Practical sessions for Introduction to Complexity and Computability - 4

November 15th, 2018

Exercises

1. (previous HW) Prove or disprove that the class of decidable (resp. partially decidable) languages is closed under symmetric difference. A symmetric difference of sets A and B is the set $A \triangle B = (A - B) \cup (B - A)$.

2. Reducibility and completeness.
   (a) (previous HW) Let A be a partially decidable language. Show that A is decidable if $A \leq_m \overline{A}$.
   (b) Let A be a decidable language. Show that $A \leq_m C$ for every nontrivial language C (i.e. $C \neq \emptyset$ and $C \neq \{0,1\}^*$).
   (c) Find examples of languages A, B such that $A \leq_m B$ and $A \not\leq_1 B$.
   (d) (previous HW) Show that $K \leq_1 Tot$ and $\overline{K} \leq_1 Tot$ where $K = \{x \mid M_x(x) \downarrow\}$ and $Tot = \{x \mid L_x = \{0,1\}^*\}$.
   (e) (previous HW) Let $D \subseteq \{0,1\}^*$ be finite. Show that $K \leq_1 \{x \mid D \subseteq L_x\}$ and $\overline{K} \leq_1 \{x \mid L_x \subseteq D\}$ where K is defined above.

3. The "s-m-n" theorem. Find injective computable total functions $f\colon \mathbb{N}^2 \to \mathbb{N}$, $g\colon \mathbb{N} \to \mathbb{N}$ such that
   (a) $\varphi_{f(x,y)}(z) \simeq \varphi_x(z) + \varphi_y(z)$ for every $x, y, z \in \mathbb{N}$,
   (b) dom $\varphi_{g(x)} = \{x\}$ for every $x \in \mathbb{N}$,
   (c) dom $\varphi_{g(x)} = \{0, 1, \ldots, x\}$ for every $x \in \mathbb{N}$.

4. Rice's theorem. Show that the following problems are undecidable.
   (a) Does the given program always accept (i.e. for every input)?
   (b) Does the given program return 0 for some input?
   (c) Do the given two programs compute the same arithmetic function?

5. Let A be a finite problem (i.e. the language of its positive instances is finite). Determine for which functions f the class TIME($f(n)$) contains A. Is A in the class **P**?

6. For a language A and $k \in \mathbb{N}$ let $A_k = \{x \in A;\ |x| = k\}$ be the restriction of A to inputs of size k. Prove or disprove that for any language A, if $A_k \in \mathbf{P}$ for every k, then $A \in \mathbf{P}$.

7. Consider a Turing machine M running in the following time $f(n)$ where $k \in \mathbb{N}$. In which cases the language $L(M)$ belongs to **P**? In which cases it belongs to **EXP**? Explain why.
   (a) $f(n) = n^k$
   (b) $f(n) = k^n$
   (c) $f(n) = n^n$
   (d) $f(n) = k^{k \log n}$
   (e) $f(n) = n^{\log \log n}$
   (f) $f(n) = k^{n^k}$
   (g) $f(n) = k^{n^n}$
   (h) $f(n) = n^{n^k}$
   (i) $f(n) = n^{n^n}$

Homework

Problems 5, 6 (both for 1 point), and 7 (all subproblems, up to 2 points).