Practical sessions for Introduction to Complexity and Computability - 6

December 13th, 2018

Exercises

1. (previous HW) Let A be the language over the alphabet $\{(,)\}$ of all words that have properly nested parentheses. For example, $(())$ is in A, but $)($ is not. Let B be the language over the alphabet $\{(,),[,]\}$ of all words that have properly nested parentheses and brackets. For example, $([()]()[])$ is in B but $([)]$ is not. Show that both A and B are in the class $\mathbf{L}$.

2. (previous HW) For each pair of complexity classes below decide if there is any inclusion between them (by using theorems and propositions from the lecture). Note that some relations may not be known (by our knowledge from the lecture).

   (a) $\mathrm{SPACE}(n)$ and $\mathrm{SPACE}(n^3)$

   (b) $\mathrm{SPACE}(n^3)$ and $\mathrm{TIME}(2^{n^4})$

   (c) $\mathrm{TIME}(2^{n^3})$ and $\mathrm{NSPACE}(n \log n)$

   (d) $\mathrm{NSPACE}(n \log n)$ and $\mathrm{NTIME}(n \log n)$

   (e) $\mathrm{NTIME}(n \log n)$ and $\mathrm{SPACE}(n)$

   (f) $\mathrm{SPACE}(n)$ and $\mathrm{TIME}(2^{n^3})$

   (g) $\mathrm{SPACE}(n^3)$ and $\mathrm{NSPACE}(n \log n)$

   (h) $\mathrm{TIME}(2^{n^3})$ and $\mathrm{NTIME}(n \log n)$

   (i) $\mathrm{NSPACE}(n \log n)$ and $\mathrm{SPACE}(n)$

   (j) $\mathrm{NTIME}(n \log n)$ and $\mathrm{SPACE}(n^3)$

3. (previous HW) In the cases above where there is some inclusion, decide if it is a proper inclusion (i.e. $\subsetneq$) (by using theorems and propositions from the lecture). Note that it may not be known (by our knowledge from the lecture).

4. Consider the following argument. The problem SAT of deciding whether a given CNF (conjunctive normal form) formula is satisfiable is $\mathbf{NP}$-complete. Every CNF formula can be easily transformed to an equivalent DNF (disjunctive normal form) formula using distribution laws:

$$\varphi \wedge (\psi \vee \chi) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$
$$(\psi \vee \chi) \wedge \varphi \leftrightarrow (\psi \wedge \varphi) \vee (\chi \wedge \varphi)$$

   Then we can decide whether the obtained DNF formula is satisfiable in polynomial time. Thus SAT is in $\mathbf{P}$, and consequently $\mathbf{P} = \mathbf{NP}$. Is it a correct argument? Explain why (not)?

5. Consider the following argument. Suppose for a contradiction that $\mathbf{P} = \mathbf{NP}$. Then the problem SAT of deciding whether a given CNF formula is satisfiable is in $\mathbf{P}$, so there exists k such that $SAT \in \mathrm{TIME}(n^k)$ where n is the length of the formula. Since every language in $\mathbf{NP}$ is polynomially reducible to SAT (i.e. in time $O(n^l)$ for some l), it follows that every language in $\mathbf{NP}$ is solvable in time $O((n^l)^k)$, and consequently $\mathbf{NP} \subseteq \mathrm{TIME}(n^{kl})$. However, time hierarchy theorem implies that there exists a language $A \in \mathrm{TIME}(n^{kl+1}) \setminus \mathrm{TIME}(n^{kl})$, which gives a contradiction

$$\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathrm{TIME}(n^{kl}) \subsetneq \mathrm{TIME}(n^{kl+1}) \subseteq \mathbf{P}.$$

   Therefore $\mathbf{P} \neq \mathbf{NP}$. Is it a correct argument? Explain why (not)?

6. Show that the following problem CLIQUE is **NP**-complete by polynomial reduction of SAT.

   Instance (CLIQUE): A graph G and an integer $k \geq 1$.

   Question (CLIQUE): Does G contain a clique of size k as a subgraph? (A *clique* of size k is the complete graph on k vertices.)

7. Find polynomial reductions between CLIQUE, VC, and IND where VC (vertex cover) and IND (independent set) are the following problems.

   Instance (the same for VC and IND): A graph G and an integer $k \geq 1$.

   Question (VC problem): Does G contain a vertex cover of size k? (A *vertex cover* of a graph G is a set of vertices in G that are incident to all edges of G.)

   Question (IND problem): Does G contain an independent set of size k? (An *independent set* in a graph G is a set of vertices in G with no edges between them.)

8. Show that the problems DOM and CC are **NP**-complete by polynomial reduction of VC.

   Instance (DOM): A graph G and an integer $k \geq 1$.

   Question (DOM): Does G contain a dominating set of size k? (A set D of vertices in a graph G is *dominating* if every vertex of G that is not in D has a neighbor in D.)

   Instance (CC): A directed graph G and an integer $k \geq 1$.

   Question (CC): Does G contain a cycle cover of size k? (A *cycle cover* of G is a set of vertices containing at least one vertex from each cycle in G.)

9. We know that the following problem HC (Hamiltonian cycle) is **NP**-complete.

   Instance (HC): An (undirected) graph G.

   Question (HC): Does G have a Hamiltonian cycle, i.e. a cycle through all the vertices?

   Show that also the following problems DHC, HP, HP+ are **NP**-complete.

   Instance (DHC): A directed graph G.

   Question (DHC): Does G have a directed Hamiltonian cycle?

   Instance (HP): A (undirected) graph G.

   Question (HP): Does G have a Hamiltonian path, i.e. a path through all the vertices?

   Instance (HP+): A (undirected) graph G and two vertices s, t.

   Question (HP+): Does G have a Hamiltonian path between s and t?

10. Show that the following problems IP (integer programming) and BIP (binary integer programming) are **NP**-hard by polynomial reductions of some of the above mentioned problems.

    Instance (the same for IP and BIP): An $m \times n$ matrix A over integers and a vector b of m integers.

    Question (IP): Is there a vector x of n integers such that $Ax \geq b$?

    Question (BIP): Is there a vector $x \in \{0,1\}^n$ such that $Ax \geq b$?

Homework

Exercises 5, 7, 8 (choose one of the problems DOM or CC), 9 (choose two problems from DHC, HP, HP+). If the task is to show that the problem is NP-complete, do not forget to also show that it belongs to NP. Each exercise is for 1 point, totally for 4 points.