Practical sessions for Introduction to Complexity and Computability - 2

October 18th, 2018

Exercises

1. Turing machines: motivation, definition, computation, recognized language, examples.

   (e) (previous HW) Let $J(n)$ for $n \geq 1$ be the last remaining person when n people gather in a circle and every second person goes out, started by skipping the first person. For example $J(5) = 3$. Construct TM that computes $J(n)$ for $n \geq 1$ (in binary encoding).

2. Modifications of Turing machines.

   (a) (previous HW) Show that for every TM there is an equivalent TM with only a right-sided tape. Assume that the first cell contains a special symbol # which cannot be rewritten and the head cannot move left from it.

   (b) Show that for every TM there is an equivalent TM that performs at most two operations in each step (out of three operations: change state, move head, write on tape); that is, it has no instructions of type $(p, a) \to (q, b, M)$ where $p \neq q$, $a \neq b$, and $M \in \{L, R\}$).

   (c) Show that for every TM there is an equivalent TM that performs at most one operation in each step (see above).

   (d) Show that for every TM there is an equivalent TM (up to encoding of the input) with only a single symbol alphabet (aside from λ).

   (e) Show that for every multi-tape TM there is an equivalent (single-tape) TM.

   (f) Show that every TM is equivalent to some pushdown automata with two stacks (and without the input tape).

   (g) Show that for every TM there is an equivalent TM with only two active (i.e. nonaccepting) states. [*****]

3. Decidable and partially decidable languages, Post's theorem, Gödel number, universal TM.

   (a) Show that the class of partially decidable languages is closed under union, intersection, concatenation, Kleene star. Show that the class of decidable languages is moreover closed under complement.

   (b) Find encodings between decision problems, languages, sets of natural numbers, real numbers (between 0 and 1).

   (c) What happens if we run universal TM U on itself, i.e. what is the result of $U(\langle U, U\rangle)$?

   (d) Let $h\colon \mathbb{N} \to \mathbb{N}$ be a Turing computable permutation. Show that the set $D = \{i \mid i \in L_{h(i)}\}$ is partially decidable but not decidable. (Recall that $L_e = L(M_e)$, i.e. L_e is the language (corresponding to a set of natural numbers) accepted by TM with Gödel number e.)

   (e) Show that the language $L_H = \{\langle M, x\rangle \mid M(x)\downarrow\}$ is partially decidable but not decidable.

   (f) Show that the language $L_{\overline{Z}} = \{w \in \{0,1\}^* \mid M_w(x)\downarrow \text{ for some input } x\}$ is partially decidable but not decidable.

   (g) Show that the language $L_{diff} = \{\langle a, b\rangle \mid M_a(x)\downarrow, M_b(x)\downarrow \text{ and } M_a(x) \neq M_b(x) \text{ for some input } x\}$ is partially decidable but not decidable.

Homework

Problem 3(d) (for 1 point) and one of Problems 3(f), 3(g) depending on your choice (for 2 points).