}) \setminus \{\{x,r\}, \{y,s\}, \{u,a\}\}\rangle$. □

Note that the presentation of the proof provides a description of a recursive algorithm, which, given a set of prescribed edges and two vertices of $Q_n$, satisfying the assumptions of Theorem 5.3.1, constructs the desired hamiltonian path.

As an easy corollary we obtain a similar result on hamiltonian cycles with prescribed edges, originally proved in [15]:

**Corollary 5.3.2.** *Let $n \geq 2$ and $\mathcal{P} \subseteq E(Q_n)$ be such that $|\mathcal{P}| \leq 2n-3$. Then $Q_n$ contains a hamiltonian cycle passing through $\mathcal{P}$ if and only if $\langle \mathcal{P} \rangle$ consists of pairwise vertex-disjoint paths.*

*Proof.* Choose $\{u,v\} \in \mathcal{P}$ and apply Theorem 5.3.1 to find a hamiltonian path of $Q_n$ between $u$ and $v$ passing through $\mathcal{P} \setminus \{\{u,v\}\}$. □

## 5.4 Hamiltonian paths avoiding forbidden edges

The existence of hamiltonian paths passing through prescribed edges is obviously related to the problem of hamiltonian paths avoiding forbidden edges. Indeed, if two prescribed edges are incident with the same vertex $x$, any path passing through them must avoid all the remaining edges incident with $x$. With this idea it is easy to show that Theorem 5.3.1 implies some classical results on edge fault tolerance of hypercubes with respect to hamiltonian paths and cycles.

**Corollary 5.4.1** ([34])**.** *Let $n \geq 2$, $\mathcal{F} \subseteq E(Q_n)$ and $u, v \in V(Q_n)$ be such that $|\mathcal{F}| \leq n-2$ and $d(u,v)$ is odd. Then there exists a hamiltonian path of $Q_n$ between $u$ and $v$ avoiding $\mathcal{F}$.*

*Proof.* Let us call the edges of $\mathcal{F}$ *forbidden*. Let $V_1$, $V_2$ be the partite sets of $Q_n$ and $A$ be the set of endvertices of forbidden edges in $V_1$. Assume that $u \in V_1$ and $v \in V_2$. In the construction described below, we will find for each $x \in A$ two distinct edges $e_1^x, e_2^x \in E(Q_n) \setminus \mathcal{F}$ incident with $x$ such that $\langle \bigcup_{x \in A} \{e_1^x, e_2^x\} \rangle$ consists of pairwise vertex-disjoint paths. Given that, put $\mathcal{P} = \{e_1^x, e_2^x \mid x \in A\}$, but in case $u \in A$ remove one of $e_1^u$, $e_2^u$ from $\mathcal{P}$ (choosing $\{u,v\}$ if possible). Then $|\mathcal{P}| \leq 2|A| \leq 2|F| \leq 2n-4$ and so by Theorem 5.3.1 there is a hamiltonian path passing through $\mathcal{P}$ and, therefore, avoiding all forbidden edges. The special cases for $n \in \{3,4\}$ cannot occur in our construction.

For $x \in A$, let $Z_x$ be the set of dimensions of all but one forbidden edges incident with $x$. It is irrelevant which forbidden edge we exclude, let it be the one with the least dimension. If $x$ is incident with only one forbidden edge then $Z_x$ is empty. Set $Z = \bigcup_{x \in A} Z_x$ and observe that $|A| \leq |F| - |Z|$.

Now choose distinct dimensions $a, b \in \{1, 2, \ldots, n\} \setminus Z$ such that there is no 4-cycle of edges with dimensions $a$ and $b$ containing two vertices from $A$. Such $a$ and $b$ exist since there are $\binom{n-|Z|}{2}$ pairs of dimensions not in $Z$, $\binom{|A|}{2} \leq \binom{|F|-|Z|}{2} \leq \binom{n-2-|Z|}{2}$ pairs of vertices in $A$, and each pair of vertices can disable at most one pair of dimensions. In the first step, for $x \in A$, let $e_1^x$ and $e_2^x$ be the edges of dimensions $a$ and $b$ incident with $x$. The paths induced by $\{e_1^x, e_2^x\}$ must be pairwise vertex-disjoint for all $x \in A$, for otherwise they would form a 4-cycle. However, we are not finished yet, since one edge of $e_1^x$, $e_2^x$ can be forbidden.

In the second step, iteratively consider all $x \in A$ such that $e_1^x$ or $e_2^x$ is forbidden, say it is $e_1^x$. For a new $e_1^x$, choose an edge that is not incident with $e_1^y$, $e_2^y$ for all $y \in A \setminus \{x\}$. Such an edge exists since there are $|A| - 1 \leq n - 3 - |Z|$ vertices in $A$ different from $x$, $n - 2 - |Z|$ edges of dimension not in $\{a, b\} \cup Z$, and every vertex can disable at most one edge of dimension not in $\{a, b\} \cup Z$. Now the desired paths are induced by edges $\{e_1^x, e_2^x\}$ for all $x \in A$. $\square$

A similar well-known result for hamiltonian cycles follows directly from Corollary 5.4.1, but it can be also proved from Corollary 5.3.2, using exactly the same argument as for hamiltonian paths.

**Corollary 5.4.2** ([9])**.** *For $n \geq 2$ and $\mathcal{F} \subseteq E(Q_n)$ such that $|\mathcal{F}| \leq n - 2$ there exists a hamiltonian cycle of $Q_n$ avoiding $\mathcal{F}$.*

Both these results are optimal since $n - 1$ forbidden edges incident with the same vertex $x$ disable any hamiltonian cycle or path not starting at $x$.

We conclude this chapter with an open problem. The connection between hamiltonicity of hypercubes with forbidden and prescribed edges, explored in the previous corollaries, suggests that there may be a relation between the complexity of both problems. The problem of existence of a hamiltonian path of $Q_n$ between $u$ and $v$ avoiding a given set of forbidden edges is known to be NP-complete ([7], even in case that $u$ and $v$ are adjacent). Does a similar result hold for the variant with prescribed edges?

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