Co-Ideal Properties (Definitions)

We say that a co-ideal $\mathcal{H}$ has property

SEL iff for each disjoint partition $\{A_n : n < \omega\}$ of the set $A \in \mathcal{H}$ consisting of sets **not in** $\mathcal{H}$, there is a set $B \in \mathcal{H}$ which meets each A_n in a **one point** set.

Q iff for each disjoint partition $\{A_n : n < \omega\}$ of the set $A \in \mathcal{H}$ consisting of **finite** sets, there is a set $B \in \mathcal{H}$ which meets each A_n in a **one point** set.

Dg iff for each $\subseteq$-descending chain $\{A_n : n < \omega\}$ of sets from $\mathcal{H}$ with $A_n \setminus A_{n+1}$ *not* **in** $\mathcal{H}$, there is a *diagonal* $B \in \mathcal{H}$ (i.e. $(\forall n \in B)(B \cap (n, \infty) \subseteq A_n)$).

$\mathbf{Q}_{dg}$ iff for each $\subseteq$-descending chain $\{A_n : n < \omega\}$ of sets from $\mathcal{H}$ with $A_n \setminus A_{n+1}$ **finite**, there is a diagonal $B \in \mathcal{H}$.

P iff for each disjoint partition $\{A_n : n < \omega\}$ of the set $A \in \mathcal{H}$ consisting of sets **not in** $\mathcal{H}$, there is a set $B \in \mathcal{H}, B \subseteq A$ which meets each A_n in a **finite** set.

Ramsey iff for each $A \in \mathcal{H}$ and each coloring $\chi : [A]^2 \to 2$ of A by two colors, there is a homogeneous set $B \in \mathcal{H}$ (i.e. $|\chi''[B]^2| = 1$).

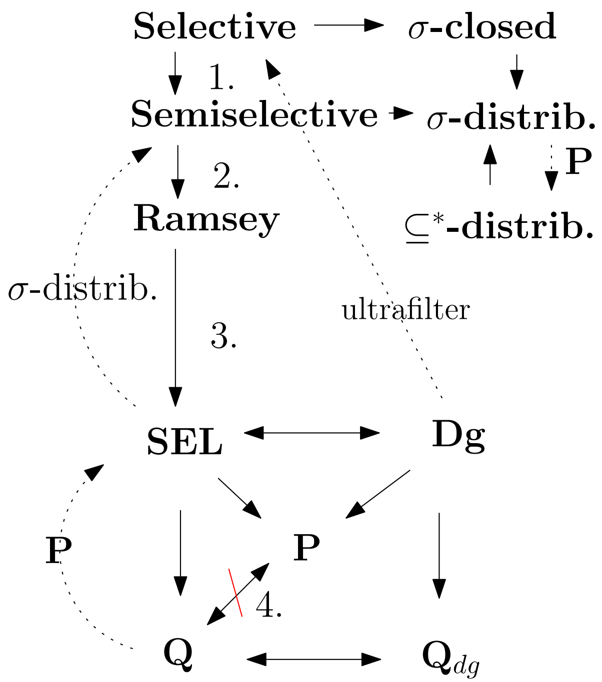
σ-distributive iff the boolean algebra $\mathcal{P}(\omega)/\mathcal{I}_{\mathcal{H}}$ is $(\omega, \cdot, 2)$-distributive.

σ-closed iff the boolean algebra $\mathcal{P}(\omega)/\mathcal{I}_{\mathcal{H}}$ is σ-closed.

$\subseteq^*$-distributive iff the intersection of a countable system of dense open subsets of the partial order $(\mathcal{H}, \subseteq^*)$ is dense.

Selectivity (Definitions)

We say that a co-ideal $\mathcal{H}$ is **SEMISELECTIVE** if it has property **Q** and is **$\subseteq^*$-distributive**. If each $\subseteq$-descending chain $\{A_n : n < \omega\} \subseteq \mathcal{H}$ has a diagonal it is called **SELECTIVE** (Mathias: Happy Family).



Examples

1. If $\mathcal{I} = \{A \subseteq \omega \times \omega : (\exists f \in \omega^\omega)(A \leq f)\}$, then I^+ (also denoted by $\mathcal{H}_{\mathcal{I}}$) is semiselective but not selective (where we take $A \leq f$ to mean $(\forall n < \omega)(A \cap \{n\} \times \omega \subseteq f(n))$).

2. Suppose $\{\mathcal{A}_n : n < \omega\}$ are MADs on ω such that $\mathcal{A}_{n+1} \preceq A_n$ and each $A \in \mathcal{A}_n$ is split into $2^{\aleph_0}$-many sets on the next level (i.e. $(|\{B \in \mathcal{A}_{n+1} : B \subseteq A\}| = 2^{\aleph_0})$. If we let $\mathcal{H} = \bigcup_{n<\omega} \mathcal{I}^+(\mathcal{A}_n)$, then $\mathcal{H}$ is Ramsey, but is not σ-distributive, so, in particular, is not semiselective. (If $\mathcal{A}$ is a MAD, then $\mathcal{I}^+(A) = \{B : (\exists \mathcal{B} \in [\mathcal{A}]^{<\omega})(B \subseteq^* \bigcup \mathcal{B})\}$).

3. If $\mathcal{I} = \langle \{A \subseteq \omega \times \omega : (\exists f \in \omega^\omega)(A \subseteq f)\} \cup \{A \subseteq \omega \times \omega : \exists n < \omega A \subseteq n \times \omega\} \rangle$, then I^+ (also denoted by $\mathcal{H}_{\mathcal{I}}$) has SEL but is not ramsey.

4. Whenever a coideal $\mathcal{H}$ is coanalytic and σ-distributive it cannot both have P and Q. Moreover any G_δ coideal is already σ-closed.

4′. If $\mathcal{I} = \langle \{n \times \omega : n < \omega\} \cup \{A : (\exists f \in \omega^\omega)(A \leq f)\} \rangle$ then $\mathcal{I}^+$ does not have P but has Q.

Misc. $\mathcal{P}(\omega) \setminus FIN$ is selective as is $\mathcal{I}^+$ if $\mathcal{I} = \langle \{n \times \omega : n < \omega\} \rangle$ or $\mathcal{I} = \mathcal{I}^+(\mathcal{A})$ for any MAD $\mathcal{A}$.