Co-Ideal Properties (Definitions)

We say that a co-ideal $\mathcal{H}$ has property

**SEL** if for each disjoint partition $\{A_n : n < \omega\}$ of the set $A \in \mathcal{H}$ consisting of sets not in $\mathcal{H}$, there is a set $B \in \mathcal{H}$ which meets each $A_n$ in a **one point** set.

**Q** if for each disjoint partition $\{A_n : n < \omega\}$ of the set $A \in \mathcal{H}$ consisting of **finite** sets, there is a set $B \in \mathcal{H}$ which meets each $A_n$ in a **one point** set.

**Dg** if for each $\subseteq$-descending chain $\{A_n : n < \omega\}$ of sets from $\mathcal{H}$ with $A_n \setminus A_{n+1}$ not in $\mathcal{H}$, there is a **diagonal** $B \in \mathcal{H}$ (i.e. $(\forall n \in \mathbb{N})(B \cap (n, \infty) \subseteq A_n)$).

**Q_{dg}** if for each $\subseteq$-descending chain $\{A_n : n < \omega\}$ of sets from $\mathcal{H}$ with $A_n \setminus A_{n+1}$ **finite**, there is a **diagonal** $B \in \mathcal{H}$.

**P** if for each disjoint partition $\{A_n : n < \omega\}$ of the set $A \in \mathcal{H}$ consisting of sets not in $\mathcal{H}$, there is a set $B \in \mathcal{H}, B \subseteq A$ which meets each $A_n$ in a **finite** set.

**Ramsey** if for each $A \in \mathcal{H}$ and each coloring $\chi : [A]^2 \to 2$ of $A$ by two colors, there is a homogeneous set $B \in \mathcal{H}$ (i.e. $|\chi''[B]| = 1$).

**$\sigma$-distributive** if the boolean algebra $\mathcal{P}(\omega)/\mathcal{I}_\mathcal{H}$ is $(\omega, \cdot, 2)$-distributive.

**$\sigma$-closed** if the boolean algebra $\mathcal{P}(\omega)/\mathcal{I}_\mathcal{H}$ is $\sigma$-closed.

**$\subseteq^*$-distributive** if the intersection of a countable system of dense open subsets of the partial order $(\mathcal{H}, \subseteq^*)$ is dense.

Selectivity (Definitions)

We say that a co-ideal $\mathcal{H}$ is **SEMISELECTIVE** if it has property $\mathcal{Q}$ and is $\subseteq^*$-distributive. If each $\subseteq^*$-descending chain $\{A_n : n < \omega\} \subseteq \mathcal{H}$ has a diagonal it is called **SELECTIVE** (Mathias: Happy Family).

Examples

1. If $\mathcal{I} = \{A \subseteq \omega \times \omega : (\exists f \in \omega^n)(A \subseteq f)\}$, then $\mathcal{I}^+$ (also denoted by $\mathcal{H}_\times$) is semiselective but not selective (where we take $A \subseteq f$ to mean $(\forall n < \omega)(A \cap \{n\} \times \omega \subseteq f(n))$).

2. Suppose $\{A_n : n < \omega\}$ are MADs on $\omega$ such that $A_{n+1} \subseteq A_n$ and each $A \in A_n$ is split into $2^n_\mathcal{I}$ many sets on the next level (i.e. $(\{B \in A_{n+1} : B \subseteq A\} = 2^n_\mathcal{I}$). If we let $\mathcal{H} = \bigcup_{n<\omega} \mathcal{I}^+(A_n)$, then $\mathcal{H}$ is Ramsey, but is not $\sigma$-distributive, so, in particular, is not semiselective. (If $\mathcal{A}$ is a MAD, then $\mathcal{I}^+(\mathcal{A}) = \{B : (\exists B \in [\mathcal{A}]^{<\omega})(B \subseteq^* \bigcup \mathcal{A})\}$.

3. If $\mathcal{I} = \{A \subseteq \omega \times \omega : (\exists f \in \omega^n)(A \subseteq f)\} \cup \{A \subseteq \omega \times \omega : \exists n < \omega A \subseteq n \times \omega\}$, then $\mathcal{I}^+$ (also denoted by $\mathcal{H}_\times$) has SEL but is not ramsey.

4. Whenever a coideal $\mathcal{H}$ is coanalytic and $\sigma$-distributive it cannot both have $\mathcal{P}$ and $\mathcal{Q}$. Moreover any $G_\delta$ coideal is already $\sigma$-closed.

4'. If $\mathcal{I} = \{n \times \omega : n < \omega\} \cup \{A : (\exists f \in \omega^n)(A \subseteq f)\}$ then $\mathcal{I}^+$ does not have $\mathcal{P}$ but has $\mathcal{Q}$.

Misc. $\mathcal{P}(\omega) \setminus \text{FIN}$ is selective as is $\mathcal{I}^+$ if $\mathcal{I} = \{n \times \omega : n < \omega\}$ or $\mathcal{I} = \mathcal{I}^+(\mathcal{A})$ for any MAD $\mathcal{A}$. 