A new ultrafilter in $\omega^*$.  

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Outline

1. Motivation
2. Definitions and introduction
3. The method
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5. Why is the first question hard?
6. Summary
Homogeneity of topological spaces

Definition

A topological space is **homogeneous** if for any two points there is a homeomorphism taking one to the other.

- The real line $\mathbb{R}$ is homogeneous.
- A convergent sequence is **not** homogeneous.
Topological types

Definition

A topological type in a topological space $X$ is a subset $T \subseteq X$ which is invariant under homeomorphisms, that is: if $f : X \rightarrow X$ is a homeomorphism and $p \in T$ then $f(p) \in T$.

- The set of isolated points of a topological space is a type.
- The set of points with countable tightness is a type.
- The set of points of uncountable character is a type.
Nonhomogeneity of $\omega^*$

A natural question when given a topological space is: What are the topological types in the space? If you look at $\omega^*$, the space of free ultrafilters on the natural numbers, you can ask: what types of ultrafilters are there? The following is a selective summary of the results:

1953 Walter Rudin proved that, assuming the continuum hypothesis, $\omega^*$ is not homogeneous because it contains P-points.

1967 Z. Frolík proved that there are actually $2^\mathfrak{c}$-many types in $\omega^*$. His proof was combinatorial in nature and didn’t yield a “topological” definition of any of the types.

1978 Kenneth Kunen has proved in ZFC that $\omega^*$ contains weak P-points.

1982 Saharon Shelah proved that it is consistent with ZFC that there are no P-points in $\omega^*$.

1982 Jan van Mill found, in ZFC, 16 topological types of $\omega^*$. 
In his 1982 paper van Mill proved the existence of, among others, the following two types of points:

- Points, which are not in the closure of a (countable) discrete set. They are called *discretely untouchable*.
- Points, which are limit points of “essentially” only a single countable set. More precisely: whenever such a point is a limit point of two countable sets, then it is a limit point of their intersection.

**Question**

Can we have both properties at the same time?
Uniquely accessible points

Definition

A point \( p \) in a space \( X \) is **uniquely \( \omega \)-accessible**, if

(i) it is not a limit point of a countable discrete set, but
(ii) it **is** a limit point of a countable crowded set and
(iii) (uniqueness) whenever it is a limit point of two countable crowded sets it is a limit point of their intersection

Question

Do uniquely \( \omega \)-accessible points exist in \( \omega^* \)?

Question

Is there at least a subspace of \( \omega^* \) with a uniquely \( \omega \)-accessible point.
Suppose $p \in \omega^*$ is uniquely $\omega$-accessible and $S$ is a countable crowded (without isolated points) set whose limit point $p$ is. The following is immediate:

**Observation**

- $p$ is not a limit point of a nowhere dense subset of $S$.
- $p$ is not a limit point of a countable subset of $S^*$.
- Whenever $D, H$ are two dense subsets of $S$ then their intersection is nonempty.

These properties lead to the following definitions:
Remote points, P-points

**Definition (Fine, Gillmann)**

A point \( p \in X^* \) is a **remote point** iff it is not a limit point (in \( \beta X \)) of a nowhere dense subset of \( X \).

**Definition**

A point \( p \in X \) is a **weak P-point** iff it is not a limit point of a countable set.
Irresolvable spaces

**Definition (Hewitt)**

A crowded topological space is **irresolvable** if it does not contain disjoint dense subsets. It is **open hereditarily irresolvable** (OHI for short) if every open subspace is irresolvable.
The idea behind my constructions is based on the previous observation:

**Building blocks**

- Find the point in the Čech-Stone remainder of some countable space $S$ and then suitably embed this space into $\omega^*$ (we use a theorem of P. Simon which allows us to embed any extremally disconnected space of weight $\leq c$ into $\omega^*$ as a closed weak P-set.)

- The point must be a weak $P$-point of $S^*$.

- If the point is a remote point of $S^*$ then it cannot be in the closure of two disjoint open subsets of $S^*$ (van Douwen)

- If space $S$ is OHI, then using the previous point we get the required uniqueness.
The plan — theorems

The following theorem can actually be proved:

**Theorem**

ω* contains an uniquely ω-accessible point iff there is a countable, extremally disconnected OHI space S with a remote point, which is a weak P-point of S*.

The iff part says, that our method is completely general. Now If p is not a weak P-point then by a similar proof we have a weaker result:

**Theorem**

If there is a countable, extremally disconnected OHI space X with a remote point, then there is a countable subset S of ω* such that ω* \ S* contains an uniquely ω-accessible point.

This will give us an answer to the easier question.
Building OHI spaces

We need a method to build OHI spaces. E. Hewitt used the axiom of choice to construct irresolvable topologies by refining them. By modifying Hewitt's technique slightly we proved the following theorem which is sufficient:

**Theorem**

If $\tau$ is a zerodimensional topology on $X$ then there is a finer OHI, extremally disconnected topology on $X$.

So the plan is:

**Plan**

Take a countable zerodimensional space with a remote point and refine the topology to get an extremally disconnected OHI space with a remote point.
Strongly remote points

Unfortunately it is not so easy. When refining a topology one could get new nowheredense sets and thus lose the remote point. To overcome this difficulty we introduce the following stronger notion:

**Definition**

A point $p \in X^*$ is a **strongly remote point** iff it is not a limit point (in $\beta X$) of a subset of $X$ which has empty interior in $X$. A closed filter $\mathcal{F}$ on $X$ is **strongly remote** iff for each $N \subseteq X$ with empty interior there is $F \in \mathcal{F}$ disjoint from $N$.

Since a strongly remote filter will remain strongly remote in any finer topology we modify our plan and we now want to find a countable, crowded zerodimensional space with a strongly remote filter.
Constructing strongly remote points

We shall simultaneously construct the filter (with base $CL_\alpha$) and the topology $\tau_\alpha$ on $\omega$. We shall need the following objects:

**Crutches**

- An ideal $I$ on $\omega$ such that $\mathcal{P}(\omega)/I$ has hereditary independence $c$. (To keep the induction going.)
- The topology $\tau_\alpha$, generated by $\{C, \omega \setminus C : C \in CL_\alpha\} \cup I^*$.
- A family $CL_\alpha$ with $\{[C]_I : C \in CL_\alpha\}$ independent in $\mathcal{P}(\omega)/I$ and $|CL_\alpha| \leq \omega \cdot \alpha$. (To avoid isolated points.)
- An enumeration $\{A_\alpha : \alpha < c\}$ of $\mathcal{P}(\omega)$ (Potential witnesses to nonremotness.)
- An enumeration $(n_\alpha, G_\alpha)$ of $\{(n, G) : n \in G \& G \in I^*\}$ (To ensure that the $I^*$ part of the topology is zerodimensional)
Inductive conditions

We shall impose the following conditions for the inductive process:

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<tr>
<th>Conditions</th>
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<tr>
<td>(i) The $CL_\alpha$’s are increasing in $\alpha$ and “independent” in $\mathcal{P}(\omega)/I$.</td>
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<tr>
<td>(ii) For each $\alpha$ there is a finite intersection of sets from $CL_{\alpha+1}$ which misses $A_\alpha$ or $A_\alpha$ has nonempty interior in $\tau_{\alpha+1}$.</td>
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<td>(iii) There is an $C \in CL_{\alpha+1}$ with $n_\alpha \in C \subseteq G_\alpha$.</td>
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<td>(iv) The topology $\tau_\alpha$ is hausdorff for all $\alpha &lt; c$.</td>
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It is clear that if we can carry on up to $c$ we will have achieved what we needed. The crucial part is, of course, the successor step, but we shall not go into the details. The construction uses the fact that $CL_\alpha$ is small and because $\mathcal{P}(\omega)/I$ has hereditary independence $c$, we can always enlarge it.
First a definition (a bit technical):

**Definition (Kunen)**

A point \( p \in X \) is a \( \kappa \)-O.K. point iff for any sequence \( \langle U_n : n < \omega \rangle \) of neibourhoods of \( p \) there are \( \kappa \) neibourhoods \( \{ V_\alpha : \alpha < \kappa \} \) such that for each \( k < \omega \) the intersection of any \( k \) of them is contained in \( U_k \).

**Fact**

*If \( A \) is ccc and is disjoint from a closed \( \omega_1 \)-O.K. set \( F \) then the closure of \( A \) is also disjoint from \( F \).*
Question

Why is the first question considerably harder.

If the answer is true, then there is a countable crowded space with a weak P-point. Any countable crowded space $X$ is nowhere locally compact so $X^*$ is dense in $\beta X$. But then $X^*$ is ccc, so the weak P-point is not an O.K. point by the previous fact. So we would need a ZFC construction of weak P-points which are not O.K. Unfortunately the only known methods of construction of weak P-points yield variations of O.K. points. (Van Mill has some methods, but I have no idea how to adapt them to the problem.)
Main results

We have been able to find a positive answer to the second question:

**Theorem**

There is a countable crowded subset $S$ of $\omega^*$ and a point $p \in \omega^*$ which is uniquely $\omega$-accessible in $\omega^*$ minus the boundary of $S$.

We have also proved the following equivalence

**Theorem**

$\omega^*$ contains an uniquely $\omega$-accessible point iff there is a countable extremally disconnected OHI space $X$ with a remote point which is a weak P-point of $X^*$. This point cannot be an O.K. point of $X^*$. 
The first question remains open:

**Question**

Does $\omega^*$ contain an uniquely $\omega$-accessible point?
References


van Mill, J., **Sixteen types in** $\beta\omega - \omega$, Topology Appl. **13** (1982), 43–57.


Thank you ...

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