Practical sessions for Introduction to Complexity and Computability - 3

November 1st, 2018

Exercises

1. Decidable and partially decidable languages, Post's theorem, Gödel number, universal TM.
   (a) (previous HW) Let $h\colon \mathbb{N} \to \mathbb{N}$ be a Turing computable permutation. Show that the set $D = \{i \mid i \in L_{h(i)}\}$ is partially decidable but not decidable. (Recall that $L_e = L(M_e)$, i.e. L_e is the language (corresponding to a set of natural numbers) accepted by TM with Gödel number e.)
   (b) Show that the language $L_H = \{\langle M, x\rangle \mid M(x)\downarrow\}$ is partially decidable but not decidable.
   (c) (previous HW) Show that the language $L_{\overline{Z}} = \{w \in \{0,1\}^* \mid M_w(x)\downarrow \text{ for some input } x\}$ is partially decidable but not decidable.
   (d) (previous HW) Show that the language $L_{diff} = \{\langle a, b\rangle \mid M_a(x)\downarrow, M_b(x)\downarrow \text{ and } M_a(x) \neq M_b(x) \text{ for some input } x\}$ is partially decidable but not decidable.
   (e) Prove or disprove that the class of decidable (resp. partially decidable) languages is closed under symmetric difference. A symmetric difference of sets A and B is the set $A \triangle B = (A - B) \cup (B - A)$.
2. Enumeration of (partially) decidable languages by computable functions.
   (a) Show that for every partially decidable language B, the language $A = \{x \mid \exists y\ \langle x, y\rangle \in B\}$ is also partially decidable.
   (b) Let $D \subseteq \{0,1\}^*$ be finite. Show that the language $\{x \mid D \subseteq L_x\}$ is partially decidable.
   (c) Show that for every $x \in \mathbb{N}$ the language $A_x = \bigcup_{y \in L_x} L_y$ is enumerable by some computable function.
3. Reducibility and completeness.
   (a) Let A be a partially decidable language. Show that A is decidable if $A \leq_m \overline{A}$.
   (b) Let A be a decidable language. Show that $A \leq_m C$ for every nontrivial language C (i.e. $C \neq \emptyset$ and $C \neq \{0,1\}^*$).
   (c) Find examples of languages A, B such that $A \leq_m B$ and $A \not\leq_1 B$.
   (d) Show that $K \leq_1 Tot$ and $\overline{K} \leq_1 Tot$ where $K = \{x \mid M_x(x)\downarrow\}$ and $Tot = \{x \mid L_x = \{0,1\}^*\}$.
   (e) Let $D \subseteq \{0,1\}^*$ be finite. Show that $K \leq_1 \{x \mid D \subseteq L_x\}$ and $\overline{K} \leq_1 \{x \mid L_x \subseteq D\}$ where K is defined above.
4. The "s-m-n" theorem. Find computable bijections $f\colon \mathbb{N}^2 \to \mathbb{N}$, $g\colon \mathbb{N} \to \mathbb{N}$ such that
   (a) $\varphi_{f(x,y)}(z) \simeq \varphi_x(z) + \varphi_y(z)$ for every $x, y, z \in \mathbb{N}$,
   (b) $\operatorname{dom} \varphi_{g(x)} = \{x\}$ for every $x \in \mathbb{N}$,
   (c) $\operatorname{dom} \varphi_{g(x)} = \{0, 1, \ldots, x\}$ for every $x \in \mathbb{N}$.
5. Rice's theorem. Show that the following problems are undecidable.
   (a) Does the given program always halt (i.e. for every input)?
   (b) Does the given program return 0 for some input?
   (c) Do the given two programs compute the same arithmetic function?

Homework

Problems 1(e), 3(a) (both for 1 point), and one of 3(d), 3(e) depending on your choice (for 2 points).