**FACULTY
OF MATHEMATICS
AND PHYSICS**
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# HABILITATION THESIS

Petr Gregor

# Combinatorial Structures in Hypercubes

Computer Science - Theoretical Computer Science

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# Contents

# Synopsis of the thesis

The thesis is compiled as a collection of 12 selected publications on various combinatorial structures in hypercubes accompanied with a commentary in the introduction. In these publications from years between 2012 and 2018 we solve, in some cases at least partially, several open problems or we significantly improve previously known results. The list of publications follows after the synopsis.

The thesis is organized into 8 chapters. Chapter 1 is an umbrella introduction that contains background, motivation, and summary of the most interesting results.

Chapter 2 studies queue layouts of hypercubes. A queue layout is a linear ordering of vertices together with a partition of edges into sets, called queues, such that in each set no two edges are nested with respect to the ordering. The results in this chapter significantly improve previously known upper and lower bounds on the queue-number of hypercubes associated with these layouts. The improvements are achieved by a novel construction for the upper bound and a new technique of out-in representations and contractions for the lower bound.

Chapter 3 studies simultaneous broadcasting of multiple messages in synchronous networks under certain communication model. First we consider the case when each node may receive and send at most one message in each step, which leads to the concept of mutually independent Hamiltonian paths and cycles. For hypercubes we improve previously known results on the possible number of mutually independent Hamiltonian paths and cycles in (faulty) hypercubes. Then we introduce a concept of level-disjoint partitions for the general communication model and we present a structural characterization of all graphs that admit two level-disjoint partitions with a given root. Finally we show that hypercubes, as well as many other graph classes, have optimal number of level-disjoint partitions of optimal height, which affirmatively answers a conjecture from [77].

Chapter 4 studies incidence colorings of graphs. In this type of coloring, colors are assigned to incidences between vertices and edges so that every two adjacent incidences receive distinct colors. For a precise definition of *adjacent incidences* see Section 1.5 in the introduction. We provide a sufficient condition for a Cartesian product of two graphs to have incidence chromatic number at most the maximal degree plus 2. Applying this result we confirm a conjecture of Pai et al. [141] on the exact value of incidence chromatic number of hypercubes. Furthermore, we show that every graph of degree at most 4 has an incidence coloring with at most 7 colors, which improves the previous upper bound of 8 colors.

Chapter 5 studies distance magic labelings of hypercubes. In this type of labeling, vertices of a $k$-vertex graph are bijectively labeled by integers from 1 to $k$ so that the sum of labels on neighbors of each vertex is always the same. We show that the hypercube $Q_n$ has a distance magic labeling for every $n \equiv 2 \pmod 4$, which completely resolves a conjecture of Acharya et al. [1].

Chapter 6 studies parity vertex colorings of trees, in particularly of binomial trees, which are spanning trees of hypercubes. In this type of coloring, colors are assigned to vertices so that each path contains some color with odd number of occurrences on this

path. We show that the binomial tree $B_n$ has a parity vertex coloring with at most $\left\lceil \frac{2n+3}{3} \right\rceil$ colors, which disproves a conjecture of Borowiecki et al. [14] on relation between parity vertex colorings and vertex rankings of trees.

Chapter 7 studies several Gray code type problems. A Gray code for some class of combinatorial objects is an enumeration of its elements such that consecutive elements differ only in a constant "amount". These codes ultimately lead to loopless generating algorithms which allow to generate each new object from the previous object in constant time. A prime example is the binary reflected Gray code for cyclic enumeration of all subsets of an $n$-element set by adding or removing a single element, which corresponds to a Hamiltonian cycle in $Q_n$.

First we consider the problem of generating all subsets of an $n$-element set with size in some fixed interval $[k, l]$ where $0 \le k \le l \le n$ by adding or removing a single element, or by exchanging a single element if necessary. This is a common generalization of the binary reflected Gray code, the well-known middle levels problem, and a Gray code for all $k$-element subsets of an $n$-set.

For this problem we show that the subgraph of the hypercube $Q_n$ induced by levels between $k$ and $l$ has a saturating cycle, as well as a tight enumeration, up to the cases covered by the generalized middle level conjecture which are still open. Both a saturating cycle and a tight enumeration are in a sense optimal Gray codes for this problem.

Then we consider a generalized middle level conjecture [148, 81] asserting that the subgraph of $Q_n$ induced by middle $2l$ levels is Hamiltonian for any $l \le n + 1$. We confirm this conjecture for middle 4 levels and for the remaining open cases we find at least a cycle factor built from two edge-disjoint symmetric chain decompositions of $Q_n$. Furthermore, we show that $Q_n$ has four pairwise edge-disjoint symmetric chain decompositions for any $n$ large enough.

Finally, we consider the problem of finding an almost Hamiltonian cycle in hypercubes with faulty vertices. We prove a conjecture of Castañeda and Gotchev [23] asserting that for any set $F$ of at most $\binom{n}{2} - 2$ vertices in $Q_n$ there is a cycle of length at least $2^n - 2|F|$ in $Q_n - F$. This quadratic number of tolerable vertices is tight for such a cycle. Previous results were only linear in the number of tolerable vertices. Similar results are obtained also for long paths between prescribed endvertices.

Chapter 8 studies linear extension diameter of certain subposets of the Boolean lattice. A linear extension diameter of a given poset is the diameter of the graph on all linear extensions of the poset as vertices, with edges between any two extensions that differ in a single adjacent transposition. We determine the linear extension diameter of the subposets of the Boolean lattice $\mathcal{B}_n$ induced by the 1st and $k$th levels for any $1 < k \le n$. This partially answers a question of Felsner and Massow [56]. We also describe all diametral pairs of extensions.

# Acknowledgements

# List of publications in the thesis

Chapter 2 - Queue layouts

[76] P. Gregor, R. Škrekovski, and V. Vukašinović, *Queue layouts of hypercubes*, SIAM J. Discrete Math 26 (2012), 77–88.

Chapter 3 - Level-disjoint partitions

[175] V. Vukašinović, P. Gregor, and R. Škrekovski, *On the mutually independent hamiltonian cycles in faulty hypercubes*, Inform. Sciences 236 (2013), 224–235.

[78] P. Gregor, R. Škrekovski, and V. Vukašinović, *Modelling simultaneous broadcasting by level-disjoint partitions*, Appl. Math. Comput. 325 (2018), 15–23.

[79] P. Gregor, R. Škrekovski, and V. Vukašinović, *Broadcasting multiple messages in the 1-in port model in optimal time*, J. Combin. Optim. 36 (2018), 1333–1355.

Chapter 4 - Incident colorings

[71] P. Gregor, B. Lužar, and R. Soták, *On incidence coloring conjecture in Cartesian products of graphs*, Discrete Applied Math 213 (2016), 93–100.

[72] P. Gregor, B. Lužar, and R. Soták, *Note on incidence chromatic number of subquartic graphs*, J. Combin. Optim. 34 (2017), 174–181.

Chapter 5 - Distance magic labelings

[73] P. Gregor and P. Kovář, *Distance magic labelings of hypercubes*, Proc. of Combinatorics 2012, Elect. Notes in Disc. Math. 40 (2013), 145–149.

Chapter 6 - Parity vertex colorings

[80] P. Gregor and R. Škrekovski, *Parity vertex coloring of binomial trees*, Discuss. Math. Graph Theory 32 (2012), 177–180.

Chapter 7 - Gray codes

[74] P. Gregor and T. Mütze, *Trimming and gluing Gray codes*, Theor. Comp. Sci. 714 (2018), 74–95.

[70] P. Gregor, S. Jager, T. Mütze, J. Savada, and K. Wille, *Gray codes and symmetric chains*, Proc. 45th Int. Colloq. on Autom., Lang., and Program. (ICALP 2018), LIPIcs, vol. 107 (2018), 66:1–66.14. *Full version available at arXiv:1802.06021.*

[60] J. Fink and P. Gregor, *Long cycles in hypercubes with optimal number of faulty vertices*, J. Combin. Optim. 24 (2012), 240–265.

Chapter 8 - Linear extension diameter

[61] J. Fink and P. Gregor, *Linear extension diameter of level induced subposets of the boolean lattice*, European J. Combin. 35 (2014), 221–231.

# Chapter 1

# Introduction

The possibility to encode all finitary objects (and discretely approximate the continuous ones) into finite sequences of 0's and 1's allows us to describe and model our world by computers and lies in the heart of computer science. No matter what information or a high level structure is encoded by binary strings, for study of low level complexity of our algorithms as the cost of our operations performed on binary strings we measure some edit distance between them.

The simplest metric to compare binary strings $u$ and $v$ of the same length is the number of bits in which they differ, called a *Hamming distance* and denoted by $d_H(u, v)$. This distance measures the write complexity in analysis of data structures in the bit probe model, the error in transmission of messages via a noisy channel, distance in genomics, and various other things depending on the context. Numerous problems across wide areas of computer science include the task to minimize the Hamming distance between considered strings.

## 1.1 Hypercubes

In a graph theory setting, these problems can often be formulated as graph problems on hypercubes. This is motivated by a prolific research and development of graph theory over the past decades.

Formally, the *hypercube of dimension $n$*, denoted by $Q_n$, is an undirected graph on all binary strings of length $n$ as vertices, and edges joining vertices with Hamming distance equal to 1. That is, the vertex set is $V(Q_n) = \mathbb{Z}_2^n = \{0, 1\}^n$ and the edge set is

$$E(Q_n) = \{uv \mid d_H(u, v) = 1\} = \{uv \mid u \oplus v = e_i \text{ for some } i \in [n]\}$$

where $e_i$ denotes the vector with 1 exactly in the $i$th coordinate. It is useful to define also the *hypercube of dimension* 0 as an isolated vertex corresponding to the empty string. Note that this is a graph theoretical rather than geometric notion and in different settings the hypercube is called a *discrete cube*, a *Boolean cube*, or shortly an *n-cube*.
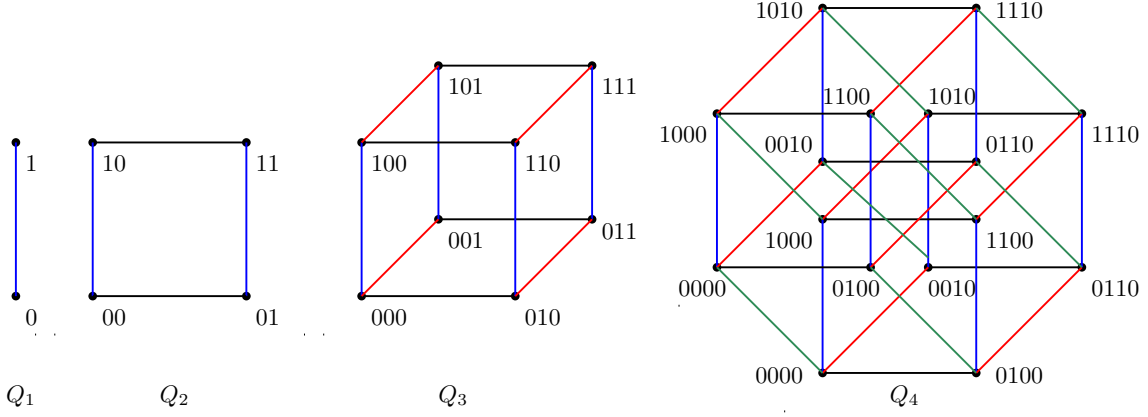
Figure 1.1: Hypercubes of dimensions $n = 1, 2, 3, 4$.

Hypercubes can be equivalently defined in several alternative ways:

- $Q_n$ is the $n$-fold Cartesian product of the complete graph $K_2$ on two vertices, i.e.

$$Q_n = K_2^n = \underbrace{K_2 \,\square\, K_2 \,\square\, \cdots \,\square\, K_2}_{n-\text{times}} .$$

  The *Cartesian product* $G \,\square\, H$ of graphs $G$ and $H$ is the graph with the vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and the edge set

$$E(G \,\square\, H) = \{(u, v)(u', v) \mid uu' \in E(G)\} \cup \{(u, v)(u, v') \mid vv' \in E(H)\}.$$

- $Q_n$ is the covering graph of the *Boolean lattice* $\mathcal{B}_n = (\mathcal{P}(X), \subseteq)$ with $|X| = n$, i.e. the poset of all subsets over an $n$-element set $X$ ordered by inclusion. The *covering graph* of a poset is the graph of its Hasse diagram.

- $Q_n$ is the Cayley graph of $\mathbb{Z}_2^n$ generated by the standard basis; that is

$$Q_n = \mathrm{Cay}(\mathbb{Z}_2^n, \{e_1, \ldots, e_n\}).$$

  A *Cayley graph* of a group $\Gamma$ generated by a set $S \subseteq \Gamma$ that is closed under inverses and does not contain the neutral element of $\Gamma$ is $\mathrm{Cay}(\Gamma, S) = (\Gamma, \{uv \mid vu^{-1} \in S\})$. Note that the condition of $S$ being closed under inverses ensures that the graph is undirected and the condition of not containing the neutral element of $\Gamma$ ensures that the graph has no loops.

- $Q_n$ is the 1-skeleton of the polytope $[0, 1]^n$. The vertices (edges) of $Q_n$ are 0-faces (1-faces, resp.) of $[0, 1]^n$.

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Furthermore, there are about 20 equivalent characterizations of hypercubes, see a survey [69], which illustrates their versatility.

Hypercubes posses many elegant properties. The hypercube of dimension $n$ has a (regular) degree $n$, diameter $n$, radius $n$, both vertex and edge connectivity $n$. Hypercubes are bipartite with partite classes formed by vertices of even and odd weight, respectively, where the *weight* of a vertex $u$ is $|u| = d_H(u, \mathbf{0})$, i.e. the number of 1's in $u$. They have a so called $(0, 2)$-*property*; that is, every two vertices have either 0 or 2 common neighbors, which implies that they are $K_{2,3}$-free.

Hypercubes are highly symmetrical, each automorphism of $Q_n$ is a composition of a unique permutation of coordinates and a unique translation, and the automorphism group of $Q_n$ is the octahedral group $Aut(Q_n) \simeq S_n \ltimes \mathbb{Z}_2^n$. They are not only vertex transitive, but also edge transitive and moreover distance transitive; that is, every ordered pair of vertices can be mapped by some automorphism to any ordered pair of vertices of the same distance.

Hypercubes have a recursive structure in the sense that $Q_n$ is composed of two copies of $Q_{n-1}$ joined by a perfect matching. Moreover, this decomposition can be chosen in $n$ possible ways corresponding to each coordinate. Further properties of hypercubes can be found in a survey [89].

### 1.1.1 Hypercubes in various areas of computer science

Hypercubes naturally occur in study of many problems from different areas of computer science. Here we provide only three examples and their motivation: from coding theory, extremal set theory, and study of Boolean functions.

In coding theory, one of the key questions is how many bits are needed to safely transfer a message consisting of $k$ bits over a noisy channel that can flip at most $d$ arbitrary bits. This includes also a question how to encode the messages. These are central questions in the design of error-correcting codes, which allowed development of space flights, DVD's, solid-state drives, and many other real-world applications. In terms of hypercubes, this question can be equivalently formulated as finding the smallest dimension $n$ of the hypercube $Q_n$ that admits packing of $2^k$ disjoint balls of radius $d$. A *ball of radius $d$* centered at a vertex $u$ in $Q_n$ is $B(u) = \{v \in V(Q_n) \mid d_H(u, v) \leq d\}$. The centers of the packed balls then correspond to the desired error-correcting code. More on this question can be found in any coding theory book, e.g. [124].

In extremal set theory, one of important problems is a design of so called covering arrays. A *covering array* $n \times k$ of strength $t$ is a binary matrix $n \times k$ such that its projection into any $t$ columns contains all $2^t$ possible rows. Let $can(k, t)$ be the minimal number of rows in a covering array of strength $t$ with $k$ columns. This has the following application in software testing. Assume that we have a software with $k$ binary inputs and we would like to verify that no combination of $t$ inputs causes an error. Then $can(k, t)$ is the minimal number of test runs of the software that we need to perform, each row of the corresponding covering array specifying the inputs for one test run.

This concept can be equivalently captured in terms of transversal sets, or Turán numbers of hypercubes. A set $S \subseteq V(Q_k)$ is an *s-face transversal* if any $s$-face (i.e. an $s$-dimensional subcube) of $Q_k$ contains an element of $S$. Let $tr(k, s)$ be the minimal size of an $s$-face transversal in $Q_k$. The *vertex Turán number* of a graph $H$ in a graph $G$ is the maximal number $ex_v(G, H)$ of vertices in an induced subgraph of $G$ not containing $H$. Observe that for any $0 \leq s \leq k$, it holds

$$can(k, k - s) = tr(k, s) = 2^k - ex_v(Q_k, Q_s).$$

Hence these problems are indeed equivalent. For a survey on covering arrays we refer to [120].

In study of Boolean functions, restricting propositional formulas that represent Boolean functions gives interesting classes of functions. One of the well-understood classes are the functions represented by 2-CNF formulas, i.e. formulas in conjunctive normal form with each clause containing at most 2 literals. This is a notorious example of class for which the satisfiability problem is solvable in polynomial (even linear) time. It is less known that 2-CNF formulas without equivalent variables correspond to median graphs and retractions of hypercubes as explained below. Two variables of a 2-CNF formula are said to be *equivalent* if they both are non-trivial and in each satisfying assignment of the formula, one variable determines the other. A variable of a propositional formula is *trivial* if it has the same value in all satisfying assignments.

A graph $G$ is a *median graph* if for every three vertices $x, y, z$ there is a unique vertex $m$, called a *median*, such that $m \in I(x, y) \cap I(y, z) \cap I(x, z)$ where $I(u, v)$ is the set of vertices on shortest paths between $u$ and $v$, called the *interval between* $u$ and $v$, i.e. $I(u, v) = \{w \in V(G) \mid d(u, v) = d(u, w) + d(w, v)\}$. This interesting class of graphs can be informally thought of as graphs between trees and hypercubes [128] and appears in study of stable matchings in the roommate problem, configurations of non-expansive networks, or chemical graph theory [29, 62, 86].

A *retraction* of a graph $G$ is an edge-preserving map $f \colon V(G) \to V(G)$ with $f(f(u)) = f(u)$ for every $u \in V(G)$, i.e. it is an idempotent endomorphism of $G$. The image of a retraction induces a so called *retract*. It can be shown that (the satisfying assignments of) any 2-CNF formula without equivalent variables induces in the hypercube a median subgraph, and vice versa, any median graph is an subgraph of hypercube induced by some 2-CNF formula [62]. Furthermore, any retract of the hypercube is a median graph, and vice versa, any median graph except the isolated vertex, is a retract of a hypercube [6].

These connections illustrate the interplay of hypercubes between different areas.

## 1.1.2 Hypercube architecture of interconnection networks

The elegant properties of hypercubes attracted designers to use hypercubes as an underlying topology of early parallel computers. In this topology, processors are represented by vertices and links between the processors are represented by edges. This made hypercubes one of the most popular and well-studied architectures in the advent of parallel comput-

ing [106, 121, 33]. Here is a brief excerpt from history of interconnection networks that involved hypercubes:

**1983-87** Cosmic Cube – Caltech ($n = 2, 6, 7$)

**1983-87** Connection Machine CM-1, CM-2, CM-200 – MIT[1] ($n = 16, 9, 13$)

**1985-90** Intel iPSC/1, iPSC/2, iPSC/860 ($n = 7$)

**1986-89** nCUBE-1, nCUBE-2 – nCUBE Corporation ($n = 10, 13$)

**1980's** other manufacturers: Floating Point Corporation (T series), Ametek

**1997** SGI Origin 2000 – partly involves hypercubes [33]

**2002** HyperCuP – p2p networks [151]

**2006** BlueCube – Bluetooth networks [25]

**2011** HyperD – dynamic distributed databases [172]

One of the major concerns in the design of interconnection networks is their robustness, i.e. tolerance to occurrence of faults. Failures can happen in hardware, software or even because of lost transmitted messages. Furthermore, the part of the network that is currently overloaded can be considered as faulty by other tasks. Hence the concept of fault tolerance can be applied in a broader sense than for actual failures.

Processor failures and connection failures in interconnection network correspond to *faulty vertices and faulty edges*, respectively, in the underlying graph. It is important that network stays functional even if multiple failures appear. This motivates the study of robustness of hypercubes with respect to the maximal number of arbitrarily chosen faulty vertices and/or edges.

### 1.1.3 Preliminaries

Let us introduce some notation and definitions used throughout the thesis. By $n$ we denote a positive integer and by $[n]$ we denote the set $\{1, 2, \ldots, n\}$. A path in the graph $G$ is a sequence $P = (v_1, v_2, \ldots, v_k)$ of distinct vertices such that every two consecutive vertices are adjacent. For a path $P = (v_1, v_2, \ldots, v_k)$ we say that $v_1$ and $v_k$ are the *endvertices* of $P$, and that $P$ is a $v_1v_k$-path, which is denoted by $P[v_1, v_k]$. A path in $G$ is *Hamiltonian* if it contains all vertices of $G$. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph $G$, respectively.

An open neighborhood of a vertex $u$ in a graph $G$ is denoted by $N_G(u)$, the degree of $u$ by $\deg_G(u)$, the distance between vertices $u$ and $v$ by $d_G(u, v)$. The *eccentricity* of a vertex

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[1]Richard Feynman, an American theoretical physicist was involved in its design [34].

$u$, i.e. the maximal distance from $u$ to other vertices, is denoted by $ecc_G(u)$. The subscript $G$ is omitted whenever the graph is clear from context.

A cycle is a sequence $C = (v_1, v_2, \ldots, v_k)$ of $k \geq 3$ distinct vertices such that every two consecutive vertices, including the first and the last vertex of the sequence are adjacent. We say that the cycle $C = (v_1, v_2, \ldots, v_k)$ is $v_1$-*starting* to emphasize the first vertex $v_1$ and we denote it by $C[v_1]$. A cycle $C$ in a graph $G$ is *Hamiltonian* if it contains all vertices of $G$.

Let $v$ be a vertex of a connected graph $G$. We denote by $G - v$ the graph obtained by removing $v$ and all incident edges from $G$. If $G - v$ is disconnected, the vertex $v$ is called a *cut-vertex*. A *bridge* of $G$ is an edge whose removal disconnects $G$. A maximal subgraph without a cut-vertex is called a *block*. Clearly, every block is 2-connected, formed by a bridge, or an isolated vertex.

A subgraph $H$ of a graph $G$ is *isometric* if it preserves distances from $G$; that is, $d_H(u,v) = d_G(u,v)$ for all $u, v \in V(H)$. For other standard graph theoretical terminology we refer to [38].

A *partially ordered set* (a *poset*) $\mathcal{P}$ is a set $P$ equipped with a binary relation on $P$ that is reflexive, antisymmetric, and transitive. An *antichain* is a poset with empty relation. A *subposet* $\mathcal{S}$ of a poset $\mathcal{P}$ is the poset induced on the set $S \subseteq P$ by the relation of $\mathcal{P}$. Two elements of a poset $\mathcal{P}$ are *comparable* if they are in the relation of $\mathcal{P}$, otherwise they are *incomparable*.

## 1.2 Queue layouts

There are many combinatorial optimization problems that can be formulated as graph layout problems. They include problems on VLSI circuit layout, network reliability, graph drawing, information retrieval, numerical analysis, parallel and distributed processing, embeddings. For detailed references and other applications we refer to a survey [37].

The first combinatorial structures that we consider in this thesis are queue layouts. A queue layout of a graph is a particular example of a linear layout together with a partition of its edges into sets such that in each set no two edges are nested. More formally, let $\sigma$ be a linear ordering of vertices in a graph $G$. Two edges $uv, xy \in E(G)$ are *nested* (with respect to $\sigma$) if $\sigma(u) < \sigma(x) < \sigma(y) < \sigma(v)$, see Figure 1.2. A set $S \subseteq E(G)$ is a *queue* if no two of its edges are nested with respect to $\sigma$. A *k-queue layout* of the graph $G$ is a pair of a linear ordering $\sigma$ of $V(G)$ and a partition of $E(G)$ into $k$ queues. The queue-number $\mathsf{qn}(G)$ of the graph $G$ is the minimum $k$ such that $G$ has a $k$-queue layout. A graph $G$ is a *k-queue graph* if $\mathsf{qn}(G) \leq k$.

Queue layouts were first introduced by Heath et al. [95, 99]. This concept is analogous to the concept of stack layouts, also known as book embeddings, in which no two edges in the same set are allowed to cross. Applications of queue layouts include sorting permutations, parallel process scheduling, matrix computations, graph drawings, and queue-based computers. See [142, 22, 45] for a comprehensive list of references. If the vertex ordering is fixed, the optimal queue layout can be efficiently determined [45, 99].
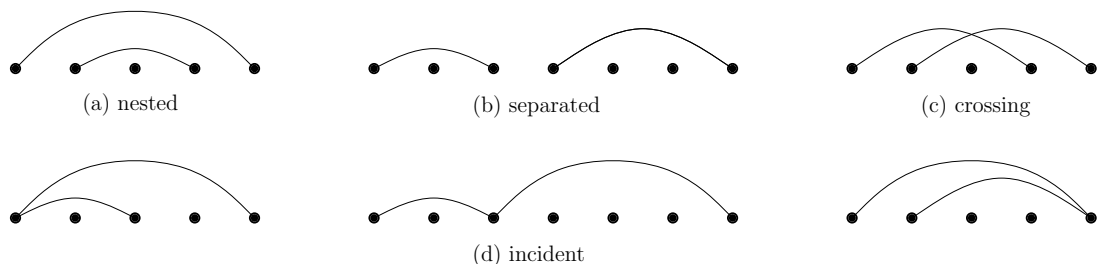
Figure 1.2: All possible relations between two edges in a fixed vertex ordering.

But in general, this problem is believed to be intractable. In particular, recognizing $k$-queue graphs is NP-complete even for $k = 1$ [99]. The class of 1-queue graphs coincides with the class of so called *arched leveled-planar* graphs [99] whereas the analogous class of 1-stack layout graphs coincides with the class of outerplanar graphs. Another characterization of 1-queue graphs based on track layouts is given in [43]. Queue layouts of directed graphs [9, 142, 98, 97], posets [96, 142], and several special graph classes [87, 88, 99, 95, 180, 179, 144, 44, 45, 46, 66, 139, 138, 137, 153] have also been investigated.

## 1.2.1 Previous results

For hypercubes, Heath and Rosenberg [99] showed that $Q_n$ has a layout into $n - 1$ queues, that is $\mathsf{qn}(Q_n) \leq n - 1$, for all $n \geq 2$. Hasunuma and Hirota [88] improved it to $\mathsf{qn}(Q_n) \leq n - 2$ for all $n \geq 5$. Subsequently, Pai et al. [137] showed that the same upper bound holds also for $n = 4$. Then Pai et al. [139] further decreased it to $\mathsf{qn}(Q_n) \leq n - 3$ for all $n \geq 8$. On the other hand, Heath and Rosenberg [99] showed that the queue-number of every graph is larger than half of its density. In particular, for hypercubes it follows that $\mathsf{qn}(Q_n) > n/4$ [139, 140]. Interestingly, the analogously defined stack-number (better known as the pagenumber) of the hypercube is known exactly to be $\mathsf{pn}(Q_n) = n - 1$ for all $n \geq 2$ [30, 113].

## 1.2.2 Improving the upper bound

Heath et al. [95] noticed that $\mathsf{qn}(G \,\square\, K_2) \leq \mathsf{qn}(G) + 1$ for every graph $G$ where $\square$ denotes the Cartesian product. In [76] we show that a queue layout of $G \,\square\, Q_k$ for $k \geq 2$ can be constructed (with the same additional cost of $k$ queues) from a queue layout of $G - A$ for every set $A$ of $k - 1$ independent vertices of $G$. This is the key idea in our improvements. It then only suffices to find a feasible set $A$ such that $\mathsf{qn}(G - A) < \mathsf{qn}(G)$.

**Lemma 1** ([76]). *Let $A$ be an independent set of vertices in a graph $G$ and $k = |A| + 1 \geq 2$. Then,*

$$\mathsf{qn}(G \,\square\, Q_k) \leq \mathsf{qn}(G - A) + k.$$

Our construction is based on a particular recursive interlaced ordering of the vertices from $A$ together with a careful distribution of incident edges into the previous known layout

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of $(G - A) \,\square\, Q_k$ into $\mathsf{qn}(G - A) + k$ queues. It was inspired by the construction of Pai et al. [139] where only the vertex $\mathbf{1} = (1, 1, \ldots, 1)$ was removed from $G = Q_{n-2}$ and it was shown that $\mathsf{qn}(Q_n) = \mathsf{qn}(Q_2 \,\square\, Q_{n-2}) \leq \mathsf{qn}(Q_{n-2} - \{\mathbf{1}\}) + 2$.

Applying Lemma 1 for certain dimensions and a composition result on strict queue layouts by Wood [180] for the remaining dimensions we obtain the following result. This is the first non-constant improvement.

**Theorem 1** ([76]). *For all $n \geq 3$,*

$$\mathsf{qn}(Q_n) \leq n - \lceil \log_2(n - \lceil \log_2(n - 1) \rceil) \rceil .$$

It is remarkable that Theorem 1 attains all previously [139] known bounds for $3 \leq n \leq 12$ except $\mathsf{qn}(Q_4) = 2$ [137]. For $n \geq 13$ we obtain better layouts. Altogether, the previously known and new results can be simplified as follows.

**Corollary 1** ([76]). *For all $n \geq 1$,*

$$\mathsf{qn}(Q_n) \leq n - \lfloor \log_2 n \rfloor .$$

Moreover, similar improvements were obtained also for a $2k$-ary hypercube $Q_n^{2k}$ [76]; that is, the $n$th Cartesian power of the $2k$-cycle. It is also worth noting that Theorem 1 also provides a partition of $Q_n$ into $n - \lceil \log_2(n - \lceil \log_2(n - 1) \rceil) \rceil$ leveled planar graphs with the same induced ordering. A graph $G$ is *leveled planar* [99] if it has a planar embedding such that vertices are mapped on vertical lines and edges are mapped to straight segments between two vertices on consecutive vertical lines. The *induced ordering* of a leveled planar graph orders its vertices by consecutive vertical lines, and from top to bottom on each line. An example for $Q_5$ is depicted on Figure 1.3.

### 1.2.3 Improving the lower bound

For improving the lower bound we extend the concept of rainbows and midpoints from [99, 45] with a new technique of out-in representations and contractions. A *k-rainbow* with respect to a vertex ordering $\sigma$ is a matching $\{u_i v_i \in E(G); 1 \leq i \leq k\}$ such that

$$\sigma(u_1) < \sigma(u_2) < \cdots < \sigma(u_k) < \sigma(v_k) < \sigma(v_{k-1}) < \cdots < \sigma(v_1).$$

Heath and Rosenberg [99] and then Dujmović and Wood [45] in a simpler argument showed that the size of a largest rainbow determines the number of queues in a queue layout of $G$ with the ordering $\sigma$. Moreover, they noticed that if $k$ edges share the same midpoint, they form a $k$-rainbow. The *midpoint* of an edge $uv$ is $(\sigma(u) + \sigma(v))/2$. This leads to the lower bound $\mathsf{qn}(G) > |E(G)|/2|V(G)|$ for every graph $G$ [99].

Our improvement is based on two tools. The first tool is the following representation of a linear layout of the graph $G$ which is equivalent regarding nesting of edges. Let $G'$ denote the graph obtained from $G$ by replacing every vertex $u$ with a pair of vertices $u_{\mathsf{out}}$, $u_{\mathsf{in}}$, and every edge $uv$ with the edge $u_{\mathsf{out}} v_{\mathsf{in}}$ if $\sigma(u) < \sigma(v)$. Furthermore, let $\sigma'$ be the
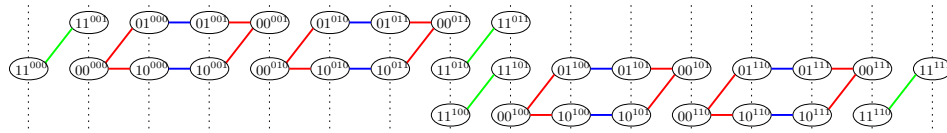
(a) A layout of $E_1$
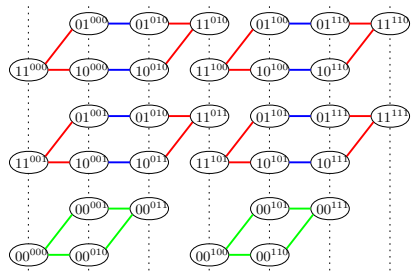


(b) A layout of $E_2$
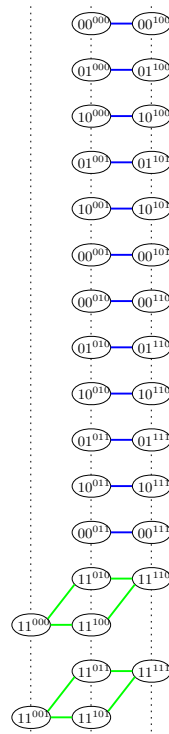


(c) A layout of $E_3$

Figure 1.3: A partition of $Q_5$ into three leveled planar graphs with the same induced ordering.
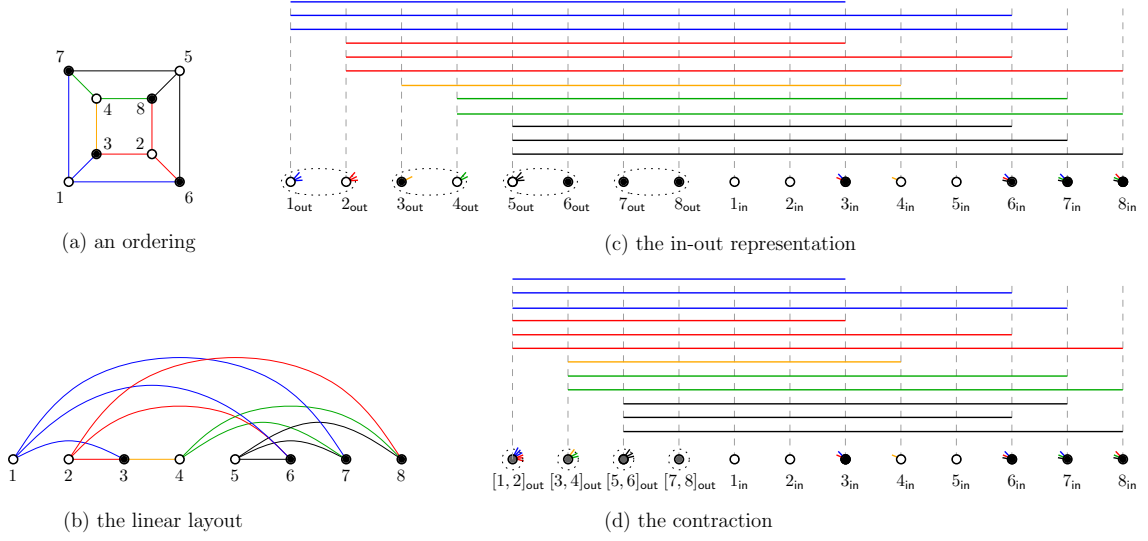
Figure 1.4: (a) An example of an ordering $\sigma$ of $Q_3$, (b) the linear layout of $Q_3$ with respect to $\sigma$, (c) the out-in representation $Q_3'$ and $\sigma'$, (d) the contraction $Q_3^*$. The colors distinguish edges from distinct out vertices.

vertex ordering of $G'$ taking all out-vertices and then all in-vertices, both according to $\sigma$. We say that the pair $(G', \sigma')$ is an *out-in representation* of $(G, \sigma)$. See Figure 1.4(a)-(c) for an illustration.

The out-in representation does not bring any improvement itself since the number of midpoints is preserved. However, it can be beneficially combined with our second tool, which is based on contractions. Let $G^*$ be a multigraph obtained by contractions of some pairwise-disjoint sets of consecutive vertices of $G'$. Here consecutive means with respect to the ordering $\sigma'$. See Figure 1.4(d) for an illustration. Clearly, if $G^*$ contains a $k$-rainbow, then $G$ contains a $k$-rainbow as well.

To improve the lower bound, the key idea is to contract large number of consecutive vertices in order to decrease the number of midpoints, but at the same time, to have only a small number of multiple edges. When pairs of consecutive out-vertices are contracted, we obtain the following preliminary lower bound.

**Proposition 1** ([76]). *For every $n \geq 1$, $\mathsf{qn}(Q_n) > (n-2)/3$.*

For a more general lower bound we employ contracting of a larger number of consecutive out-vertices together. The following result shows that we can get arbitrarily close to the factor $1/2$ instead of $1/3$ in Proposition 1.

**Theorem 2** ([76]). *For all $\varepsilon > 0$, for every sufficiently large $n$,*

$$\mathsf{qn}(Q_n) > \left(\frac{1}{2} - \varepsilon\right) n - O(1/\varepsilon).$$

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It should be noted that in the proof of the lower bound we actually only used $K_{2,3}$-free property of hypercubes. Therefore this new technique may be applied to larger graph classes. Still, the gap between the lower and upper bound leaves the question: Is it true that $\mathsf{qn}(Q_n) = n - \Theta(\log_2 n)$?

## 1.3 Level-disjoint partitions

A massive amount of traffic in communication networks that flows from providers of large data (such as video streaming services) to many clients at once leads to various optimization problems for broadcasting of multiple messages. Similar types of problems arise in master/workers parallel computations on interconnection networks when multiple tasks are simultaneously distributed from one node (master) to all other nodes (workers). This has been subject of research for many years. For surveys on broadcasting and other communication protocols in various kinds of networks see e.g. [82, 83, 100, 103, 104].

For simplicity we restrict ourselves to synchronous networks, where at each time unit messages can be sent from nodes to their neighbors in one unit of time. Since networks have limited capacity of links, any larger data to be broadcast needs to be split into multiple messages and sent individually. This leads to a more general variant of *broadcasting* in which several different messages need to be simultaneously transmitted from one source node, called the *originator*. The problem of multiple broadcasting was first tackled in [55] and previously studied under several different models in [8, 18, 94]. The minimal overall time needed for simultaneous broadcasting and the maximal number of messages that can be simultaneously broadcast were considered in [8, 77, 94].

We consider a scenario when each message (or task) needs to be handled (or processed) at each node in a time unit before it is sent out further to other selected neighbors, possibly to more than one. It is reasonable to demand that each node has to handle at each time unit only a single message (task). Equivalently, each node receives at most one message in each time unit. This restriction is called a *1-in-port model*. Furthermore, every received message is sent out only in the next time unit and no message is sent to already informed vertex. In other words, nodes have no buffers to store messages for delayed transmission. This is known as memoryless or queueless communication [111]. This simplification is motivated by memory or security restrictions, or a need for uninterrupted data flow.

### 1.3.1 Mutually independent hamiltonian cycles

First we consider a simplified case of simultaneous broadcasting when not only the number of received messages but also the number of outgoing messages are limited to at most one in each time step (both 1-in and 1-out-port model). In this case each broadcasted message traverses a Hamiltonian path, and the messages must never meet at the same time in the same vertex.

This leads to the following definitions. Two Hamiltonian paths $P_1 = (u_1, u_2, \ldots, u_m)$ and $P_2 = (v_1, v_2, \ldots, v_m)$ of $G$ are *independent* if $u_i \neq v_i$ for all $i \in [m]$. A set $S$ of

Hamiltonian paths of $G$ is *mutually independent* if every two paths from $S$ are independent. Two $v$-starting Hamiltonian cycles $C_1 = (v, u_2, \ldots, u_m)$ and $C_2 = (v, v_2, \ldots, v_m)$ are *independent* if $v_i \neq u_i$ for all $2 \leq i \leq m$. A set $S$ of $v$-starting Hamiltonian cycles of $G$ is *mutually independent* if every two cycles from $S$ are independent.

## Previous results

Mutually independent Hamiltonian paths and cycles have been previously studied in particular for hypercubes and in the presence of faults. Sun et al. [168] proved that for any vertex $s$, the $n$-dimensional hypercube $Q_n$ contains $n-1$ mutually independent $s$-starting Hamiltonian cycles if $n = 2, 3$; and $n$ mutually independent $s$-starting Hamiltonian cycles if $n \geq 4$. They also proved that for any set of $n-1$ distinct pairs of adjacent vertices, $Q_n$ contains $n-1$ mutually independent Hamiltonian paths with these pairs of vertices as endvertices. Hsieh and Yu [107] claimed that the $n$-dimensional hypercube $Q_n$ with at most $f \leq n-2$ faulty edges contains a set of $n-1-f$ mutually independent Hamiltonian paths and a set of $n-1-f$ mutually independent $s$-starting Hamiltonian cycles for any vertex $s$. However, Kueng et al. and Hsu [117] noticed a flaw in their proof and published the correction. Hsieh and Weng [105] proved that for $n \geq 3$, $Q_n$ with at most $f \leq n-2$ faulty edges contains a set of $n-1-f$ mutually independent Hamiltonian paths between any two vertices of different parity. Shih et al. [158] studied mutually independent paths of different lengths in $Q_n$. Mutually independent Hamiltonian cycles have been also studied in alternating group graphs [166] and in toroidal graphs [181].

## Our results

In [175] we improve previously known results by showing that $Q_n$ for every $n \geq 2$ contains $n$ mutually independent Hamiltonian paths with prescribed endvertices if they form a matching, see Theorem 3.

**Theorem 3** ([175]). *Let* $M = \{w_1 b_1, w_2 b_2, \ldots, w_n b_n\} \subseteq E(Q_n)$ *be a matching of* $Q_n$ *($n \geq 2$) where* $w_1, w_2, \ldots, w_n$ *are vertices of the same parity. Then,* $Q_n$ *has* $n$ *mutually independent Hamiltonian paths* $P_1[w_1, b_1], P_2[w_2, b_2], \ldots, P_n[w_n, b_n]$.

We also prove that $Q_n$ for every $n \geq 4$ and for every set of at most $f \leq n-2$ faulty edges contains $n-f$ mutually independent $s$-starting Hamiltonian cycles for any vertex $s$, see Theorem 4. This is an optimal result both in the number of faulty edges and in the number of cycles since $s$ may be incident with $f$ faulty edges.

**Theorem 4** ([175]). *Let* $F \subseteq E(Q_n)$, $n \geq 4$, $f = |F| \leq n-2$, *and* $s \in V(Q_n)$. *Then,* $Q_n - F$ *has* $n-f$ *mutually independent $s$-starting Hamiltonian cycles.*

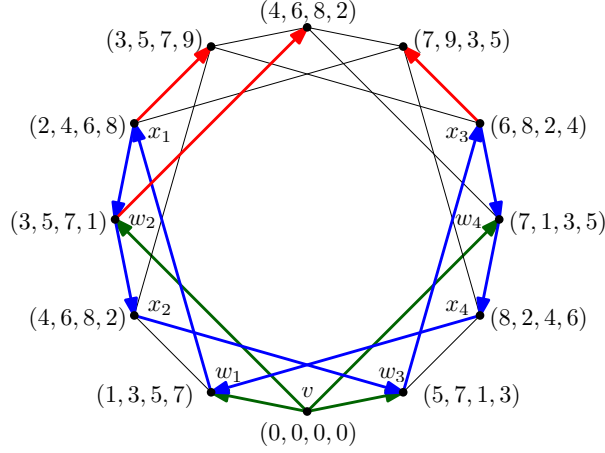The same result as in Theorem 4 appeared independently in [119].

Figure 1.5: Four level-disjoint partitions of the circulant graph $C_{12}(1,3)$ rooted at $v$.

## 1.3.2 Level-disjoint partitions - the concept

In general simultaneous broadcasting, the restriction on the number of outgoing messages is lifted, so each node may send its message in each time step to more (possibly all) neighbors at once (thus we have a so called *1-in and all-out port model*). For this scenario we developed the concept of level-disjoint partitions in [77], which is explained below, as a generalization of mutually independent Hamiltonian cycles. For a different scenario of multiple message routing when different messages can arrive in the same vertex at the same time but always via edge-disjoint paths there is a close concept of independent spanning trees [26].

A *level partition* of a graph $G$ is a partition $\mathcal{S} = (S_0, \ldots, S_h)$ of $V(G)$ into a sequence of sets, called *levels*, such that $S_i \subseteq N(S_{i-1})$ for every $1 \leq i \leq h$; that is, every vertex has some neighbor in the previous level. The number $h = h(\mathcal{S}) = |\mathcal{S}| - 1$ is called the *height* of $\mathcal{S}$. The broadcasting starts at all vertices from the level $S_0$: at each time unit the same message is sent from all vertices of the current level to all vertices in the next level through edges of the graph, till the $h$-th time unit, when the message is spread to all vertices of $G$. Note that we do not care which particular edges are used. In the case when the starting level $S_0$ is a singleton, say $S_0 = \{v\}$, we say that the level partition is *rooted* at $v$ (or $v$-rooted) and the vertex $v$ is called the *root* of $\mathcal{S}$.

Two level partitions $\mathcal{S} = (S_0, \ldots, S_{h(\mathcal{S})})$ and $\mathcal{T} = (T_0, \ldots, T_{h(\mathcal{T})})$ are said to be *level-disjoint* if $S_i \cap T_i = \emptyset$ for every $1 \leq i \leq \min(h(\mathcal{S}), h(\mathcal{T}))$. Note that we allow $S_0 \cap T_0 \neq \emptyset$ since we consider the case when different messages have the same originator. Level partitions $\mathcal{S}^1, \ldots, \mathcal{S}^k$ are said to be (mutually) level-disjoint if any two partitions are level-disjoint. Then we say that $\mathcal{S}^1, \ldots, \mathcal{S}^k$ are *level-disjoint partitions*. If every partition is rooted at the same vertex $v$ and they are level-disjoint (up to the starting level $\{v\}$), we say that $\mathcal{S}^1, \ldots, \mathcal{S}^k$ are *level-disjoint partitions with the same root $v$*. For an example of four $v$-rooted level-disjoint partitions of a circulant graph, see Figure 1.5. Note that the 4-tuple at a vertex denotes its levels in each partition.

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The number of level-disjoint partitions determines how many messages can be broadcast simultaneously while their maximal height determines the overall time of the broadcasting. Hence a general aim is to construct for a given graph

- as many as possible (mutually) level-disjoint partitions; and

- with as small maximal height as possible.

Clearly these two criteria are in a sense complementary. The following necessary conditions on the number of $v$-rooted level-disjoint partitions as well as on their maximal height were observed in [78].

**Proposition 2** ([78]). *Let $\mathcal{S}^1, \ldots, \mathcal{S}^k$ be level-disjoint partitions of a graph $G$ with the same root $v$. Then,*

$$k \leq \deg(v) \tag{1.1}$$

$$\max_{1 \leq i \leq k} h(\mathcal{S}^i) \geq \begin{cases} \operatorname{ecc}(v) + k - 1 & \text{if } G \text{ is not bipartite,} \\ \operatorname{ecc}(v) + 2k - 2 & \text{if } G \text{ is bipartite.} \end{cases} \tag{1.2}$$

In particular for the hypercube $Q_n$, it follows from Proposition 2 that $Q_n$ can have at most $n$ level-disjoint partitions with the same root and their maximal height must be in this case at least $3n - 2$. In [77] we conjectured that this is indeed attainable for every $n \geq 3$ and we verified it for certain (infinitely many) values of $n$.

**Conjecture 1** ([77]). *For every $n \geq 3$ there exist $n$ level-disjoint partitions of $Q_n$ with the same root and with the maximal height $3n - 2$.*

In simultaneous broadcasting one often needs to send out at the same time as many messages as possible without limitations on the overall time for such a task. In our communication model this scenario leads to the following question. Given a graph $G$, $v \in V(G)$, and $k \geq 2$, are there $k$ level-disjoint partitions of $G$ rooted in $v$?

First we showed that it suffices to find level-disjoint partitions locally "around" the root $v$ on some suitable subgraph $H$ of $G$. Then they can be extended to level-disjoint partitions with the same root to the entire graph $G$. Since $v$ could be a cut-vertex of $G$, we need that $H$ meets each component of $G - v$.

**Lemma 2** ([78]). *Let $v$ be a vertex of a graph $G$ and $H$ be a subgraph of $G$ containing $v$ and some vertex from each component of $G - v$. Then any $k$ $v$-rooted level-disjoint partitions of $H$ can be extended to $k$ $v$-rooted level-disjoint partitions of $G$.*

For $k = 2$, it is easy to see that, for example, odd cycles have two level-disjoint partitions with the same root whereas even cycles do not. For a full structural characterization of graphs $G$ admitting two level-disjoint partitions rooted in $v$, we need the following definitions. A cycle containing a vertex $v$ is called a *$v$-cycle*. Let us denote by $\bar{v}_C$ the opposite vertex to $v$ on an even cycle $C$. We say that a path $P[u, w]$ is *chordal* to a cycle

$C$ if $V(P) \cap V(C) = \{u, w\}$. We say that a chordal path $P[u, w]$ to a cycle $C$ *separates* $x, y \in V(C)$ if $x$ and $y$ belong to different subpaths of $C - \{u, w\}$.

We characterized graphs admitting two level-disjoint partitions rooted in a given vertex as follows.

**Theorem 5** ([78]). *Let $v$ be a vertex in a graph $G$. Then $G$ has two level-disjoint partitions rooted in $v$ if and only if for every block $B$ of $G$ containing $v$ it holds that*

(*a*) *$B$ is 2-connected, and*

(*b*) *$B$ is non-bipartite, or $B$ has a $v$-cycle $C$ with a chordal path that separates $v, \overline{v}_C$.*

It would be interesting to find a similar structural characterization for existence of $k$ level-disjoint partitions with the same root as in Theorem 5 also for $k \geq 3$.

### 1.3.3 (Bi)perfect level-disjoint partitions

Let us now focus on finding level-disjoint partitions with as small maximal height as possible. This determines the overall time of the broadcasting. We say that $v$-rooted level-disjoint partitions $\mathcal{S}^1, \ldots, \mathcal{S}^k$ of a graph $G$ are

- *perfect* if every vertex $u$ except $v$ belongs in $\mathcal{S}^1, \ldots, \mathcal{S}^k$ to levels $\{d(u, v), d(u, v) + 1, \ldots, d(u, v) + k - 1\}$ (i.e. not necessarily in this order),

- *biperfect* if every vertex $u$ except $v$ belongs in $\mathcal{S}^1, \ldots, \mathcal{S}^k$ to levels $\{d(u, v), d(u, v) + 2, \ldots, d(u, v) + 2k - 2\}$ (i.e. not necessarily in this order).

Informally, this means that each vertex is in the smallest levels possible. The latter definition is an adjustment for bipartite graphs. Clearly, the maximal height of perfect (or biperfect for bipartite graphs) level-disjoint partitions attains the lower bound (1.2). Hence broadcasting via perfect level-disjoint partitions of $G$ attains optimal time (even locally on each vertex). If $G$ is bipartite, broadcasting via biperfect level-disjoint partitions of $G$ attains optimal time (even locally on each vertex).

First we have shown that the subgraph extension technique from Lemma 2 can be adopted for finding (bi)perfect level-disjoint partitions as well if we include the following isometric condition on the subgraph. We say that a subgraph $H$ of a graph $G$ containing a vertex $v$ *preserves distances to $v$* if $d_H(u, v) = d_G(u, v)$ for every vertex $u$ of $H$.

**Theorem 6** ([79]). *Let $v$ be a vertex of a graph $G$ and $H$ be a subgraph of $G$ containing $N(v) \cup \{v\}$ and preserving distances to $v$. Then any $k$ (bi)perfect $v$-rooted level-disjoint partitions of $H$ can be extended to $k$ (bi)perfect, respectively, $v$-rooted level-disjoint partitions of $G$.*

This raised a question for which simple graphs one can find (bi)perfect level-disjoint partitions. It turned out that so called wheels and biwheels are very useful for this purpose. A *$k$-wheel $W_k$* for $k \geq 0$ *centered at* a vertex $v$ is the graph on vertices $v, w_1, \ldots, w_k$ with
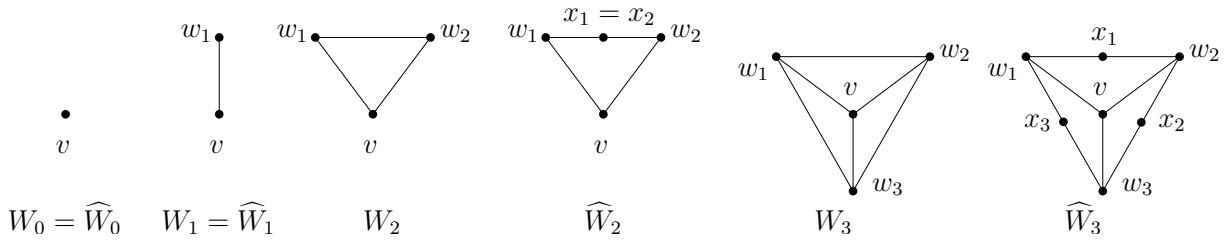
Figure 1.6: $k$-wheels and $k$-biwheels centered at $v$ for $k = 0, 1, 2, 3$.

edges joining $v$ to all $w_i$'s and edges joining $w_i$ and $w_{i+1}$ for every $1 \leq i \leq k$ where $w_{k+1}$ is identified as $w_1$. A *$k$-biwheel* $\widehat{W}_k$ for $k \geq 0$ *centered at* a vertex $v$ is the subdivision of $W_k$ centered at $v$ obtained by inserting a new vertex $x_i$ to the edge between $w_i$ and $w_{i+1}$ for every $1 \leq i \leq k$. Note that we define wheels and biwheels also for $k \leq 2$ only for technical reasons. See Figure 1.6 for an illustration of small wheels and biwheels.

By applying Theorem 6 for wheels and biwheels we obtain the following sufficient condition on finding (bi)perfect level-disjoint partitions. Note that the number $k$ of partitions is optimal.

**Theorem 7** ([79])**.** *Let $v$ be a vertex of degree $k \geq 1$ in a graph $G$. If $G$ has a $k$-wheel centered at $v$, then $G$ has $k$ perfect level-disjoint partitions rooted at $v$. If $G$ is bipartite, $k \geq 3$, and $G$ has a $k$-biwheel centered at $v$, then $G$ has $k$ biperfect level-disjoint partitions rooted at $v$.*

Biwheels naturally occur in Cartesian product graphs, but they can be also found in other extensively studied networks such as circulant graphs or Knödel graphs. This gives us tight constructions for large classes of graphs. In particular for hypercubes we obtain the following result, affirmatively answering a Conjecture 1.

**Corollary 2** ([79])**.** *For every $n \geq 3$ there exist $n$ (biperfect) level-disjoint partitions of $Q_n$ with the same root and with the optimal height $3n - 2$.*

Similar tight results were obtained for bipartite tori, meshes, Knödel graphs, and circulant graphs [79].

## 1.4   Incidence colorings

An incidence coloring is one of many variants of graph colorings. An *incidence* in a graph $G$ is a pair $(v, e)$ where $v$ is a vertex of $G$ and $e$ is an edge of $G$ incident to $v$. Two incidences $(v, e)$ and $(u, f)$ are said to be *adjacent* if at least one of the following holds: (a) $v = u$, (b) $e = f$, or (c) $vu \in \{e, f\}$. See Figure 1.7 for an illustration. An *incidence coloring* of $G$ is a coloring of its incidences such that adjacent incidences are assigned distinct colors. The least $k$ such that $G$ admits an incidence coloring with $k$ colors is called the *incidence chromatic number of $G$*, and is denoted by $\chi_i(G)$.
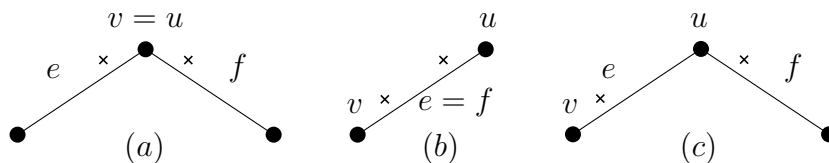
Figure 1.7: Three types of adjacent incidences.

This notion has an application for conflict-free communication in the following situation. Consider a synchronous communication network represented by a graph $G$. The following restrictions seem to apply in many real cases and are reasonable also for human communication. In each round,

- every node can either listen or talk (or do nothing),

- every node can listen to at most one neighbor,

- every node can talk to more neighbors.

Our aim is to design a scheme of communication with a minimal number of rounds during which all pairs of neighbors can communicate between each other (i.e. in both directions) according to the above restrictions. It is easy to see that an incidence coloring of $G$ with colors $[n]$ corresponds to such communication scheme. Indeed, a color $i$ at incidence $(u, uv)$ indicates that in round $i$, the vertex $v$ should talk (possibly to more neighbors at once) and the vertex $u$ should listen to $v$. Thus, the incidence chromatic number $\chi_i(G)$ specifies the minimal number of rounds needed for such communication. See Figure 1.8 for an illustration.
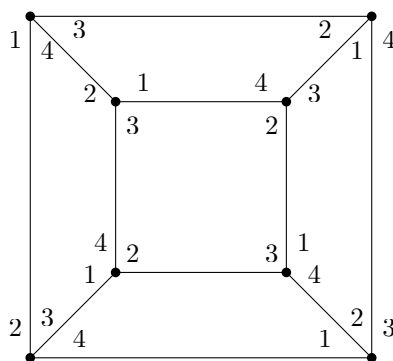


Figure 1.8: An incidence coloring of $Q_3$ with 5 colors.

The incidence coloring of graphs was defined by Brualdi and Massey [17] and attracted considerable attention as it is related to several other types of colorings. As already observed in [17], it is directly connected to *strong edge-coloring*, i.e. a proper edge-coloring such that the edges at distance at most two receive distinct colors. Indeed, an incidence coloring of a graph $G$ corresponds to a strong-edge coloring of the graph obtained by

subdividing each edge of $G$ with a single vertex. Furthemore, Guiduli [85] observed that incidence coloring is a special case of directed star arboricity, introduced by Algor and Alon [3]. For a survey on incidence colorings we refer to [162].

### 1.4.1 Incidence colorings of Cartesian products

It is easy to see that $\chi_i(G) \geq \Delta(G) + 1$ for every (nontrivial) graph. It is known that equality holds for Halin graphs with maximum degree at least 5 [177], outerplanar graphs with maximum degree at least 7 [161], planar graphs with girth at least 14 [13], and square, honeycomb and hexagonal meshes [108]. Sun [169] observed that for any $n$-regular graph $G$, $\chi_i(G) = n + 1$ if and only if $G$ has a partition into $n + 1$ (perfect) dominating sets. A prime example of a graph having this property is the hypercube of dimension $n = 2^p - 1$ where $p$ is an integer. In this case such partition of $Q_n$ exists by the well-known Hamming code and its cosets.

As for the upper bound, Brualdi and Massey [17] proved that $\chi_i(G) \leq 2\Delta(G)$ for every graph $G$ and they conjectured that $\chi_i(G) \leq \Delta(G) + 2$ for every graph $G$. This conjecture has been disproved by Guiduli [85] who showed that Paley graphs have incidence chromatic number at least $\Delta + \Omega(\Delta)$. Furthermore, he proved that $\chi_i(G) \leq \Delta(G) + 20 \log \Delta(G) + 84$ for every graph $G$, which is currently the best upper bound for general graphs.

Although the "$\Delta + 2$ conjecture" of Brualdi and Massey has been disproved in general, it has been confirmed for many graph classes, e.g. for cubic graphs [126], partial 2-trees (and thus also outerplanar graphs) [102], and powers of cycles (with a finite number of exceptions) [135]. In [71] we gave the following sufficient condition for a Cartesian product graph to have the incidence chromatic number at most $\Delta + 2$.

**Theorem 8** ([71]). *Let $G$ be a graph with $\chi_i(G) = \Delta(G) + 1$ and let $H$ be a subgraph of a regular graph $H'$ such that $\chi_i(H') = \Delta(H') + 1$ and $\Delta(G) + 1 \geq \Delta(H') - \Delta(H)$. Then,*

$$\chi_i(G \,\Box\, H) \leq \Delta(G \,\Box\, H) + 2\,.$$

In [71] we also introduce two classes of graphs such that the Cartesian product of factors from each of them has the incidence chromatic number at most $\Delta + 2$.

For hypercubes, Pai et al. [141] showed that $\chi_i(Q_n) = n + 2$ if $n = 2^p - 2$ and $p \geq 2$, or $n = 2^p + 2^q - 1$, or $n = 2^p + 2^q - 3$ and $p, q \geq 2$ and they conjectured the following.

**Conjecture 2.** *[141] $\chi_i(Q_n) = n + 2$ if $n = 2^p - 1$ for no integer $p$.*

By applying Theorem 8 we confirmed Conjecture 2. Hence the incidence chromatic number of hypercubes is as follows.

**Corollary 3** ([71]). *For every $n \geq 1$,*

$$\chi_i(Q_n) = \begin{cases} n + 1 & \text{if } n = 2^p - 1 \text{ for some integer } p \geq 0, \\ n + 2 & \text{otherwise.} \end{cases}$$

Conjecture 2 has been independently confirmed also by Shiau et al. [157].

### 1.4.2 Incidence colorings of subquartic graphs

Many real-world applications lead to graphs of small degrees, for example a city road map. For cubic graphs, it is known that the $\Delta + 2$ conjecture holds, i.e. they have an incidence coloring with 5 colors [126]. On the other hand, the smallest graph $G$ in terms of the maximum degree and the number of vertices that does not admit an incidence coloring with at most $\Delta(G) + 2$ colors is, up to our knowledge, a 6-regular graph on 11 vertices introduced in [32].

For graphs of maximum degree 5 the current best upper bound [17] is 10. For graphs of maximum degree 4 the previous best upper bound [17] was 8. In [72] we improved the upper bound for subquartic graphs, i.e. graphs with the maximum degree 4.

**Theorem 9** ([72]). *Let $G$ be a graph with the maximum degree $\Delta(G) = 4$. Then $\chi_i(G) \leq 7$.*

In our proof we first decompose a subquartic graph $G$ into a subcubic graph (which admits an incidence coloring with at most 5 colors by [126]) and two disjoint matchings. Then we show how to modify the incidence coloring of the subcubic graph so that, using two additional colors, one can color all the incidences of $G$ using 7 colors altogether.

It still remains an open problem whether the bound can be decreased to 6 colors. In [54], the authors verified that some special classes of subquartic graphs have an incidence coloring with at most 6 colors. Additionally, using computer analysis, we verified that it holds also for all 4-regular graphs on at most 14 vertices.

## 1.5 Distance magic labelings

There are many problems that lead to study of various assignments of integers to vertices or edges (or both) in a given graph under certain conditions. Such assignments are called *graph labelings*. They include problems from coding theory, circuit design, frequency assignment etc. For references see the (dynamic) survey on graph labelings by Galian [65].

Distance magic labelings are graph labelings defined as follows. Let $G$ be an (undirected) graph on $k$ vertices. A bijection $f : V(G) \to [k]$ is a *distance magic labeling* of $G$ if there is (a so called *magic constant*) $m$ such that

$$\sum_{v \in N(u)} f(v) = m$$

for every vertex $u$. For an example see Figure 1.9.

Distance magic labelings are also known as $\Sigma$-labelings [174], 1-vertex magic vertex labelings [127], or neighborhood magic labelings [1]. Miller et al. [127] refer to Sedláček [152] for a first introduction of the notion of magic labelings in 1963 (although it was defined for edges at that time). The term distance magic labeling for this concept was introduced by Sugeng et al. [167].

The concept of distance magic labelings of graphs is motivated by the well-known magic rectangles. An $m \times n$ array containing entries $\{1, 2, \ldots, mn\}$ is a *magic rectangle* if its row
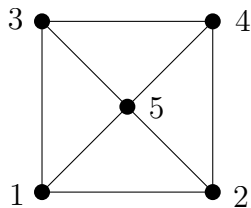
Figure 1.9: A distance magic labeling of a 4-wheel.

sums are equal and its column sums are equal. Given a magic rectangle $m \times n$, we obtain a distance magic labeling of the complete $m$ partite graph with each partite set of size $n$ by labeling the vertices of each part by the rows of the rectangle.

Furthermore, distance magic labelings have applications in design of incomplete tournaments. Assume we have $k$ players with assigned ranks from 1 to $k$ that express their strength, for instance in some long-term ranking. If they play a complete round robin tournament, a player with rank $r$ plays $k-1$ matches against all $k-1$ possible opponents with their total rank $t-r$ where $t = (1+k)k/2$ is the sum of all ranks. Instead, the players may decide to play a (shorter) incomplete tournament such that each player plays against the same number $n$ of opponents and the sum of ranks of opponents is the same number $m$ for each player. This is called an *equal strength incomplete tournament*. Clearly, such tournament exists if and only if there is an $n$-regular graph on $k$ vertices that has a distance magic labeling. Moreover, if we organize a tournament according to the complement of a such labeled graph, then a player with rank $r$ plays $k-1-n$ matches against different opponents with their total rank $t-m-r$, which mimics the complete tournament. This is called a *fair incomplete tournament*. For results on existence of such graphs we refer to the survey [5].

In their survey, Arumugam et al. [5] posed the following conjecture on distance magic labelings of hypercubes that originally appeared in Acharya et al. [1]. It is easy to see that an $n$-regular graph admits a distance magic labeling only if $n$ is even.

**Conjecture 3** ([1]). *For any even integer $n \geq 4$, the $n$-dimensional hypercube $Q_n$ does not have a distance magic labeling.*

Fronček et al. [7] proved that the statement holds for every $n \equiv 0 \pmod 4$. Furthermore, he found an ad-hoc distance magic labeling of $Q_6$, contradicting Conjecture 3. In our work [73], we completely settle the remaining cases.

**Theorem 10** ([73]). *$Q_n$ has a distance magic labeling if and only if $n \equiv 2 \pmod 4$.*

Our construction of distance magic labelings of $Q_n$ for $n \equiv 2 \pmod 4$ is based on elementary linear algebra. We work with labels as with the vectors in $\mathbb{F}_2^n$, i.e. in their binary representation. Then we consider bijective mappings $f \colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ such that the set of vectors

$$L(u) = \{f(u \oplus e_i) \mid i \in [n]\}$$

28

has the same number of 0's as 1's in each coordinate for every $u \in \mathbb{F}_2^n$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{F}_2^n$. This kind of balance condition implies that $f$ is a distance magic labeling. The key idea relies in showing that such a mapping $f$ can be obtained by multiplication with a regular matrix that has a balanced set of columns. Then it suffices to observe that such matrix exists if $n \equiv 2 \pmod 4$.

Moreover, our construction gives a labeling of $Q_n$ that is in general $d$-distance magic for every odd $1 \leq d \leq n$. That is, for every odd $1 \leq d \leq n$, if we sum the labels at all vertices at distance $d$ from a vertex $u$, we obtain the same value for each vertex $u$.

Applying Theorem 10, Cichacz and Nikodem [31] showed that $tQ_n$, $G \times Q_n$, and $G \circ Q_n$ have a distance magic labeling for any integer $t \geq 1$ and any regular graph $G$ if $n \equiv 2 \pmod 4$, where $tQ_n$ denotes the disjoint union of $t$ copies of $Q_n$ and $G \times Q_n$, $G \circ Q_n$ denote the direct and lexicographic (respectively) products of $G$ and $Q_n$.

## 1.6 Parity vertex colorings

Many problems across different areas can be studied in terms of homomorphisms and in particular embeddings. An *embedding* of a graph $G$ into a graph $H$ is an injective map $f \colon V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ for every edge $uv \in E(G)$, i.e. an injective homomorphism of $G$ to $H$. It is an NP-complete problem to decide whether a given graph admits an embedding into $Q_n$ [116], even when restricted for trees [176].

Embeddings into hypercubes are motivated by parallel processing on hypercube architectures or by simulations of other architectures. Although this problem has been studied since 1950 [154], there are still many open problems, for example, Havel's conjecture [90] asserting that every balanced ternary tree on $2^n$ vertices is a spanning tree of $Q_n$.

Havel and Morávek [92] came with a concept of so called $B$-valuations to characterize all graphs embeddable into hypercubes. Bunde et al. [20] introduced a simplified notion of parity edge colorings. A *parity path* in a colored graph is a path with every color occurring even number of times. A *parity edge coloring* is an edge coloring without a parity path. The minimum number of colors in a parity edge coloring of $G$ is denoted by $\chi_p'(G)$. Bunde et al. [20] showed that a tree $T$ embeds into $Q_n$ if and only if $\chi_p'(T) \leq n$.

Borowiecki et al. [14] considered a vertex version of parity colorings. A *parity vertex coloring* of a graph $G$ is a coloring of $V(G)$ such that each path in $G$ contains some color odd number of times. The minimum number of colors in a parity vertex coloring of $G$ is denoted by $\chi(G)$. This type of coloring is closely related to vertex rankings, conflict free colorings, or unique maximum colorings [28, 163].

In particular, a *vertex ranking* of a graph $G$ is a proper vertex coloring by a linearly ordered set of colors such that every path between vertices of the same color contains some vertex of a higher color. The minimum numbers of colors in a vertex ranking of $G$ is denoted by $\chi_r(G)$. This parameter is also known as a *tree-depth* [136].

Clearly, every vertex ranking is also parity vertex coloring, so $\chi_p(G) \leq \chi_r(G)$ for every graph $G$. Borowiecki et al. [14] conjectured that for trees these parameters behave almost the same.

**Conjecture 4** ([14]). *For every tree $T$ it holds $\chi_r(T) - \chi_p(T) \leq 1$.*

In [80] we show that the above conjecture is false for every binominal tree of order $n \geq 5$. A *binomial tree $B_n$* of order $n \geq 0$ is a rooted tree defined recursively. $B_0 = K_1$ with the only vertex as its root. The binomial tree $B_n$ for $n \geq 1$ is obtained by taking two disjoint copies of $B_{n-1}$ and joining their roots by an edge, then taking the root of the second copy to be the root of $B_n$.

Binomial trees have been under consideration also in other areas. For example, $B_n$ is a spanning tree of the $n$-dimensional hypercube $Q_n$ that has the minimum average congestion (also known as total distance or Wiener index) among all spanning trees of $Q_n$ [4].

In [14] it was shown that $\chi_r(B_n) = n+1$ for all $n \geq 0$. We show that $\chi_p(B_{3k}) \leq 2k+1$ for every $k \geq 1$, which hence disproves the above conjecture.

**Theorem 11** ([80]). *For every $k \geq 1$ the binomial tree $B_{3k}$ has a parity vertex coloring with $2k + 1$ colors.*

From Theorem 11 we obtain the following general upper bound.

**Corollary 4** ([80]). *$\chi_p(B_n) \leq \left\lceil \frac{2n+3}{3} \right\rceil$ for every $n \geq 0$.*

On the other hand, Borowiecki et al. [14] showed that $\chi_p(P_n) = \lceil \log_2(n+1) \rceil$ for every $n$-vertex path $P_n$. This gives a trivial lower bound $\chi_p(B_n) \geq \lceil \log_2(2n+1) \rceil$ as $B_n$ contains a $2n$-vertex path. This gap between the lower bound and our upper bound remains open.

As for the hypercube, Borowiecki et al. [14] determined values $\chi_p(Q_0) = 1$, $\chi_p(Q_1) = 2$, $\chi_p(Q_2) = 3$, $\chi_p(Q_3) = 5$, $\chi_p(Q_4) = 8$ and they strongly believe that $\chi_p(Q_n) = F_n + 2$, where $F_n$ is the $n$th Fibonacci number, which still remains an open question. Soukup [163] checked that $13 \leq \chi_p(Q_5) \leq 15$.

## 1.7 Gray codes

Generating all objects in a combinatorial class such as permutations, subsets, combinations, partitions, trees, strings etc. is one of the oldest and most fundamental algorithmic problems, and such generation algorithms appear as core building blocks in a wide range of practical applications, see the survey [149]. In fact, half of the most recent volume [115] of Knuth's seminal series *The Art of Computer Programming* is devoted entirely to this fundamental subject. The ultimate goal for algorithms that efficiently generate each object of a particular combinatorial class exactly once is to generate each new object in constant time. Such optimal algorithms are sometimes called *loopless algorithms*, a term coined by Ehrlich in his influential paper [52]. Note that a constant-time algorithm requires in particular that consecutively generated objects differ only in a constant amount, for example in a single transposition of a permutation, in adding or removing a single element from a set, or in a single tree rotation operation. These types of orderings have become known as *combinatorial Gray codes*.

They are named after Frank Gray, a physicist and researcher at Bell Labs, who invented a method [68] to generate all $2^n$ many subsets of $[n]$ by repeatedly adding or removing a single element, now known as the binary *reflected Gray code*. In graph theory terms this corresponds to a Hamiltonian cycle in $Q_n$. The binary reflected Gray code found widespread use, for example in circuit design and testing, signal processing and error correction, data compression etc.; many more applications are mentioned in the survey [149]. This code is also implicit in the well-known *Towers of Hanoi* puzzle and the *Chinese ring* puzzle, and it also appears in music theory [40]. It has many interesting properties, see [115, Section 7.2.1.1], and there is a simple loopless algorithm to compute it [52, 10].

Since the discovery of the binary reflected Gray code, there has been continued interest in developing Gray codes for bitstrings of length $n$ that satisfy various additional constraints. For instance, a Gray code with the property that each bit is flipped (almost) the same number of times was first constructed by Bakos [2]. Goddyn and Gvozdjak constructed an $n$-bit Gray code in which any two flips of the same bit are almost $n$ steps apart [67], which is best possible. These are only two examples of an extensive work on possible Gray code transition sequences; see also [21, 170, 39]. Savage and Winkler [150] constructed a Gray code that generates all $2^n$ bitstrings such that all bitstrings with Hamming weight $k$ appear before all bitstrings with weight $k + 2$, for each $0 \leq k \leq n - 2$.

They used this construction to tackle the *middle two levels problem*, which asks for a cyclic listing of all bitstrings of length $2n+1$ with weights in the interval $[n, n+1]$ by flipping a single bit in each step; that is, a Hamiltonian cycle in the subgraph $Q_{2n+1,[n,n+1]}$ of $Q_{2n+1}$ induced by the middle two levels. The existence of such Hamiltonian cycle for any $n \geq 1$ was conjectured independently in the 80's by Havel [91] and Buck and Wiedemann [19]. The conjecture has also been attributed to Dejter, Erdős, Trotter [112] and various others, and also appears in the popular books [178, 115, 36]. The middle levels conjecture has attracted considerable attention over the last 30 years [148, 58, 150, 110, 42, 112, 41, 101, 81, 133, 159, 160], and a positive solution, i.e. an existence proof for a Hamiltonian cycle in $Q_{2n+1,[n,n+1]}$ for any $n \geq 1$, has been found only recently by Mütze [132]. A shorter and much accessible new proof that avoids almost all technical details was presented in [75], and an algorithm for computing the cycle using $O(1)$ amortized time and $O(n)$ space was presented in [130].

### 1.7.1 Trimming and gluing Gray codes

Another combinatorial generation problem of similar importance is to list all $\binom{n}{k}$ many $k$-element subsets of $[n]$ by repeatedly exchanging a single element. Also for this problem, loopless algorithms are well-known [171, 52, 10, 50, 51, 146, 27, 109] (see also [115, Section 7.2.1.3]).

A common generalizations of the binary reflected Gray code, the middle levels problem, and the above problem is to generate all, or almost all, subsets of $[n]$ whose size is in some interval $[k, l]$, where $0 \leq k \leq l \leq n$, by repeatedly adding or removing a single element, or by exchanging a single element if necessary. The binary reflected Gray code corresponds to the case $k = 0$ and $l = n$ in $Q_n$, the middle levels problem corresponds to the case $k = n$,

$l = n + 1$ in $Q_n$, and the above problem corresponds to case $l = k$ in $Q_n$.

Since the graph $Q_{n,[k,l]}$ induced by all levels between $k$ and $l$ in $Q_n$, $0 \leq k \leq l \leq n$, is bipartite, it has a Hamiltonian cycle only if the two partition classes have the same size, which happens only for odd dimension $n$ and between two symmetric levels $k$ and $l = n - k$, or for even dimension $n$ and $[k, l] = [0, n]$. However, we may at least ask for a cycle that covers a smaller partition class, which is called a *saturating cycle*. Clearly, a saturating cycle in a balanced bipartite graph is a Hamiltonian cycle. Hence saturating cycles naturally generalize Hamiltonian cycles for unbalanced bipartite graphs.

In our work [74] we asked for which weight ranges $[k, l]$ does the graph $Q_{n,[k,l]}$ have a saturating cycle, and we positively resolved this question for all values of $k$ and $l$ except the cases covered by the generalized middle level conjecture (Conjecture 5 below), which is still open. The case $l = k + 1$ was already covered in [131].

**Theorem 12** ([74]). *For any $n \geq 3$ the graph $Q_{n,[k,l]}$ has a saturating cycle in the following cases:*

*(i) If $0 = k < l \leq n$ or $0 \leq k < l = n$, and $l - k \geq 2$.*

*(ii) If $1 \leq k < l \leq n - 1$ and $l - k \geq 2$ is even.*

*(iii) If $1 \leq k < l \leq \lceil n/2 \rceil$ or $\lfloor n/2 \rfloor \leq k < l \leq n - 1$, and $l - k \geq 3$ is odd.*

*(iv) If $1 \leq k < l \leq n - 1$ and $l - k \geq 3$ is odd, under the additional assumption that $Q_{2m+1,[m-c,m+1+c]}$, $c := (l - k - 1)/2$, has a Hamiltonian cycle for all $m = c, c+1, \ldots, (\min(k + l, 2n - k - l) - 1)/2$.*

If the graph $Q_{n,[k,l]}$ is unbalanced, i.e. is has $\delta > 0$ more vertices in the larger than in the smaller partite set, a saturating cycle in $Q_{n,[k,l]}$ necessarily omits $\delta$ vertices from the larger partite set. However, if we insist on all vertices of $Q_{n,[k,l]}$ to be included in our listing, then this can be achieved by allowing steps where instead of only a single bitflip, two bits are flipped. This can be viewed as augmenting the underlying graph $Q_{n,[k,l]}$ by adding distance-2 edges. In this case we may ask for a cyclic enumeration of all vertices of $Q_{n,[k,l]}$ with only $\delta$ of these distance-2 steps. Such an enumeration is called a *tight enumeration*. A tight enumeration can be seen as a travelling salesman tour through all vertices of $Q_{n,[k,l]}$ of minimal total distance, where distances are measured by Hamming distance. An example of such tight enumeration is the above mentioned listing of all $k$-element subsets of $[n]$ by repeatedly exchanging a single element [171].

In [74] we also asked for which weight ranges $[k, l]$ does the graph $Q_{n,[k,l]}$ have a tight enumeration. Similarly as in Theorem 12, we positively resolved this question for all values of $k$ and $l$ except the cases covered by the generalized middle level conjecture (Conjecture 5 below), which is still open. The case $l = k$ was already covered in [171].

**Theorem 13** ([74]). *For any $n \geq 3$ there is a tight enumeration of the vertices of $Q_{n,[k,l]}$ in the following cases:*

*(i) If $0 = k < l \leq n$ or $0 \leq k < l = n$.*

*(ii)* If $1 \le k < l \le n$ and $l - k \ge 2$ is even.

*(iiia)* If $1 \le k < l \le n - 1$ and $l - k = 1$.

*(iiib)* If $1 \le k < l \le \lceil n/2 \rceil$ or $\lfloor n/2 \rfloor \le k < l \le n - 1$, and $l - k \ge 3$ is odd.

*(iv)* If $1 \le k < l \le n - 1$ and $l - k \ge 3$ is odd, under the additional assumption that $Q_{2m+1,[m-c,m+1+c]}$, $c := (l - k - 1)/2$, has a Hamilton cycle for all $m = c, c + 1, \ldots, (\min(k + l, 2n - k - l) - 1)/2$.

Proofs of Theorems 12 and 13 (cases *(i)* and *(ii)*) are based on 'trimming' the reflected Gray code to consecutive levels in $Q_n$, as well as 'gluing' saturating cycles or tight enumerations together (other cases). From an algorithmic point of view, this approach leads to corresponding loopless algorithms that generate each bitstring of a saturating cycle or a tight enumeration in time $O(1)$ (cases *(i)* and *(ii)*) or in time $O(1)$ on average (case *(iii)*).

### 1.7.2 Gray codes and symmetric chains

The middle levels problem can be generalized for middle $2l$ levels as follows, suggested independently by Savage [148], by Gregor and Škrekovski [81], and by Shen and Williams [156].

**Conjecture 5.** *For any $n \ge 1$ and $1 \le l \le n + 1$, the graph $Q_{2n+1,[n+1-l,n+l]}$ has a Hamiltonian cycle.*

The special case $\ell = 1$ of Conjecture 5 is the middle two levels problem resolved by Mütze [132], as mentioned before. The other boundary case $\ell = n + 1$ is solved by the binary reflected Gray code. Moreover, the cases $\ell = n$ and $\ell = n-1$ were settled in [53, 123] and [81], respectively.

In our work [70] we solve the case $\ell = 2$ of Conjecture 5, i.e., we construct a cyclic listing of all bitstrings of length $2n+1$ with Hamming weights in the interval $[n-1, n+2]$.

**Theorem 14** ([70])**.** *For any $n \ge 1$, the subgraph of $Q_{2n+1}$ induced by the middle four levels has a Hamiltonian cycle.*

The proof of Theorem 14 uses similarly involved techniques as the proof of the middle two levels problem [132, 75]. Combining Theorem 14 with Theorem 12 shows more generally that the subgraph of the $n$-cube induced by any four consecutive levels has a saturating cycle.

As another partial result towards Conjecture 5, we show that the subgraph of the $(2n + 1)$-cube induced by the middle $2\ell$ levels has a cycle factor. A *cycle factor* is a collection of disjoint cycles which together visit all vertices of the graph. In particular, a Hamilton cycle is a cycle factor consisting only of a single cycle.

**Theorem 15** ([70])**.** *For any $n \ge 1$ and $1 \le \ell \le n + 1$, the subgraph of $Q_{2n+1}$ induced by the middle $2\ell$ levels has a cycle factor.*

Our proof of Theorem 15 uses a result of Shearer and Kleitman [155] on existence of two edge-disjoint symmetric chain decompositions in $Q_n$ for any $n \geq 2$. A *symmetric chain* in $Q_n$ is a path $(x_k, x_{k+1}, \ldots, x_{n-k})$ in $Q_n$ where $x_i$ is from level $i$ for all $k \leq i \leq n-k$, and a *symmetric chain decomposition*, or SCD for short, is a partition of the vertices of $Q_n$ into symmetric chains. Two SCDs are *edge-disjoint* if the corresponding paths in the graph $Q_n$ are edge-disjoint, i.e., if there are no two consecutive vertices in one chain of the first SCD that are also contained in one chain of the second SCD.

Apart from building Gray codes, symmetric chain decompositions are used in construction of rotation-symmetric Venn diagrams for $n$ sets when $n$ is a prime number [84, 147], and they also appear in the solution the Littlewood-Offord problem on sums of vectors [12].

This approach motivates the search for a large collection of pairwise edge-disjoint SCDs in the $n$-cube. So far, we know that there $Q_n$ has four pairwise edge-disjoint SCDs except in few cases for small dimension $n$. This extends the only previously known construction of SCD that dates back to 50's [15].

**Theorem 16** ([70]). *$Q_n$ contains four pairwise edge-disjoint SCDs for any even $n \geq 6$, for $n = 7$, and for any odd $n \geq 13$.*

Note that four edge-disjoint SCDs are best possible for $Q_6$, as they use up all edges incident with the middle level. The cases $n = 9$ and and $n = 11$ are excluded in the statement as our proof technique does not cover these cases. However, we believe that the statement holds in these cases as well. In fact, we conjecture that the $Q_n$ has $\lfloor n/2 \rfloor + 1$ pairwise edge-disjoint SCDs, but so far we only know that this holds for $n \leq 7$. Clearly, finding this many edge-disjoint SCDs would be best possible, as they use up all middle edges of the cube.

Recently, Däubel et al. [35] showed that $Q_n$ has five edge-disjoint SCDs for any $n \geq 90$. There is also a stronger concept of so called *orthogonal symmetric chain decompositions* introduced by Shearer and Kleitman [155] and studied recently by Spink [164].

### 1.7.3 Long paths and cycles in hypercubes with faulty vertices

Applications of the hypercube in the theory of interconnection networks inspired many questions related to its robustness. In particular, if some faulty (or busy) vertices $F \subseteq V(Q_n)$ and all incident edges are removed from $Q_n$, is there a cycle in the remaining graph, denoted by $Q_n - F$, which covers 'almost' all vertices? And how many vertices in the worst-case can be removed? This is another approach to Gray code type problems.

Clearly, if all vertices of $F$ are from the same bipartite class of $Q_n$, the length of any cycle in $Q_n - F$ cannot exceed $2^n - 2|F|$. This leads to the following definition. A cycle of length at least $2^n - 2|F|$ in $Q_n - F$ is called a *long F-free cycle in $Q_n$*. Let $f(n)$ be the maximum integer such that $Q_n - F$ has a long $F$-free cycle for every set $F$ of at most $f(n)$ vertices in $Q_n$.

The study of this parameter has a numerous literature. Firstly, Chan and Lee [24] showed that $f(n) \geq (n-1)/2$. Then, Yang et al. [182] improved it to $f(n) \geq n-2$, and Tseng [173] to $f(n) \geq n-1$. Next, Fu [63] significantly increased it to $f(n) \geq 2n-4$ for

$n \geq 3$, and Castañeda and Gotchev [23] strengthened it further to $f(n) \geq 3n - 7$ for $n \geq 5$. Fink and Gregor [59] obtained the first quadratic lower bound $f(n) \geq n^2/10 + n/2 + 1$ for $n \geq 15$.

On the other hand, Koubek [114] and independently Castañeda and Gotchev [23] noticed that for every $n \geq 4$ there is a set $F$ of $\binom{n}{2} - 1$ vertices such that $Q_n - F$ contains no cycle of length at least $2^n - 2|F|$, so $f(n) \leq \binom{n}{2} - 2$. An example of a such set $F$ consists of all but one vertex of weight 2. Indeed, since all vertices of $F$ have even weight, any long $F$-free cycle in $Q_n$ must visit all the remaining vertices of even weight. Namely, it has to visit the vertex $\mathbf{0} = (0, \ldots, 0)$ and some vertex of weight 4, which is clearly impossible as they are in different 2-connected components of $Q_n - F$.

From the previous results it follows that the above upper bound is sharp for $n = 4$ [63] and for $n = 5$ [23]. Castañeda and Gotchev [23] conjectured that it is sharp for all $n \geq 4$.

**Conjecture 6** ([23]). *For every $n \geq 4$ it holds $f(n) = \binom{n}{2} - 2$.*

In [60] we confirm Conjecture 6 by the following result.

**Theorem 17** ([60]). *For every set $F$ of at most $\binom{n}{2} - 2$ vertices in $Q_n$ and $n \geq 4$, the graph $Q_n - F$ contains a cycle of length at least $2^n - 2|F|$.*

To prove Theorem 17, we needed to consider a modification of this problem for long paths with prescribed endvertices. Similarly as above, a path in $Q_n - F$ between vertices $u$ and $v$, and of length at least $2^n - 2|F| - 2$ is called a *long $F$-free $uv$-path in $Q_n$*. Note that in case $u$ and $v$ are from different bipartite classes, the length of any long $F$-free $uv$-path is at least $2^n - 2|F| - 1$. Also note that in the case when $F \cup \{u, v\}$ is in the same bipartite class of $Q_n$, the length of any $uv$-path in $Q_n - F$ cannot exceed $2^n - 2|F| - 2$, and hence a long $F$-free $uv$-path has optimal length.

Fu [64] showed that $Q_n - F$ contains a long path between any two vertices if $|F| \leq n - 2$ and $n \geq 3$. To improve this result for larger sets $F$, one needs to introduce additional conditions on the neighbors of prescribed endvertices. Kueng et al. [118] strengthened the number of tolerable faults to $|F| \leq 2n - 5$ under the condition that the minimal degree of $Q_n - F$ is at least 2. Fink and Gregor [59] showed that a much weaker condition is both necessary and sufficient for sets $F$ with $|F| \leq 2n - 4$. Namely, for every two vertices $u$ and $v$ of $Q_n - F$, there exists a long $F$-free $uv$-path in $Q_n$ if and only if $N(u) \not\subseteq F \cup \{v\}$ and $N(v) \not\subseteq F \cup \{u\}$, where $N(x)$ denotes the set of neighbors of a vertex $x$ in $Q_n$.

In [60] we show that $F$ can be as large as $f(n+1)/2$ if both prescribed endvertices have only few neighbors in $F$.

**Theorem 18** ([60]). *For every set $F$ of at most $(n^2 + n - 4)/4$ vertices in $Q_n$ and $n \geq 5$, the graph $Q_n - F$ contains a path of length at least $2^n - 2|F| - 2$ between every two vertices such that each of them has at most 3 neighbors in $F$.*

The general difficulty with quadratic bounds on $|F|$ in Theorems 17 and 18 is that the hypercube cannot be always split into subcubes so that the bounds hold in each subcube.

Thus, the standard induction technique fails. We introduce up to our knowledge a new technique of so called *potentials* which allows us to effectively deal with such situations.

From Theorem 17 it follows that the decision problem whether the hypercube $Q_n$ for the given set $F$ of faulty vertices contains an $F$-free cycle has a trivial answer if $|F| \leq \binom{n}{2} - 2$. On the other hand, Dvořák and Koubek [49] showed that this problem is NP-hard if $|F|$ is unbounded. Moreover, they [49] presented a function $\phi(n) = \Theta(n^6)$ such that the problem remains NP-hard even if $|F| \leq \phi(n)$. Later, Dvořák et al. [47] showed that this problem remains NP-hard even for a certain function $\phi(n) = \Theta(n^4)$. Furthermore, Dvořák and Koubek [48] described a polynomial algorithm for the similar decision problem of long $F$-free paths between given vertices in $Q_n$ if $|F| \leq n^2/10 + n/2 + 1$.

Li et al. [122] considered a variant of the long path problem in the hypercube $Q_n$ with $f$ faulty vertices for multiple paths. They showed that for any integer $k$ with $1 \leq k \leq n-1$ and any two sets $S$ and $T$ of $k$ fault-free vertices in different partite sets of $Q_n$ ($n \geq 2$), if $f \leq 2n - 2k - 2$ and each fault-free vertex has at least two fault-free neighbors, then there exist $k$ fully disjoint fault-free paths linking $S$ and $T$ which contain at least $2n - 2f$ vertices. Note that their bound on the number $f$ of faulty vertices is linear whereas we have quadratic bounds in Theorems 17 and 18.

## 1.8 Linear extension diameter

As for other classes of combinatorial Gray codes, Pruesse and Ruskey [143] considered the problem of generating all linear extensions of a given poset $\mathcal{P}$ by adjacent transpositions. This corresponds to finding a Hamiltonian path in a so called linear extension graph of $\mathcal{P}$, which they first introduced.

A *linear extension* $L$ of a poset (a partially ordered set) $\mathcal{P}$ is a linear order on the elements of $\mathcal{P}$ that preserves the relation from $\mathcal{P}$; that is, $x \leq_P y$ implies $x \leq_L$ for all $x, y \in \mathcal{P}$. By an *adjacent transposition* in $L$ we mean swapping the order of two consecutive elements in $L$. The *linear extension graph* $G(\mathcal{P})$ of $\mathcal{P}$ has all its linear extensions as vertices, two of them being adjacent whenever they differ in a single adjacent transposition. For example, the linear extension graph of an antichain is the permutahedron.

An explicit study of structural properties of linear extension graphs was started by Björner and Wachs [11] and by Reuter [145]; see also [134]. Among its properties, let us mention that the linear extension graph of any poset is a *partial cube*; that is, an isometric subgraph of a hypercube. Incomparable pairs of the poset correspond to directions in the minimal hypercube into which the linear extension graph isometrically embeds, which also correspond to the $\Theta$-classes of the so called *Djoković-Winker relation* $\Theta$.

The *linear extension diameter* of a finite poset $\mathcal{P}$, denoted by $\mathrm{led}(\mathcal{P})$, is the diameter of $G(\mathcal{P})$. It equals the maximum number of pairs of $\mathcal{P}$ that appear in a reversed order in two linear extensions of $\mathcal{P}$. In other words, it is the maximum number of incomparable pairs in a 2-dimensional extension of $\mathcal{P}$. The linear extension diameter was introduced by Felsner and Reuter [57] who investigated its relation to other poset parameters such as height, width, fractional dimension and other properties. They also conjectured that the

linear extension diameter of the Boolean lattice $\mathcal{B}_n$ is

$$\text{led}(\mathcal{B}_n) = 2^{2n-2} - (n+1)2^{n-2}.$$

Felsner and Massow [56] proved this conjecture by an (elegant) combinatorial argument and characterized all diametral pairs of linear extensions of $\mathcal{B}_n$. They are formed by a reversed lexicographical order with respect to some permutation $\sigma$ of atoms (shortly $\sigma$-revlex) and a $\overline{\sigma}$-revlex order where $\overline{\sigma}$ denotes the reverse of $\sigma$. Moreover, they extended this characterization to a more general class of downset lattices of 2-dimensional posets.

Brightwell and Massow [16] show that determining the linear extension diameter of a given poset is NP-complete problem. Interestingly, diametral pairs can be used to obtain optimal drawings of the poset [56]. For further properties of linear extension graphs and the linear extension diameter we refer to a dissertation of Massow [125] and the references within.

In our work [61] we determine the linear extension diameter of the subposet $\mathcal{B}_n^{1,k}$ of the Boolean lattice $\mathcal{B}_n$ induced by the 1st and $k$th levels and we describe an explicit construction of all diametral pairs of linear extensions. This partially solved a question of Felsner and Massow [56] on diametral pairs of subposets of the Boolean lattice induced by two levels.

**Theorem 19** ([61]). *For every* $1 < k \le n$,

$$\text{led}(\mathcal{B}_n^{1,k}) = \binom{\binom{n}{k}}{2} + 2\binom{n}{k+1} + \binom{n}{2} - \sum_{\substack{i=k \\ i \equiv n \ (\text{mod } 2)}}^{n-2} \binom{i}{k}.$$

Almost all diametral pairs are formed by two linear extensions that reverse all pairs of atoms, all pairs of $k$-sets and certain pairs of an atom and a $k$-set that correspond to a minimal vertex cover of so called *dependency graph*. For a precise characterization of all diametral pairs see [61]. Our approach in fact allows to determine the maximal distance between two linear extensions with fixed orders of atoms in terms of the minimal size of a vertex cover of the respective dependency graph. The concept of dependency graphs is new and may be of independent interest.

# Summary

The thesis is compiled as a collection of 12 publications from years between 2012 and 2018 selected from the total of 43 publications since 2000. It is divided into the introduction with commentary as the first chapter and 7 following chapters that correspond to combinatorial structures studied in these publications: queue layouts, level-disjoint partitions, incidence colorings, distance magic labelings, parity vertex colorings, Gray codes, and linear extension diameter.

In these publications several open problems and conjectures have been solved, or previously known results were significantly improved. The improvements were often achieved by novel constructions or by development of new techniques. In some of these publications, new concepts have been introduced and explored. The most interesting results are listed in the synopsis of the thesis.

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