Introduction to Artificial Intelligence

English practicals 6: Probability in AI

Marika Ivanová

Department of Theoretical Computer Science and Mathematical Logic (KTIML)
Faculty of Mathematics and Physics

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Reasoning under uncertainty

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- Reasons for why we reason under uncertainty:
  1. **laziness:** modeling every detail is expensive
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- Example: deploy a network of smoke sensors to detect a fire in a building
  1. We are too lazy to model what, other than fire, triggers the sensors
  2. We do not know how exactly smoke triggers the sensors (smoke intensity,...)

Probability

- **Probability space** represents our uncertainty regarding a random experiment
- It consists of
  1. **Sample space** Ω - set of outcomes
  2. **Probability measure** P - real function of the subsets of Ω
- **Event** - a set of outcomes $A \subseteq \Omega$. $P(A)$ represents how likely is it that the experiment's actual outcome belongs to A.

The three axioms of probability

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Several simple consequences:

- $P(A) = 1 - P(\Omega \setminus A)$
- $P(\emptyset) = 0$
- if $A \subseteq B$ then $P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B) \leq P(A) + P(B)$
- ...

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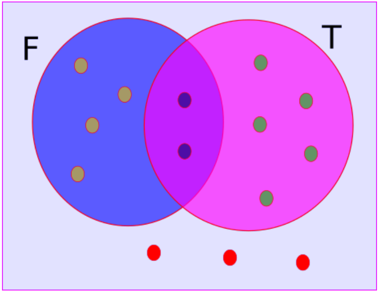
  $P(A \vee B) = P(A) + P(B)$
- **Inclusive events** can happen at the same time

  $P(A \vee B) = P(A) + P(B) - P(A \& B)$

Conditional probability - reasoning with partial information



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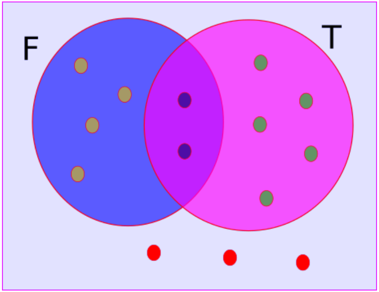

	F	$\neg F$	**Sum row**
T	2 2/14	5 5/14	7 7/14
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Sum column	6 6/14	8 8/14	

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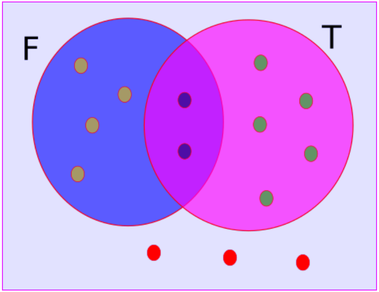

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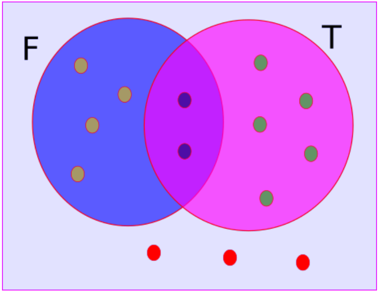

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Bayes rule

$$P(A|B)P(B) = P(B|A)P(A)$$

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Chain rule

$P(A_1\&\dots\&A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1,A_2)\dots P(A_k|A_1,\dots,A_{k-1})$

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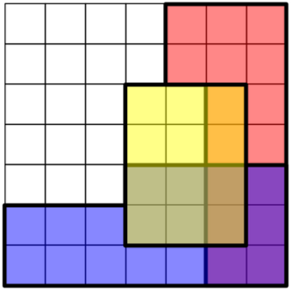
Example:



- Three events: R, B, and Y
- R and B are conditionally independent given Y
- but not conditionally independent given ¬Y (complement of Y)

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- Height and math skills are dependent events.

  After knowing that two people are 18 years old, there is no reason to think that taller is better in math (conditionally independent)

Exercises

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  $1/3$

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Company A supplies 40% of the computers sold and is late 5% of the time. Company B supplies 30% of the computers sold and is late 3% of the time. Company C supplies another 30% and is late 2.5% of the time. A computer arrives late - what is the probability that it came from Company A?

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$$\frac{0.05 * 0.4}{0.05 * 0.4 + 0.03 * 0.3 + 0.025 * 0.3} = 0.548$$

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$$\frac{(3/4)^3 * (1/3)}{(3/4)^3 * (1/3) + (1/2)^3 * (2/3)} = 0.6279$$

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