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Ultrafilters and Independent Systems

Dissertation

Ultrafilters and Independent Systems

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Above all I would like to thank God for the beautiful world he has created. Dei gloria!

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Abstrakt: Práce podává přehled různých konstrukcí ultrafiltrů. V první části uvádí konstrukce, které nepotřebují dodatečné axiomy teorie množin. Je předvedena metoda nezávislých systémů pocházející od K. Kunena. Dále je předvedeno její použití v topologickém zkoumání prostoru ω^* (důkaz existence šestnácti topologických typů J. van Milla). Tato část je zakončena předvedením nové konstrukce a důkazem autorovy věty o existenci ultrafiltrů, které mají speciální topologické vlastnosti (důkaz existence 17 typu): V ω^* existuje bod, který není hromadným bodem spočetné diskrétní množiny, je hromadným bodem spočetné množiny a spočetné množiny, v jejichž je hromadným bodem tvoří filtr.

Druhá část se zabývá konstrukcemi ultrafiltrů vyžadujícími dodatečné množinové axiomy, resp. teorii forcingu. Je předvedena klasická konstrukce P-bodů, pocházející od J. Ketonena, a konstrukce Q-bodu, pocházející od A. R. D. Mathiase. Další dvě kapitoly se zabývají silnými P-body, které zavedl C. Laflamme. V první z těchto kapitol je dokázána nová charakterizační věta (výsledek autora společně s A. Blassem a M. Hrušákem): Ultrafiltr je Canjarův právě když je silný P-bod. Je též uveden nový důkaz věty M. Canjara o existenci non-dominating filtrů (Canjarovy ultrafiltry), který využívá zmíněnou charakterizační větu a dále je dokázána charakterizační věta pro Canjarovy ultrafiltry M. Hrušáka a H. Minamiho. Druhá z těchto kapitol se zabývá generickými ultrafiltry na $\mathcal{P}(\omega)/I$, kde I je definovatelný ideál. Je ukázáno, jak lze tyto ideály charakterizovat pomocí vlastností jejich generických ultrafiltrů. Kapitola zároveň obsahuje odpověď na Laflammovu otázku o Canjarových ultrafiltrech (výsledek autora společně s A. Blassem a M. Hrušákem): Existence ultrafiltru který není silným P-bodem, je P-bodem a nemá rapid předchůdce v Rudin-Keislerově uspořádání je konzistentní s teorií množin.

Klíčová slova: $\beta\omega$, irresolvable prostory, topologický typ, lonely points, strong P-points, Canjar ultrafilters

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Abstract: This work presents an overview of several different methods for constructing ultrafilters. The first part contains constructions not needing additional assumptions beyond the usual axioms of Set Theory. K. Kunen's method using independent systems for constructing weak P-points is presented. This is followed by a presentation of its application in topology (the proof of the existence of sixteen topological types due to J. van Mill). Finally a new construction due to the author is presented together with a proof of his result, the existence of a seventeenth topological type: ω^* contains a point which is discretely untouchable, is a limit point of a countable set and the countable sets having it as its limit point form a filter.

The second part looks at constructions which use additional combinatorial axioms and/or forcing. J. Ketonen's construction of a P-point and A. R. D. Mathias's construction of a Q-point are presented in the first two sections. The next sections concentrate on strong P-points introduced by C. Laflamme. The first of these contains a proof of a new characterization theorem due jointly to the author, A. Blass and M. Hrušák: An ultrafilter is Canjar if and only if it is a strong P-point. A new proof of Canjar's theorem on the existence of non-dominating filters (Canjar filters) which uses the characterization is presented as is a new theorem characterizing Canjar filters (due to M. Hrušák and H. Minami). The second section investigates generic ultrafilters on $\mathcal{P}(\omega)/I$ where I is a definable ideal on ω. It is shown how these ideals may be classified according to the properties of the generic ultrafilter. Several examples are presented including an example which answers a question of Laflamme about Canjar ultrafilters (due jointly to the author, A. Blass and M. Hrušák): It is consistent with ZFC that there is a P-point with no rapid Rudin-Keisler predecessors which is, nevertheless, not a strong P-point.

Keywords: $\beta\omega$, irresolvable spaces, topological type, lonely points, strong P-points, Canjar ultrafilters

Contents

Ἐν ἀρχῇ ἦν ὁ λόγος, καὶ ὁ λόγος ἦν
πρὸς τὸν θεόν, καὶ θεὸς ἦν ὁ λόγος.

Κατα Ιωαννην

Master Theodoric: **St. Catherine**

Introduction

THE starting point for the study of the nonhomogeneity of ω^* — the space of nonprincipal ultrafilters on ω — was W. Rudin's proof ([Rud56]) that, under CH, there are P-points in ω^*. Since there are obviously points in ω^* which are not P-points this shows, supposing CH, that ω^* is not homogeneous. The continuum hypothesis in his proof can be weakened to Martin's axiom (and in fact, much less is needed, see theorem 3.8) but, by a deep and hard result of S. Shelah ([Wim82]), it is consistent with ZFC that there are no P-points. This leads to a split in the study of ultrafilters.

One branch of inquiry asks what can be proven in ZFC. Following this line of research, in 1967 Z. Frolík was able to show in ZFC, using an ingenious combinatorial argument, that ω^* is not homogeneous ([Fro67a], [Fro67b]). In fact, there are $2^{\mathfrak{c}}$ pairwise "topologically different" points (i.e. there is no homeomorphism taking one to another) in ω^*. In his paper he actually showed even more.

Theorem (Frolík). *If X is non-pseudocompact then X^* is not homogeneous.*

From the point of view of topology the problem with his proof was that it was based on cardinality arguments and did not yield a "topological" description of even two different ultrafilters. The next major step forward was K. Kunen's proof of the existence of weak P-points in ZFC ([Kun80]):

Theorem (Kunen). *There is a weak P-point ultrafilter on ω.*

Weak P-points are points in ω^* which are not limit points of any countable set. Obviously not every point of ω^* is a weak P-point, so this also gives a proof of the nonhomogeneity of ω^*. And it actually shows two concrete topologically distinct points (a weak P-point and a non-weak P-point) witnessing the nonhomogeneity. Hence it is an "effective" proof in the sense of E. van Douwen [vD81]. The next result was J. van Mill's description of sixteen distinct topological "types" in ω^* ([vMill82a]). Continuing in this direction, we were able to find a seventeenth topological "type" in ω^*.

One of the points van Mill constructed had the property that it was in the closure of a countable set without isolated points but not in the closure of any discrete set. What happens if you add the requirement, that the countable set is in some sense unique, e.g. that the countable sets having it as a limit point must form a filter?

Definition (Simon). A point $p \in \omega^*$ is *lonely* if it is discretely untouchable, a limit point of some countable subset of ω^* and all countable subsets whose limit point it is form a filter.

These points are called lonely since there is essentially only one way they can be accessed (see also [VD93] for a somewhat related concept). I was able to prove ([Ver08]) that large subspaces of ω^* contain lonely points. However I was unable to show their existence in ω^* because of the difficulty of constructing weak P-points in ccc remainders of countable spaces. Later A. Dow suggested an elegant way of overcoming this difficulty and based on some of his work I was able to prove a theorem.

Theorem. *ω^* contains a lonely point.*

The proof of the theorem is contained in section 2.3. The main idea is to start with a countable zerodimensional space having an $\aleph_0$-bounded remainder, then refine the topology to get an irresolvable topology and finally embed this into ω^*.

The other branch of inquiry about ultrafilters looks beyond ZFC. The oldest result is probably Rudin's already mentioned construction of a P-point under CH. This was later significantly extended by J. Ketonen. Other other combinatorial properties (selective ultrafilters, Q-points, rapid ultrafilters, etc.) going beyond ZFC were also considered. One particular class of ultrafilters was motivated by considerations coming from the theory of forcing. A natural way one can "destroy" an ultrafilter using forcing is to force a pseudointersection. This is typically done using a version of Mathias forcing with an ultrafilter parameter. Standard Mathias forcing adds a dominating real and most of the time this is also the case with the ultrafilter version. M. Canjar showed that, consistently, this is not always the case:

Theorem (Canjar). *Assume $\mathfrak{d} = \mathfrak{c}$. There is an ultrafilter such that $\mathbb{M}_{\mathcal{U}}$ does not add a dominating real.*

Later C. Laflamme in [Laf89] introduced the notion of a strong P-point and noted, without proof, that if $\mathbb{M}_{\mathcal{U}}$ does not add dominating reals, then $\mathcal{U}$ must be a strong P-point.

Together with A. Blass and M. Hrušák we were able to show ([BlHrVe11]) that being a strong P-point actually characterizes the situation when $\mathbb{M}_{\mathcal{U}}$ does not add dominating real. The proof of this is presented in section 3.3. We also constructed a consistent counterexample to a conjecture of Laflamme about a possible characterization of strong P-points.

Example. Assume $cov(\mathcal{M}) = \mathfrak{c}$. Then there is a P-point which has no Rudin-Keisler rapid predecessor but which is, nevertheless, *not* a strong P-point.

In section 3.4 we present another construction of such an example which is based on [HrVer11].

The thesis is split into three chapters. The first chapter contains some preliminary material and standard definitions and theorems. It may be safely skipped and used only as a reference for notions which the reader is unfamiliar with.

The second chapter concentrates on ZFC constructions of ultrafilters. First, Frolík's constructions are given, then a section is devoted to an exposition of Kunen's method for constructing ultrafilters using independent matrices. After that van Mill's proof of the existence of sixteen topological types is presented in some detail and finally I prove the existence of lonely points in ω^*.

The third chapter is devoted to constructions with additional assumptions. First Ketonen's result is given and related questions are mentioned. Then we look at selective ultrafilters and Q-points and present A. R. D. Mathias's proof of the existence of a Q-point from $\mathfrak{d} = \omega_1$. The next section presents a combinatorial characterization of Canjar ultrafilters via strong P-points and the last section contains a study of the forcings $\mathcal{P}(\omega)/\mathcal{I}$ for definable $\mathcal{I}$ together with some interesting examples.

Chapters are mostly independent. Chapter two assumes the reader is comfortable with general topology while chapter three requires some acquaintance with forcing. It is not unlikely that the reader, if there will be one, will feel intimidated by intricacies of the techniques presented, especially those in the second chapter. The authors presentation style will probably not help her either. She is advised to find an edition of some tabloid magazine and be comforted in the fact, that — intellectually — she could be much worse off. If it is of any consolation, in moments of utter despair, the author himself resorted to [Ter08] which offered at least some reassurance of his mental abilities.

Proofs are ended with $\square$, proofs of claims inside other proofs are ended with the $\blacksquare$ symbol. The statement of theorems whose proofs are not given and trivial observations will be closed with the $\square$ to indicate this. The numbering of statements and definitions consists of the number of the chapter followed by a dot and a number which grows in increments of one starting from one in each chapter.

The main results of this thesis are theorems 2.33, 3.39 and example 3.61.

Chapter 1

Preliminaries

THE reader should be familiar with basic topological and set-theoretical concepts and language (including forcing). In this chapter we include some of the ones we will be using but its purpose is rather to fix notation then to introduce the reader into the subject. A reader looking for introduction to topology is referred to [Eng]. A standard reference work in topology is [TopEnc]. An excellent introduction to set theory is [BŠ] or [Kun80] which also includes a lot of material concerning independence and Martin's axiom, [Jech] can serve as a reference for more advanced results. The standard reference for ultrafilters is [ComNeg74].

Another purpose of this chapter is to state some theorems so that we may refer to them in later chapters. We do not provide references as all results (unless otherwise stated) are standard. In the cases where we do not provide proofs they may be found in the cited works. Parts of this chapter are taken from my masters thesis [Ver07] and from the unpublished notes [BPV09].

1.1 Topology

For a topological space X, denote by $\tau(X)$ the topology of X. Let $\overline{A}^X$ be the closure of A in X and $\mathrm{int}_X A$ the interior of A (the largest open set contained in A). If X is clear from the context, we will drop it. $Clopen(X)$ consists of subsets of X which are both closed and open. A set G is *functionally open* (also *co-zero*) in X if it is the preimage of $(0, 1)$ by some continuous map $f : X \to \mathbb{R}$. It is *functionally closed* (also *zero*) if it is the preimage of $\{0\}$ by such a map. It is *regular open* if it is equal to the interior of its closure. A subset of a topological space is *dense* if its closure is the whole space or, equivalently, if it meets any nonempty open set. It is called *nowhere dense* (n.w.d. for short) if its closure has empty interior (or, equivalently, if the complement of its closure is dense), and it is called *somewhere dense* otherwise.

All topological spaces we will consider will be (at least) Hausdorff (i.e. T_2). Recall that other separation properties are commonly used: T_0, T_1, regularity (T_3) and complete regularity ($T_{3\frac{1}{2}}$). The following holds.

$$\text{discrete} \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$$

A space is *compact* if any cover of the space by open sets contains a finite subcover or, equivalently, if any centered system of closed sets has nonempty intersection. It is *locally compact* if any point has an (open) neighbourhood with compact closure and it is *nowhere locally compact* if the closure of any nonempty open set is not compact. Note that a subset of a compact, T_2 space is compact if and only if it is closed and any compact subset of a T_2 space is closed. A space X is *pseudocompact* if every continuous real-valued function on X is bounded.

The *weight* of a space (denoted by $w(X)$) is the minimal cardinality of a *base* for the space (i.e., a system of open sets of X such that any open set is a union of sets from the system). A *π-base* for a space is a family of nonempty open sets such that any nonempty open set of the space contains a set from the π-base. The π-weight of a space (denoted $\pi w(X)$) is the minimal cardinality of a π-base. A *local base* at a point x is a system of open sets containing x such that any open set containing x contains a set from the local base. A *local π-base* at x is a system of nonempty open sets such that any open neighbourhood of x contains a set from the π-base. Define the character ($\chi(x)$) and π-character ($\pi\chi(x)$) of a point $x \in X$ as the minimal cardinality of a base and π-base respectively at x. The *pseudocharacter* of a point in p is the minimal cardinality of a system of neighbourhoods of p which contain only p in their intersection. It is denoted by $\psi(p)$.

1.1 Observation. *If $p \in X$ then $\psi(p) \leq |X|$.* □

1.2 Fact. *If X is T_2 and compact then $\psi(p) = \chi(p)$.* □

A space is *zero-dimensional* if it has a base consisting of closed-and-open sets, *clopen* for short. It is *extremally disconnected*, ED for short, if the closure of any open set is open. A space is discrete if each one-point set is open. A discrete space is zero-dimensional, ED and satisfies all of the above separation properties. One may view zero-dimensionality (and ED) as a separation property, e.g. the following holds

1.3 Lemma. *A T_0 zero-dimensional space satisfies $T_{3\frac{1}{2}}$* □

A point p in a topological space X is *isolated*, provided that $\{p\}$ is open in X. A topological space is *crowded*, also *dense in itself*, if it has no isolated points. Note that a dense subset of X must contain all its isolated points. We say that a space X is *extremally disconnected* at $p \in X$ if p is not in the closure of two disjoint open sets.

A space is said to be κ*-cc* if every family of disjoint open sets has cardinality strictly less than κ. Instead of ω_1-cc it is customary to say just *ccc*.

A *homeomorphism* between two topological spaces is a continuous bijection which has a continuous inverse. A continuous map (function) is *open*, if the images of open sets are open. It is *closed* if the images of closed sets are closed and it is *irreducible*, if the image of a proper closed subspace of the domain is never onto. A closed map is *perfect* if the preimages of points are compact. For a space X we say that EX is its *projective cover* if and only if it is extremally disconnected and admits an irreducible perfect map onto X. EX (sometimes called the *absolute* of X) can be shown to exist for any completely regular space X (e.g. by taking the *Gleason space*, the space of ultrafilters on $RO(X)$)

A topological space is *homogeneous* if, for any two points x, y, there is a homeomorphism $f_{x,y}$ from the space onto itself such that $f(x) = y$. A *topological type* is a subset T of X such that

(i) For any two $x, y \in T$ there is a homeomorphism $f_{x,y}$ from X onto X such that $f_{x,y}(x) = y$ and

(ii) For any $x \in T$, $y \notin T$ there is no homeomorphism of X onto X taking x to y.

Note that if X contains distinct topological types then it is not homogeneous.

1.1.1 The Čech-Stone compactification

For any completely regular space X there is a compact space βX, such that X embeds densely into βX and any continuous function from X into $[0, 1]$ can be continuously extended to βX. (The Stone theorem says that this is equivalent to requiring that a continuous function into *any* compact space can be continuously extended.) The space βX is called the Čech-Stone compactification of X. The book [Wal74] is a standard reference for Čech-Stone compactifications. We refer the reader to this book for the proofs in this section which we omit. The Čech-Stone compactification can be constructed as a space of maximal filters. The idea is to add a point into the intersection of each closed filter (as required by compactness). First we need to be more precise about which filters we will take:

1.4 Definition. A *z-filter* in a topological space X is a filter which consists of functionally closed sets. If F is a functionally closed set, by $\hat{F}$ we denote the set of maximal z-filters containing F.

1.5 Theorem. *If X is completely regular. Then the set of all maximal z-filters on X with the topology generated by $\{\hat{F} : F$ is functionally closed$\}$ (as a closed subbase) is isomorphic to βX.* □

1.6 Note. In the case of zerodimensional spaces maximal z-filters coincide with ultrafilters on the algebra of clopen sets. In the case of discrete spaces, maximal z-filters are just ordinary set filters.

1.7 Note. In section 2.2 we will be dealing with *closed filters* (centered systems, etc.) on general topological spaces where closed sets need not form an algebra since they need not be closed under complements in a reasonable way. However the complement operation is only needed for ultrafilters. The definition of a filter (centered system, etc.) only needs the order structure of the closed sets (i.e. w.r.t. inclusion), so our usage will be safe.

Dealing with Čech-Stone compactifications, it is customary that X^* stands for the (Čech-Stone) *remainder* of X, i.e. $X^* = \beta X \setminus X$. We will now list some facts about Čech-Stone compactifications.

1.8 Fact ([Wal74],21.3)**.** *A space X is extremally disconnected if and only if βX is.* □

1.9 Fact. *Any countable subset of ω^* is extremally disconnected.* □

1.10 Theorem. *For $F \subseteq X^*$ we have $\overline{F} \approx \beta F$ if and only if F is C^*-embedded in X^*.*
Proof. F is certainly dense in the compact set $\overline{F}$. We only need to show that any function from F into $[0,1]$ can be extended to $\overline{F}$, but that immediately follows from the fact that F is C^*-embedded in X^*. □

1.11 Theorem ([vMill84], 1.5.2)**.** *Any countable subset of ω^* is C^*-embedded in ω^*.*

Proof. The proof is an adaptation of the proof of the Tietze extension theorem. Suppose $C \in [\omega^*]^\omega$ is countable and $f : C \to [-1,1]$ is continuous. Fix some small $1/4 > \epsilon > 0$. Since C is countable, the complement of the image of f is dense in all subintervals of $[-1,1]$. Thus we can find $r_0 \in (1/2-\epsilon, 1/2+\epsilon)$ such that both r_0 and $-r_0$ are not in the range of f. Then Let $A^0_{-1} := f^{-1\prime\prime}[-1,-r_0]$, $A^0_0 := f^{-1\prime\prime}[-r_0, r_0]$ and $A^0_1 := f^{-1\prime\prime}[r_0, 1]$.

We shall use the following claim to find $B^0_i \in \mathcal{P}(\omega)$ such that $B^{0*}_i = \overline{B^0}^{\beta\omega} \setminus B^0 = A^0_i$:

Claim. For any countable $C \in [\omega^*]^\omega$ and a clopen disjoint A_{-1}, A_O, A_1 partition of C, there are pairwise almost disjoint B_{-1}, B_0, B_1 in $[\omega]^\omega$ such that $B^*_i \cap C \subseteq A_i$.
Proof of claim. Enumerate $A_{-1} \cup A_0 \cup A_1$ as $\{x_n : n < \omega\}$. By induction construct disjoint $B^i_n, i = -1, 0, 1$ subsets of ω such that $x_n \in \bigcup_{j \le n, i=-1,0,1} B^{i*}_j$ and $B^{i*}_n \cap C \subseteq A_i$. If x_n is already covered, let $B^i_n = \emptyset$. Otherwise suppose (without loss of generality) $x_n \in A_{-1}$. There is a closed subset of ω^* such that $A_0 \cup A_1 = F \cap C$. Because $x_n \notin F \cup \bigcup_{j<n, i=-1,0,1} B^{i*}_j$ there is an open (in ω^*!) U disjoint from $F \cup \bigcup_{j<n,i=-1,0,1} B^{i*}_j$ and containing x_n. Because ω^* is zerodimensional there is a clopen (in ω^*) U' subset of U containing x_n which misses $F \cup \bigcup_{j<n,i=-1,0,1} B^{i*}_j$. Choose $B^0_n \subseteq \omega$ disjoint with $B^i_j, j < n, i = -1, 0, 1$ and $B^{0*}_n = U'$. This is possible because U' is disjoint from each $B^{i*}_j, j < n, i = -1, 0, 1$. Since $U' \cap C \subseteq A_{-1}$, $B^{-1}_n \cap C \subseteq A_{-1}$. Let $B^0_n = B^1_n = \emptyset$. Now $B_i = \bigcup_{n<\omega} B^i_n$ are as required. ■

Now the (continuous) function $f_0 : (B^0_{-1} \cup B^0_0 \cup B^0_1) \to [-1,1]$ having value $-1/2$ on B^0_{-1}, 0 on B^0_0 and $1/2$ on B^0_1 can be extended to some $F_0 : \beta\omega \to [-1,1]$. Necessarily $F_0 \restriction A^0_{-1} \equiv -1/2$, $F_0 \restriction A^0_0 \equiv 0$, $F_0 \restriction A^0_1 \equiv 1/2$. Then the supremum of $\{|F_0(x) - f(x)| : x \in C\}$ is less than $1/2+\epsilon$. By the same reasoning looking at $g = F_0 - f : C \to [-1/2-\epsilon, 1/2+\epsilon]$ we can inductively construct $F_n : \beta\omega \to [-(1/2+\epsilon)^n, (1/2+\epsilon)^n]$ such that the supremum of $\{|F_n(x) - (f(x) - \sum_{i=0}^{n-1} F_i(x))|\}$ is less than $(1/2+\epsilon)^{n+1}$. Then the sum of the functions F_n converges uniformly on ω^* so their sum F is the required continuous extension. □

1.12 Definition. A space X is an F-space if each bounded continuous real-valued function on a co-zero set can be extended to a continuous function on X.

1.13 Theorem. *Suppose X is a locally compact σ-compact space. Then X^* is an F-space.* □

1.14 Proposition. *If X a ccc F-space then X is extremally disconnected.* □

1.15 Definition. The projective cover EX of a space X is the unique extremally disconnected space which admits an irreducible perfect map onto X.

1.2 Set Theory

Our Set Theory notation is standard. The Greek letters κ, λ, θ denote infinite cardinal numbers, α, β denote ordinal numbers, and k, n, m, i, j denote natural numbers. The first infinite cardinal is denoted by ω and $\mathfrak{c}$ is the cardinality of the powerset of ω. For two sets X, Y their *symmetric difference* is denoted by $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$. If X is a set let its powerset be denoted by $\mathcal{P}(X)$. The symbols $[X]^\kappa, [X]^{<\kappa}$ denote the set of all subsets of X of cardinality κ and less than κ respectively. ${}^X Y$ denotes the set of all functions from X to Y. Occasionally, in particular if $X = Y = \omega$, we shall write Y^X for the same set, if there is no danger of confusion. The cardinality of a set X is denoted by $|X|$ and $2^{|X|}$ is the cardinality of ${}^X 2$. For subsets $A, B \subseteq \omega$ of ω and functions $f, g : \omega \to \omega$ we write $A \subseteq^* B$ if $|A \setminus B| < \omega$ and $f \le^* g$ if $|\{n : f(n) > g(n)\}| < \omega$.

1.2.1 Infinite Combinatorics

1.16 Definition. A family of sets $\mathbf{X}$ is *almost disjoint* (AD for short) if any two members of $\mathbf{X}$ have finite intersections. It is called a *maximal almost disjoint family* (MAD for short) on Y if it consists of subsets of Y and any larger family of subsets of Y is not AD. Usually the ambient Y will be clear from the context and will not be mentioned.

1.17 Theorem. *There is a MAD family of subsets of ω of size $\mathfrak{c}$.* □

1.18 Definition. A family $\{X_\alpha : \alpha \in A\}$ of subsets of Y is called *independent* if for each disjoint finite sets $B, C \subseteq A$ the following intersection is infinite

$$\bigcap_{\alpha\in B} X_\alpha \cap \bigcap_{\alpha\in C} (Y \setminus X_\alpha).$$

1.19 Definition. A system of MAD families $\{\mathcal{M}_\alpha : \alpha \in A\}$ is *independent* if for each n, distinct $\alpha_0, \dots, \alpha_n$ and each choice $A_0 \in \mathcal{M}_{\alpha_0}, \dots, A_n \in \mathcal{M}_{\alpha_n}$ the intersection

$$\bigcap_{i=0}^{n} A_i$$

is infinite.

1.20 Theorem. *There is an independent family of subsets of ω. In fact there is even an independent system of size $\mathfrak{c}$ of MAD families each of which has size $\mathfrak{c}$.* □

1.2.2 Ultrafilters and Definable Ideals

Unless otherwise stated, we shall always assume that ideals contain the ideal Fin of finite subsets of ω (and filters contain the filter of cofinite subsets of ω). Let $\emptyset$ be the empty ideal. An ideal is *tall* if each infinite $A \in [\omega]^\omega$ contains an infinite $B \subseteq A$ from the ideal. It is *Fréchet* (or *locally Fin*) if for each infinite $A \in [\omega]^\omega$ there is an infinite $B \subseteq A$ such that B contains no infinite subset from the ideal.

1.21 Definition. Suppose $\mathcal{I}, \mathcal{J}$ are ideals on ω, define an ideal $\mathcal{I} \times \mathcal{J}$ on $\omega \times \omega$ as follows

$$\mathcal{I} \times \mathcal{J} = \{A \subseteq \omega \times \omega : \{x : A_x \notin \mathcal{J}\} \in \mathcal{I}\},$$

where $A_x = \{y : (x, y) \in A\}$.

Several orders may be defined on the ideals (or filters) on ω.

1.22 Definition. Let $\mathcal{I}, \mathcal{J}$ be ideals (or filters) on ω. Recall that

(i) (Rudin-Keisler ordering, [Kat68]) $\mathcal{I} \leq_{RK} \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that

$$\mathcal{I} = f_*(\mathcal{J}) = \{A \subseteq \omega : f^{-1}[A] \in \mathcal{J}\}.$$

(ii) (Rudin-Blass ordering, [Laf89]) $\mathcal{I} \leq_{RB} \mathcal{J}$ if $\mathcal{I} \leq_{RK} \mathcal{J}$ and the function witnessing this can be chosen to be finite-to-one.

(iii) (Katětov ordering, [Kat68]) $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that preimages of $\mathcal{I}$-small sets are $\mathcal{J}$-small.

(iv) (Katětov-Blass ordering, [HrHe07]) $\mathcal{I} \leq_{KB} \mathcal{J}$ if $\mathcal{I} \leq_K \mathcal{J}$ and the witnessing function can be chosen to be finite-to-one.

1.23 Definition (Rudin). An ultrafilter $\mathcal{U}$ on ω is a *P-point* if any countable sequence of sets from $\mathcal{U}$ has a pseudointersection in $\mathcal{U}$.

1.24 Note. Compare this with the topological notion of a P-point: A point $x \in X$ is a P-point if the intersection of countably many neighbourhoods of x is again a neighbourhood of x. In the case of ω^* these notions coincide.

1.25 Fact. *An ultrafilter $\mathcal{U}$ on ω is a P-point if and only if each function $f : \omega \to \omega$ is either finite-to-one on a set in $\mathcal{U}$ or constant on a set in $\mathcal{U}$.* □

1.26 Corollary. *Any RK-predecessor of a P-point is its RB-predecessor.* □

1.27 Definition. An ideal $\mathcal{I}$ on ω is a *P-ideal* if for any countable sequence $\langle A_n : n < \omega\rangle \subseteq \mathcal{I}$ there is a set $A \in \mathcal{I}$ such that $A_n \subseteq^* A$ for each $n < \omega$.

Recall that ideals are subsets of $\mathcal{P}(\omega)$ which can be identified with the Cantor space and thus we may consider their descriptive complexity. The following definition of a summable ideal motivates the further definition of a lower semicontinuous submeasure:

1.28 Definition. Given a function $g : \omega \to \mathbb{R}_0^+$ and $A \subseteq \omega$ we let $\varphi_g(A) = \sum_{n\in A} g(n)$ and define the *summable ideal* $\mathcal{I}_g = \{A \subseteq \omega : \varphi_g(A) < \infty\}$.

1.29 Definition. A *lower semicontinuous submeasure* (lscsm. for short) on ω is a function $\varphi : \mathcal{P}(\omega) \to \mathbb{R} \cup \{\infty\}$ such that

(i) $\varphi(\emptyset) = 0$

(ii) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$

(iii) For each infinite $A \subseteq \omega$ we have $\varphi(A) = \sup\{\varphi(a) : a \in [A]^{<\omega}\}$.

These submeasures give rise to two naturally defined ideals, the ideal $Fin(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}$ of sets of φ-finite submeasure and $Exh(\varphi) = \{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0\}$ the ideal of sets on which φ is exhaustive. It is easy to verify that $Fin(\varphi)$ is F_σ and that $Exh(\varphi)$ is $F_{\sigma\delta}$. The fact that each F_σ-ideal is of this form is due to K. Mazur.

1.30 Theorem ([Maz91])**.** *An ideal I on ω is F_σ if and only if there is a lscsm. φ on ω such that $I = Fin(\varphi)$.* □

The theorem has a generalization due to S. Solecki.

1.31 Theorem ([Sol96])**.** *Suppose I is an analytic ideal. Then precisely one of the following happens*

(i) $Fin \times \emptyset \leq_{RB} I$ *or*

(ii) $Fin \times Fin \leq_{RB} I$ *and* $I = Exh(\varphi)$ *for some lscsm* φ *or*

(iii) $I = Exh(\varphi) = Fin(\varphi)$ *for some lscsm* φ.

□

1.32 Corollary. *Every analytic P-ideal is an $F_{\sigma\delta}$ subset of $\mathcal{P}(\omega)$.*

Proof. No P-ideal can be RB-above $Fin \times \emptyset$. □

1.2.3 Cichoń's diagram and cardinal characteristics

We shall now look at cardinal invariants of the continuum. Most of them are associated with some properties of the real line. They are all greater or equal to ω_1 and less or equal to $\mathfrak{c}$ but in general their value is not decided by the usual axioms of ZFC. Positing that some cardinal invariant is equal to $\mathfrak{c}$ may be seen as a weakening of the continuum hypothesis (the continuum hypothesis says that $\mathfrak{c} = \omega_1$, this is abbreviated as CH) and in fact many constructions which work under CH may be carried out under weaker assumptions of this form. For a beautiful exposition of the topic see [Bl09].

1.33 Definition. For any ideal $\mathcal{I}$ on a set X we define the following cardinal characteristics:

$$\begin{aligned}
\mathrm{add}(\mathcal{I}) &= min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \ \&\ \bigcup \mathcal{A} \notin I\} \\
\mathrm{non}(\mathcal{I}) &= min\{|A| : A \subseteq X \ \&\ A \notin \mathcal{I}\} \\
\mathrm{cov}(\mathcal{I}) &= min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \ \&\ \bigcup \mathcal{A} = \bigcup \mathcal{I}\} \\
\mathrm{cof}(\mathcal{I}) &= min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \ \&\ (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq A)\}
\end{aligned}$$

1.34 Definition. *Martin's Axiom* for κ (MA_κ) says that if $(P, \leq)$ is a ccc partial order of size at most $\mathfrak{c}$ and $\langle D_\alpha : \alpha < \kappa\rangle$ is a sequence of dense subsets of P, then there is a filter G on P which meets each D_α. Martin's Axiom is MA_{ω_1}.

The following is a summary definition of some of the important cardinal characteristics.

1.35 Definition. We define the following cardinals

$$\begin{aligned}
\mathfrak{b} &= min\{|\mathcal{A}| : \mathcal{A} \subseteq {}^\omega\omega \ \&\ (\forall f \in {}^\omega\omega)(\exists g \in \mathcal{A})(g \not\leq^* f)\} \\
\mathfrak{d} &= min\{|\mathcal{A}| : \mathcal{A} \subseteq {}^\omega\omega \ \&\ (\forall f \in {}^\omega\omega)(\exists g \in \mathcal{A})(f \leq^* g)\} \\
\mathfrak{a} &= min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \ \&\ \mathcal{A} \text{ is almost disjoint } \&\ \omega \leq |\mathcal{A}| \ \&(\forall B \in [\omega]^\omega)(\exists A \in \mathcal{A})(|A \cap B| = \omega)\} \\
\mathfrak{s} &= min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \ \&\ (\forall B \in [\omega]^\omega)(\exists A \in \mathcal{A})(|A \cap B| = |(\omega \setminus A) \cap B| = \omega)\} \\
\mathfrak{r} &= min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \ \&\ (\forall B \in [\omega]^\omega)(\exists A \in \mathcal{A})(|A \cap B| < \omega \vee |A \setminus B| < \omega)\} \\
\mathfrak{u} &= min\{|\mathcal{U}| : \mathcal{U} \subseteq [\omega]^\omega \ \&\ \mathcal{U} \text{ is centered } \&\ \omega \leq |\mathcal{U}| \ \&(\forall B \in [\omega]^\omega)(\exists A \in \mathcal{U})(|A \cap B| < \omega \vee A \subseteq^* B))\} \\
\mathfrak{h} &= min\{\kappa : \mathcal{P}(\omega)/Fin \text{ is not } (\kappa, \cdot, 2)\text{-distributive}\} \\
\mathfrak{p} &= min\{|\mathcal{U}| : \mathcal{U} \subseteq [\omega]^\omega \ \&\ (\forall \mathcal{V} \in [\mathcal{U}]^{<\omega})(|\bigcap \mathcal{V}| = \omega) \ \&\ (\forall P \in [\omega]^\omega)(\exists U \in \mathcal{U})(P \not\subseteq^* U)\} \\
\mathfrak{t} &= min\{|\mathcal{U}| : \mathcal{U} \subseteq [\omega]^\omega \ \&\ \mathcal{U} \text{ is linearly ordered by } \subseteq^* \ \&\ (\forall P \in [\omega]^\omega)(\exists U \in \mathcal{U})(P \not\subseteq^* U)\} \\
\mathfrak{i} &= min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \ \&\ \mathcal{A} \text{ is independent } \&\ (\forall \mathcal{A}' \subseteq [\omega]^\omega)(\mathcal{A} \subsetneq \mathcal{A}' \to \mathcal{A}' \text{ is not independent})\} \\
\mathfrak{m} &= min\{\kappa : \mathrm{MA}_\kappa \text{ fails}\}
\end{aligned}$$

1.36 Definition. A family $\mathcal{G} \subseteq [\omega]^\omega$ is *groupwise dense* if it is closed under almost subsets and for any infinite partition of ω into intervals there are infinitely many intervals from the partition whose union is in $\mathcal{G}$. The *groupwise density number* $\mathfrak{g}$ is now defined to be the smallest cardinality of a set $\mathcal{A}$ of groupwise dense families with $\bigcap \mathcal{A} = \emptyset$.

It turns out that it is very useful to think of these definitions in the following way (introduced and isolated explicitly in [Voj93], but implicitly appearing already in [Fre84] and in an unpublished work of Miller).

1.37 Definition. Suppose we are given a triple $\mathbf{A} = (A_-, A_+, R)$ consisting of a set A_- of challenges a set A_+ of responses and a relation $R \subseteq A_- \times A_+$. We say that $b \in A_+$ is a response to a challenge $a \in A_-$ if aRb. We define the *norm* $||\mathbf{A}|| = ||(A_-, A_+, R)||$ of this triple to be:

$$||(A_-, A_+, R)|| = \min\{|\mathcal{B}| : \mathcal{B} \subseteq A_+ \ \& \ (\forall a \in A_-)(\exists b \in \mathcal{B})(aRb)\}.$$

In other words the norm is the cardinality of the smallest set of responses which are enough to answer every challenge. We also define the *dual* $\mathbf{A}^\perp$ of $\mathbf{A}$ to be the triple $(A_+, A_-, -(R^{-1}))$, where $(x, y) \in -(R^{-1}) \equiv (y, x) \notin R$.

Note that most of the cardinal invariants considered so far are norms of some triple. The usefulness of this framework lies in the fact that the notion of duality gives a precise meaning to the intuition that some of the cardinals defined (e.g. $\mathfrak{b}, \mathfrak{d}$) are intimately connected to each other.

1.38 Definition (Tukey's connection)**.** Given two triples $\mathbf{A} = (A_-, A_+, R), \mathbf{B} = (B_-, B_+, Q)$ we say that $\mathbf{B} \leq_T \mathbf{A}$ if there are functions $\phi_- : B_- \to A_-$ and $\phi_+ : A_+ \to B_+$ which satisfy:

$$(\forall b \in B_-, a \in A_+)(\phi_-(b) R a \to b Q \phi_+(a))$$

Following Blass we call the pair $\phi = (\phi_-, \phi_+)$ a *morphism* from $\mathbf{A}$ to $\mathbf{B}$ (Vojtáš calls ϕ a generalized Galois-Tukey connection from $\mathbf{B}$ to $\mathbf{A}$) and we shall write $\phi : \mathbf{A} \to \mathbf{B}$. Note that if $\phi : \mathbf{A} \to \mathbf{B}$ then $\phi^\perp = (\phi_+, \phi_-)$ is a morphism from $\mathbf{B}^\perp$ to $\mathbf{A}^\perp$.

It is instructive to think of $\mathbf{A} \geq_T \mathbf{B}$ as saying that meeting challenges in $\mathbf{B}$ is not harder then meeting challenges in $\mathbf{A}$, (cf. the notion of a Borel reduction of two Borel Equivalence relations or the notion of Turing reducibility in Recursion theory).

1.39 Observation. *If* $\mathbf{A} \leq_T \mathbf{B}$ *then* $||\mathbf{A}|| \leq ||\mathbf{B}||$. □

The following diagram (see [Fre84]) summarizes all that can be proved in ZFC concerning the relations between cardinal invariants defined from the ideal of null and meager sets. The dashed boxes indicate, respectively, that the top cardinal and bottom cardinal are the maximum and the minimum of the other two cardinals.

$$\begin{array}{ccccccc}
\mathrm{cov}(\mathcal{N}) & \longrightarrow & \mathrm{non}(\mathcal{M}) & \longrightarrow & \mathrm{cof}(\mathcal{M}) & \longrightarrow & \mathrm{cof}(\mathcal{N}) \\
 & & \uparrow & & \uparrow & & \\
\uparrow & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & \uparrow \\
 & & \uparrow & & \uparrow & & \\
\mathrm{add}(\mathcal{N}) & \longrightarrow & \mathrm{add}(\mathcal{M}) & \longrightarrow & \mathrm{cov}(\mathcal{M}) & \longrightarrow & \mathrm{non}(\mathcal{N})
\end{array}$$

Figure 1.1: Cichoń's diagram

It is a direct consequence of the following reformulation of the cardinal invariants in terms of norms of triples:

1.40 Theorem.

$$\begin{array}{ccccccc}
(\mathbb{R}, \mathcal{N}, \in) & \longrightarrow & (\mathcal{M}, \mathbb{R}, \not\ni) & \longrightarrow & (\mathcal{M}, \mathcal{M}, \subset) & \longrightarrow & (\mathcal{N}, \mathcal{N}, \subset) \\
 & & \uparrow & & \uparrow & & \\
\uparrow & & (\omega^\omega, \omega^\omega, \not\geq^*) & \longrightarrow & (\omega^\omega, \omega^\omega, \leq^*) & & \uparrow \\
 & & \uparrow & & \uparrow & & \\
(\mathcal{N}, \mathcal{N}, \not\supset) & \longrightarrow & (\mathcal{M}, \mathcal{M}, \not\supset) & \longrightarrow & (\mathbb{R}, \mathcal{M}, \in) & \longrightarrow & (\mathcal{N}, \mathbb{R}, \not\ni)
\end{array}$$

□

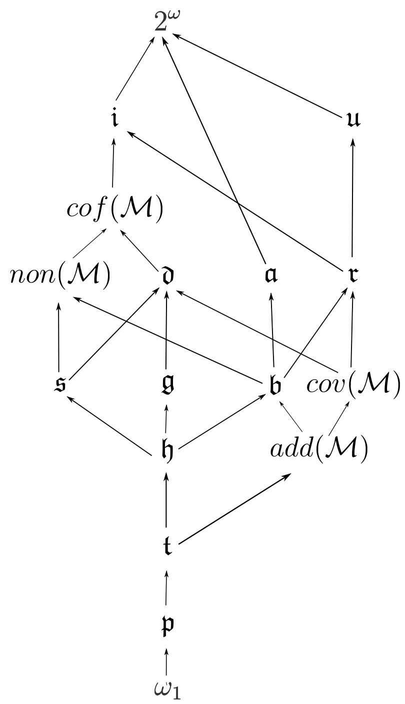
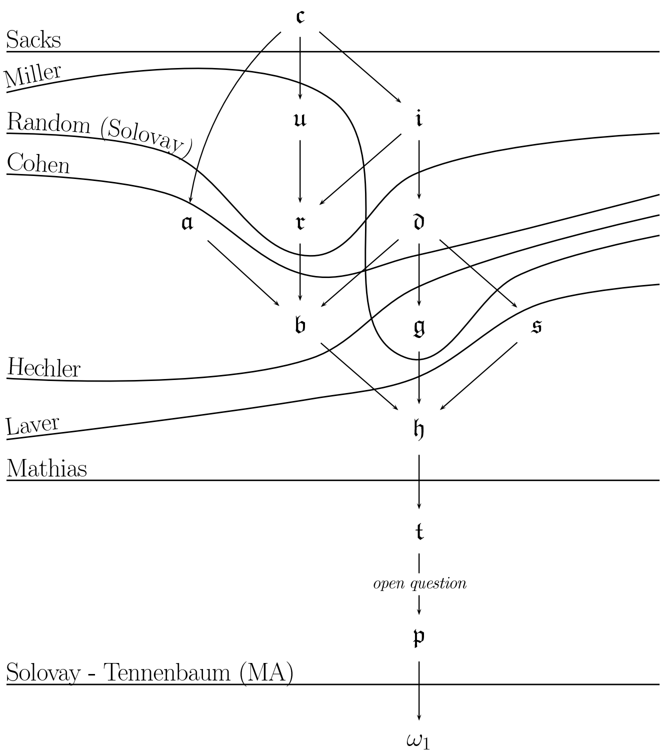


Figure 1.2: Relations between cardinal invariants of the continuum (a fusion of Cichoń's diagram with van Douwen's diagram, see [Bl09] and [Br06])



	MA	Cohen	Random	Sacks	Hechler	Laver	Mathias	Miller
$\mathfrak{a}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$
$\mathfrak{b}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$
$\mathfrak{d}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{g}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{h}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$
$\mathfrak{i}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{m}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{p}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{s}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$
$\mathfrak{t}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{u}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$
add($\mathcal{N}$)	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
cov($\mathcal{N}$)	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
non($\mathcal{N}$)	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$
cof($\mathcal{N}$)	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
add($\mathcal{M}$)	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$
cov($\mathcal{M}$)	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$
non($\mathcal{M}$)	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$
cof($\mathcal{M}$)	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$

Figure 1.3: Separating cardinal invariants (the table is taken from [Bl09], the graphical rendering was typeset by D. Chodounský from a picture of J. Flašková)

Albrecht Dürer: **Father**

Chapter 2

ZFC Constructions

This chapter will present some of the combinatorial methods used to construct special kinds of ultrafilters. The easiest way to construct objects with special properties is to use induction. Take two classical examples, Cantor's back-and-forth construction of an isomorphism between two countable dense linear orders without ends, and Hausdorff's constructions of an (ω_1, ω_1)-gap. The first induction is of length ω, while the second, which is more tricky, is of length ω_1. The fact which allows the induction to keep going is that at each step of the induction the object constructed so far is at most countable — dealing with countable objects is *much* easier than dealing with uncountable ones. This also indicates that inductions of longer length will usually be much more difficult. To illustrate this, suppose you would want to construct a sequence $\langle A_\alpha : \alpha < \omega_2 \rangle$ of infinite subsets of ω such that $A_\alpha \subseteq^* A_\beta$ for each $\alpha < \beta \leq \omega_2$. There will be no problem going up to ω_1 since at each point below ω_1 the sequence will be countable and hence will have an infinite pseudointersection. However already at step ω_1 we have a problem. For example if $\mathfrak{t} = \omega_1$ it can happen that the sequence we have constructed at step ω_1 cannot be extended further. The problem with ultrafilters is that they are objects of size $\mathfrak{c}$. In general to construct them by induction from smaller objects (filters) one will need an induction of length $\mathfrak{c}$ and $\mathfrak{c}$ may very well be bigger then ω_1. This already indicates that ZFC constructions will be very hard. Even disregarding this problem we will have another problem. The objects constructed during the induction process will, from some point on, not be countable anymore. One way to overcome these problems is to assume additional axioms which will typically say that $\mathfrak{c}$ behaves like ω_1 in some suitable sense. This approach allows us to construct very nice ultrafilters (see chapter 3), but unfortunately is not good at all if we want to work in ZFC only.

The current chapter presents an ingenious method invented by K. Kunen to overcome the problems with long inductive constructions of filters. After that we will show how van Mill was able to use these methods in a topological setting and finally we will present our construction of a special kind of ultrafilter. First, as a warm-up exercise, we shall show two clever combinatorial arguments which imply that there are many ultrafilters. The first is an old theorem of B. Pospíšil.

2.1 Theorem ([Po37]). *There are $2^{\mathfrak{c}}$ ultrafilters on ω.*

Proof. Fix an independent system $\{A_\alpha : \alpha < \mathfrak{c}\}$ of subsets of ω (see 1.20) and for each $f : \mathfrak{c} \to 2$ let $\mathcal{U}_f$ be some ultrafilter extending $\{A_\alpha : f(\alpha) = 1\} \cup \{\omega \setminus A_\alpha : f(\alpha) = 0\}$. It is easy to see that for $f \neq g$, $\mathcal{U}_f \neq \mathcal{U}_g$ and this shows that $2^{\mathfrak{c}} \leq |\beta\omega|$. $|\beta\omega| \leq |\mathcal{P}\mathcal{P}(\omega)| = 2^{\mathfrak{c}}$ is easy. □

The next theorem of Z. Frolík says that not only are there many ultrafilters, but there are many *different* ultrafilters. Frolík's theorem was the first ZFC proof of the nonhomogeneity of ω^*.

2.2 Theorem ([Fro67a]). *There are $2^{\mathfrak{c}}$ ultrafilters none of which can be mapped to another via a homeomorphism of ω^*.*

2.3 Note. In [Fro67b], Frolík generalized the theorem to show that X^* is not homogeneous for any nonpseudocompact X.

To prove the theorem Frolík introduced the following notion of a sum of ultrafilters:

2.4 Definition ([Fro67a],1.2). Suppose $\langle p_n : n < \omega \rangle$ is a sequence of ultrafilters on ω, and p is an ultrafilter on ω. We define the sum of the p_n's relative to p as follows:

$$p - \sum_{n<\omega} p_n = \{A : \{n \in \omega : A \in p_n\} \in p\}$$

Given two ultrafilters p, q we say that $p \leq_F q$ if $q = p - \sum_{n<\omega} p_n$ for some *discrete* sequence of ultrafilters $p_n \in \omega^*$.

The main ingredient of Frolík's proof of theorem 2.2 is the following proposition:

2.5 Proposition (Frolík). *Any $q \in \omega^*$ has at most $\mathfrak{c}$ predecessors in the $\leq_F$ ordering.*

Proof. We shall first prove a simple claim.

Claim. Suppose $\langle A_n : n < \omega \rangle$ is a partition of ω, $p \neq q$, $\langle p_n, q_n : n < \omega \rangle$ are ultrafilters with $A_n \in p_n \cap q_n$. Then $p - \sum_{n<\omega} p_n \neq q - \sum_{n<\omega} q_n$.

Proof of claim. Pick some $A \in p \setminus q$ and notice that

$$X = \left(\bigcup_{n \in A} A_n \right) \in p - \sum_{n<\omega} p_n$$

while

$$\omega \setminus X = \left(\bigcup_{n \in \omega \setminus A} A_n \right) \in q - \sum_{n<\omega} q_n.$$

■

For each $p \leq_F q$, we have $q = p - \sum_{n<\omega} p_n$ for some discrete sequence $\langle p_n : n < \omega \rangle$ of ultrafilters. Let $\mathcal{A}_p = \langle A_n : n < \omega \rangle$ be a partition of ω witnessing the discreteness (i.e. $A_n \in p_n$). By the previous claim this is an injective map from $\{p : p \leq_F q\}$ into $\{\mathcal{A} : \mathcal{A}$ is a partition of $\omega\}$ which has size $\mathfrak{c}$. □

Frolík's theorem 2.2 now easily follows from the previous proposition, Pospíšil's theorem 2.1 and the following observation.

2.6 Observation. *For $q \in \omega^*$ let $T_q = \{p : p \leq_F q\}$. If $f : \omega^* \to \omega^*$ is a homeomorphism then $T_q = T_{f(q)}$.* □

Proof of theorem 2.2. Suppose $X \subseteq \omega^*$ is of size $< 2^{\mathfrak{c}}$ and let $X' = \bigcup_{p \in X} T_p$. Since each T_p is of size at most $\mathfrak{c}$ by proposition 2.5, $|X'| < 2^{\mathfrak{c}} = |\omega^*|$, so we can find $q' \in \omega^* \setminus X'$ and some q RF-above q'. Then $T_q \neq T_p$ for any $p \in X$ so no $p \in X$ can be mapped onto q by a homeomorphism. Frolík's theorem now easily follows. □

Frolík proved that there must be many "topologically different" points in ω, but his proof doesn't give any concrete examples. The next section describes the work of K. Kunen who, almost 10 years later, gave a first ZFC example of two types in ω^* which differ in "natural topological properties".

2.1 Independent Systems & Their Applications

2.7 Definition. A point $x \in X$ is a weak P-point if it is not a limit point of a countable subset of X.

2.8 Fact. *If X is T_1 space then each P-point is a weak P-point.* □

This section is devoted to the techniques K. Kunen invented to prove the following theorem.

2.9 Theorem ([Kun80]). *There is a weak P-point in ω^*.*

The problem with proving the above theorem by directly constructing the point using induction lies in the fact that there are too many countable subsets of ω^* to take care of in an induction of length $\mathfrak{c}$. Kunen's first step was to introduce a stronger notion which, at first, seems unrelated weak P-points.

2.10 Definition (Kunen). A point $p \in X$ is a *κ-O.K. point* of X if for any countable sequence $\langle U_n : n \in \omega \rangle$ of neighbourhoods of p there is a system $\{V_\alpha : \alpha < \kappa\}$ of neighbourhoods of p such that for any nonempty finite $K \in [\kappa]^{<\omega}$, the following is true:

$$\bigcap_{\alpha \in K} V_\alpha \subseteq U_{|K|}$$

Note that if $\kappa < \lambda$ then any λ-O.K. point is also a κ-O.K. point and if $\mathcal{B}$ is a base for the topology of X, then the definition is equivalent if we only consider sequences of neighbourhoods from the base.

The relation with weak P-points is given in the following proposition

2.11 Proposition (Kunen). *If X is a T_1 space and p is an ω_1-O.K. point of X, then p is a weak P-point of X.*

Proof. If $\{x_n : n \in \omega\} \subseteq X \setminus \{p\}$, then because X is T_1 we can choose a descending sequence of neighbourhoods U_n of p such that U_n misses x_n. Then, because p is ω_1-O.K., we can choose $\{V_\alpha : \alpha < \omega_1\}$ neighbourhoods of p, so that the intersection of any n of them is contained in U_n. Then each x_n is contained in only finitely many of them, so there is an $\alpha < \omega_1$ such that V_α misses all of them, so p is not in the closure of $\{x_n : n \in \omega\}$. □

Since ω^* has a basis of size $\mathfrak{c}$ (and since $\mathfrak{c}^\omega = \mathfrak{c}$) there is some hope that O.K.-points could be constructed inductively. However it is still not clear how to manage the induction. It might happen that there are ultrafilters with character $< \mathfrak{c}$ so another problem is to make sure that we do not construct an ultrafilter before taking care of all the countable sequences of open subsets of ω^* and that the induction can keep going. To guarantee this K. Kunen came up with the following somewhat complicated notion generalizing the concept of an independent family:

2.12 Definition (Kunen). Suppose $\mathcal{F}$ is a filter on ω. We say a family (or a matrix)

$$\mathcal{X} = \{X^n_{\alpha,\beta} : n < \omega, \alpha \in \kappa, \beta \in \lambda\}$$

of subsets of ω is a κ *by* λ *independent linked family w.r.t.* $\mathcal{F}$ if

(i) For each α, β, n we have $X^n_{\alpha,\beta} \subseteq X^{n+1}_{\alpha,\beta}$ (i.e. the sets increase with n),

(ii) For each finite set of indices $L \in [\lambda]^{<\omega}$, for each function $n : L \to \omega$ and $A \in \prod_{\beta\in L}[\kappa]^{n(\beta)}$ and for each $F \in \mathcal{F}$ the intersection
$$F \cap \bigcap_{\beta\in L} \bigcap_{\alpha\in A(\beta)} X^{n(\beta)}_{\alpha,\beta}$$
is infinite, while for each $\beta \in \lambda, n < \omega, A \in [\kappa]^{n+1}$ the intersection
$$\bigcap_{\alpha\in A} X^n_{\alpha,\beta}$$
is finite.

By a complicated argument using trees, he was able to show that such families exist. Here we present a much simpler proof of this fact due to P. Simon.

2.13 Theorem ([Kun80]). *There is a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family with respect to the Fréchet filter.*

Proof. (due to P. Simon, see [Kun80] or [vMill82b] Lemma 2.4) We shall construct such a family consisting of subsets of the countable set $S = \{(k, f) : k \in \omega, f \in {}^{\mathcal{P}(k)}\mathcal{P}\mathcal{P}(k)\}$. Given $A, B \subseteq \omega$ and $n < \omega$ let

$$X^n_{A,B} = \{(k, f) \in S : |f(B \cap k)| \leq n \ \&\ A \cap k \in f(B \cap k)\}.$$

It is routine, if perhaps somewhat involved, to check that $\{X^n_{A,B} : n < \omega, A, B \subseteq \omega\}$ is a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family w.r.t the Fréchet filter. □

2.14 Note. To prove the above theorem one may also start with an independent family of MAD families $\{M_{\alpha,\beta} : \alpha, \beta < \mathfrak{c}\}$ (see theorem 1.20) and let

$$X^n_{\alpha,\beta} = \bigcup_{i<n} M_{\omega\cdot\alpha+i,\beta}.$$

Kunen used such a matrix to keep the induction going. At each step he added new sets to the filter to get rid of one countable sequence of open subsets of ω^* while sacrificing a finite number of rows from the matrix and keeping it independent with respect to the larger filter. The fact that at each step he had a matrix independent with respect to the filter allowed him to proceed. The main ideas of his proof are contained in the following two lemmas.

2.15 Lemma (Kunen). *Suppose $\mathcal{X}$ is an A by B independent linked family with respect to some filter $\mathcal{F}$. Suppose $\langle U_n : n < \omega\rangle$ is a $\subseteq^*$-descending sequence of elements of $\mathcal{F}$. Then there is a $\beta \in B$ and a family $\{V_\alpha : \alpha < \mathfrak{c}\}$ of subsets of ω such that for each $K \subseteq [\mathfrak{c}]^{<\omega}$*

$$(*) \qquad \bigcap_{\gamma\in K} V_\gamma \subseteq^* U_{|K|}$$

while $\mathcal{X}$ restricted to $B \setminus \{\beta\}$ is an A by $(B \setminus \{\beta\})$ independent linked family with respect to the filter generated by $\mathcal{F} \cup \{V_\alpha : \alpha < \mathfrak{c}\}$.

Proof. Fix any $\beta \in B$ and for $\alpha < \mathfrak{c}$ let

$$V_\alpha = \bigcup_{n<\omega} X^n_{\alpha,\beta} \cap U_n$$

The condition (*) follows from the fact that for any $n + 1$-many indices $\alpha_0, \dots, \alpha_n$ the intersection $\bigcap_{i=0}^n X^n_{\alpha_i,\beta}$ is finite by (ii), part 2 of the definition of an independent linked family. We need to check that the matrix will be independent w.r.t. the larger filter if we drop the βth row. But this is clear since the original matrix was independent w.r.t. $\mathcal{F}$, $U_n \in \mathcal{F}$ and $U_n \cap X^n_{\alpha,\beta} \subseteq V_\alpha$ for each $n < \omega$. □

2.16 Lemma (Kunen). *Suppose $\mathcal{X}$ is an A by B independent linked family with respect to some filter $\mathcal{F}$ and $Y \subseteq \omega$. Then there is a finite $L \subseteq B$ such that $\mathcal{X}$ restricted to $B \setminus L$ is an A by $(B \setminus L)$ independent linked family with respect to the filter generated by either $\mathcal{F} \cup \{Y\}$ or $\mathcal{F} \cup \{\omega \setminus Y\}$*

Proof. If $\mathcal{X}$ is independent linked w.r.t. the filter generated by $\mathcal{F}\cup\{Y\}$ then we are done. Otherwise there is some $L \subseteq \mathbb{B}$, $n : L \to \omega$, $a \in \prod_{\beta\in L}[A]^{n(\beta)}$ and $F \in \mathcal{F}$ such that

$$(**) \qquad Y \cap F \cap \bigcap_{\beta\in L}\ \bigcap_{\alpha\in a(\beta)} X^{n(\beta)}_{\alpha,\beta}$$

is finite. We will show that $\mathcal{X}$ restricted to $B \setminus L$ is independent linked w.r.t. the filter generated by $\mathcal{F} \cup \{\omega \setminus X\}$. So take some L', n', a' and F' as above. Let

$$Z = F' \cap F \cap \bigcap_{\beta\in L}\ \bigcap_{\alpha\in a(\beta)} X^{n(\beta)}_{\alpha,\beta} \cap \bigcap_{\beta\in L'}\ \bigcap_{\alpha\in a'(\beta)} X^{n'(\beta)}_{\alpha,\beta}.$$

Since the original matrix was independent w.r.t. $\mathcal{F}$ and since $L \cap L' = \emptyset$, Z is infinite. By (**) it is almost disjoint from Y. So $Z \cap \omega \setminus Y$ is infinite also which finishes the proof. □

We are now ready to prove Kunen's result about weak P-points:

Proof of theorem 2.9. It is instructive to compare this proof with the proof of theorem 3.2. The overall inductive structure is the same, however whereas in the proof of 3.2 the induction process is kept going by a very simple requirement (c.f. condition 3.2.iii) here the requirement is much more complicated (conditions (i), (ii)).

By proposition 2.11 it is sufficient to show that there are $\mathfrak{c}$-O.K. points in ω^*. Enumerate all countable $\subseteq^*$-descending sequences of infinite subsets of ω as $\langle\langle U^\alpha_n : n < \omega\rangle : \alpha < \mathfrak{c}\rangle$ such that each sequence appears cofinally often. Also enumerate $\mathcal{P}(\omega)$ as $\{Y_\alpha : \alpha < \mathfrak{c}\}$. Let $\mathcal{X}$ be a $\mathfrak{c}$ by $\mathfrak{c}$ independent family w.r.t. the Fréchet filter on ω. By recursion on $\mathfrak{c}$ construct filters $\mathcal{F}_\alpha$, and sets L_α such that

(i) $\mathcal{FR} = \mathcal{F}_0 \subseteq \mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ and $\emptyset = L_0 \subseteq L_\alpha \subseteq L_\beta$ for $\alpha < \beta$ and $\mathcal{X}$ restricted to $\mathfrak{c} \setminus L_\alpha$ is an $\mathfrak{c}$ by $(\mathfrak{c} \setminus L_\alpha)$-independent linked family w.r.t. $\mathcal{F}_\alpha$,

(ii) $|L_\alpha| \leq \omega \cdot |\alpha|$,

(iii) $L_\beta = \bigcup_{\alpha<\beta} L_\alpha$ and $\mathcal{F}_\beta = \bigcup_{\alpha<\beta} \mathcal{F}_\alpha$ for $\beta < \mathfrak{c}$ limit,

(iv) If $\langle U_n : n < \omega\rangle \subseteq \mathcal{F}_\alpha$ then there are $\{V_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathcal{F}_{\alpha+1}$ such that for each finite $K \subseteq [\mathfrak{c}]^{<\omega}$

$$\bigcap_{\alpha\in K} V_\alpha \subseteq^* U_{|K|}$$

and

(v) either $Y_\alpha \in \mathcal{F}_{\alpha+1}$ or $\omega \setminus Y_\alpha \in \mathcal{F}_{\alpha+1}$.

The nontrivial steps (guaranteeing (iv) and (v) at successors) are taken care of by the previous two lemmas. Finally let $\mathcal{U} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$. By (v) $\mathcal{U}$ will be an ultrafilter. By (iv) and the fact that $\mathrm{cf}(\mathfrak{c}) > \omega$ and hence each countable sequence of elements of $\mathcal{U}$ must be included in $\mathcal{F}_\alpha$ from some α_0 on we infer that $\mathcal{U}$ is a $\mathfrak{c}$-O.K. point of ω^*. (This follows from the fact that the sets $\hat{A} = \{p \in \omega^* : A \in p\}$ form a basis of ω^*) □

Kunen's machinery for constructing O.K.-points was soon ingeniously put to good use by van Mill to construct various kinds of points in different topological spaces. We will be looking at that in the next section. Before we do that let us mention that K. Kunen together with his Ph.D. student J. Baker later generalized the notion of an O.K.-point to the notion of a general hat-point. This general definition encompassed O.K.-points and also the good ultrafilters defined by Keisler ([Kei64]). They have been able to also extend the independent matrix method construction to construct these general points and get some surprising new results (see [KuBa01] or the survey [KuBa02]).

2.2 16 Topological Types

This section is an exposition of the techniques developed in van Mill's [vMill82a] which, in turn, relied on results from [Kun80] and [CS80],[vD81]. The first two subsections show the techniques and the last gives the original application.

2.2.1 Remote Filters

2.17 Definition ([FG62]). A closed filter on X (see note 1.7) is called *remote* if for any nowhere dense $D \subseteq X$ there is some $F \in X$ disjoint with D.

Remote points were first defined by Fine and Gillman who were able to prove using CH that they exist in $\beta\mathbb{R}$. They were later studied by several authors. E. van Douwen ([vD81]), S. Chae and J. Smith ([CS80]) independently proved that they exists in the remainders of nonpseudocompact spaces of countable π-weight. In the converse direction T. Terada ([Ter79]) was able to prove that pseudocompact spaces do not have remote points.

Van Mill invented a combinatorial method for constructing remote filters via a sequence of n-linked systems with increasing n. We illustrate this method proving the following theorem

2.18 Theorem ([vMill82a],1.3). *If $X = \sum_{n<\omega} X_n$ where each X_n is compact and a product of at most ω_1 spaces of countable π-weight, then there is a remote filter $\mathcal{F}$ on X such that each $F \in \mathcal{F}$ misses only finitely many X_n's.*

Before giving the proof, we will introduce some notation and prove three technical lemmas.

2.19 Notation. *Suppose $\pi : X \to Y$ is a mapping (usually a projection), and $\mathcal{F}_0, \mathcal{F}_1$ are systems of closed sets on X and Y respectively. We will write*

$$\mathcal{F}_1 \sqsubseteq \mathcal{F}_0 \equiv \pi^{-1}[\mathcal{F}_1] := \{\pi^{-1}[F] : F \in \mathcal{F}_1\} \subseteq \mathcal{F}_0,$$

and say that $\mathcal{F}_0$ extends the lift of $\mathcal{F}_1$ via π. Also, given a system $\mathcal{F}$ we define

$$c(\mathcal{F}) = \sup\left\{n < \omega : (\forall \mathcal{F}' \in [\mathcal{F}]^n)\left(\bigcap \mathcal{F}' \neq \emptyset\right)\right\},$$

the maximal n (or ω) such that $\mathcal{F}$ is n-centered. Notice that if $\mathcal{F}_1 \sqsubseteq \mathcal{F}_0$ then $c(\mathcal{F}_0) \leq c(\mathcal{F}_1)$.

2.20 Lemma ([vMill82a],1.1). *Suppose that $\pi : X \to Y$ is open, X is compact and $\mathcal{B}$ is a π-base of X closed under finite unions. If $\mathcal{F}$ is an n-centered remote system of closed sets on Y and $N \subseteq X$ is nowhere dense then there is a $B \in \mathcal{B}$ whose closure is disjoint from N such that $\pi^{-1}[\mathcal{F}] \cup \{B\}$ is n-centered.*

Proof. For $B \in \mathcal{B}$ let $U(B) = \mathrm{int} f[B]$. Define $\mathcal{C} = \{U(B) : B \in \mathcal{B}\ \&\ \overline{B} \cap N = \emptyset\}$. Since f is open and since $\bigcup\{B \in \mathcal{B} : \overline{B} \cap N = \emptyset\}$ is dense, $Y \setminus \bigcup \mathcal{C}$ is nowhere dense. Since $\mathcal{F}$ is remote, there is $F \in \mathcal{F}$ covered by $\mathcal{C}$. Since F is compact, choose a finite subcover $\mathcal{C}' \subseteq \mathcal{C}$. Then $B = \bigcup\{B \in \mathcal{B} : U(B) \in \mathcal{C}'\}$ is as required, since $\mathcal{F}$ is n-centered. □

2.21 Lemma ([vMill82a],1.2). *Suppose $\pi : X \to Y$ is open, X compact of countable π-weight and $\mathcal{F}$ is an n-centered remote system of closed sets on Y. Then there is an $(n$-1$)$-centered remote system $\mathcal{F}_0$ which extends the lift of $\mathcal{F}$, i.e. $\mathcal{F} \sqsubseteq \mathcal{F}_0$.*

Proof. Let $\mathcal{B}$ be a countable π-base for X which is closed under finite unions. For $2 \le i \le n$ define

$$\mathcal{F}(i) = \{B \in \mathcal{B} : \{\overline{B}\} \cup \pi^{-1}[\mathcal{F}_0] \text{ is } i\text{-centered}\}$$

By the previous lemma each $\mathcal{F}(i)$ is remote. Since the restriction of π to a regular closed set is still open, we can use the previous lemma to show

Claim. For each $E \in \mathcal{F}(i), n \ge i > 2$ and for each nowhere dense $N \subseteq X$ there is $F \in \mathcal{F}(i-1)$ disjoint from N with $\overline{F} \subseteq E$.

Proof of claim. Enumerate $\mathcal{F}(i)$ as $\{E^i_k : k < \omega\}$ and for a nowhere dense set $N \subseteq X$ and $2 \le i \le n$ define

$$K(N,i) = \{k < \omega : \overline{E^i_k} \cap N = \emptyset\}$$

Now for $m = n$ define $k(N,m) = \min K(N,m)$ and for $2 \le m < n$

$$k(N,m) = \min\{k < \omega : (\forall k' \le k(N,m+1))(\exists j \le k)(j \in K(N,m) \ \&\ E^m_j \subseteq E^{m+1}_{k'})\}.$$

Let

$$F(N) = \bigcup_{i=2}^{n} \bigcup \{\overline{E^i_k} : k \le k(N,i) \ \&\ k \in K(N,i)\}.$$

■

Claim. For each $i \le n-1$ and $N_1, \dots, n_i$ there is some $k \le \max\{k(N_j, n-i+1) : j = 1, \dots, i\}$ such that $E^{n-i+1}_k \subseteq \bigcap_{j=1}^{i} F(N)$.

Proof. The claim is proved by induction on i. ■

Claim. The family $\mathcal{F}_0 = \{F(N) : N \text{ is nowhere dense}\} \cup \pi^{-1}[\mathcal{F}]$ is $(n$-1$)$-centered.

■

Let $n-1 = i_0 + i_1$. Choose nowhere dense sets $N_1, \dots, N_{i_0}$ and $F_1, \dots, F_{i_1} \in \mathcal{F}_0$. The case $i_1 = 0$ is taken care of by the previous claim. Also by the previous claim $\bigcap_{j=1}^{i_0} F(N)$ contains some element of $\mathcal{F}(n - i_0 + 1) = \mathcal{F}(i_1 + 2)$ and this finishes the proof. □

2.22 Lemma ([KuMi80], Lemma 2.1). *Suppose $X = \prod_{\alpha<\omega_1} X_\alpha$ where each X_α has countable π-weight. Let $Y_\alpha = \prod_{\beta<\alpha} X_\alpha$ and $\pi_{\omega_1\alpha} : X \to Y_\alpha$ be the natural projection. Then for each nowhere dense set $N \subseteq X$ there is $\alpha < \omega_1$ such that $\pi_{\omega_1\alpha}[N]$ is nowhere dense in Y_α.* □

proof of theorem 2.18. Let $X_n = \prod_{\alpha<\omega_1} X^\alpha_n$, $Y^\alpha_n = \prod_{\beta<\alpha} X^\beta_n$ and $Y^\alpha = \sum_{n<\omega} Y^\alpha_n$. For $\alpha < \beta$ and $n < \omega$ let $\pi^n_{\beta\alpha} : Y^\beta_n \to Y^\alpha_n$ and $\pi_{\beta\alpha} : Y^\beta \to Y^\alpha$ be the projections. By induction on $\alpha < \omega_1$ we will construct remote systems $\mathcal{F}^\alpha_n$ of closed sets on Y^α_n such that

(i) $\liminf_{n\to\infty} c(\mathcal{F}^\alpha_n) = \infty$,

(ii) For each $\alpha < \beta$ there is an $n < \omega$ such that for each $m \ge n$, $\mathcal{F}^\alpha_m \sqsubseteq \mathcal{F}^\beta_m$.

Assume for a moment that the induction can be carried out. Let

$$\mathcal{F}^\alpha = \{F \subseteq Y^\alpha : F \cap Y^\alpha_n \in \mathcal{F}^\alpha_n\}$$

and finally

$$\mathcal{F} = \bigcup_{\alpha<\omega_1} \pi^{-1}_{\omega_1\alpha}[\mathcal{F}^\alpha]$$

Claim. $\mathcal{F}$ is remote.

Proof of claim. Given a nowhere dense set N use theorem 2.22 to find $\alpha < \omega_1$ such that $N_n = \pi^n_{\omega_1\alpha}[N]$ is nowhere dense in Y^α_n for each $n < \omega$. Then for $n < \omega$ choose $F_n \in \mathcal{F}^\alpha_n$ disjoint from N_n and let $F^\alpha = \bigcup_{n<\omega} F_n$. Clearly $F^\alpha \in \mathcal{F}^\alpha$ and $\pi^{-1}_{\omega_1\alpha}[F^\alpha] \in \mathcal{F}$ is disjoint from N. ■

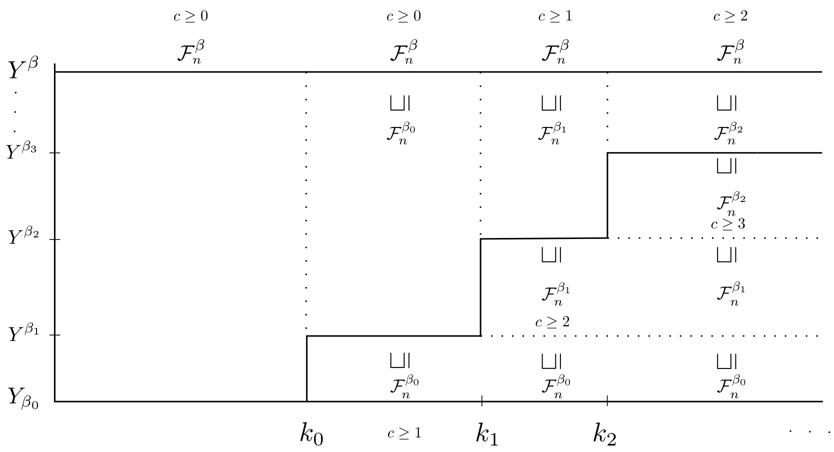


Figure 2.1: Inductive construction of $\mathcal{F}_n^\beta$'s

Claim. $\mathcal{F}$ is centered.

Proof of claim. Take $F_0, \dots, F_k \in \mathcal{F}$. Then $F_i = \pi^{-1}_{\omega_1\alpha_i}[F^{\alpha_i}]$ for some $F^{\alpha_i} \in \mathcal{F}^{\alpha_i}$. Fix $\alpha < \omega_1$ bigger than all the α_i's. Next, using (i) and (ii), find $n < \omega$ such that $c(\mathcal{F}_n^\alpha) > k$ and for each $i \leq k$, $\mathcal{F}_n^{\alpha_i} \sqsubseteq \mathcal{F}_n^\alpha$. Then $H_i := (\pi^n_{\alpha\alpha_i})[F^{\alpha_i} \cap Y_n^{\alpha_i}] \in \mathcal{F}_n^\alpha$ so, since $c(\mathcal{F}_n^\alpha) > k$, $F = \bigcap_{i \leq k} H_i$ is nonempty and $(\pi^n_{\omega_1\alpha})^{-1}[F] \subseteq \bigcap_{i\leq k} F_i$. ■

Claim. If $\mathcal{F}'$ is the closed filter generated by $\mathcal{F}$ then for each $F \in \mathcal{F}'$ the set $\{n : F \cap X_n = \emptyset\}$ is finite.

Proof of claim. This is proved in the same way as the previous claim. ■

So it remains to be shown that the induction can indeed be carried out. To construct $\mathcal{F}_n^0$ we need only realize that Y^0 is a nonpseudocompact space of countable π-weight and use the results of Chae and Smith ([CS80], theorems 1 and 3) or van Douwen ([vD81]) that there are remote points in βY^0. So suppose we have constructed $\mathcal{F}_n^\alpha$ for $\alpha < \beta$. If β is a successor, apply lemma 2.21.

If β is limit let $\{\beta_n : n < \omega\}$ be a strictly increasing cofinal subset of β. We will construct the $\mathcal{F}_n^\beta$'s by induction in consecutive blocks of n's (see figure 2.1).

Choose k_0 such that $1 \leq c(\mathcal{F}_{k_0}^{\beta_0})$ and $\mathcal{F}_n^{\beta_0} \sqsubseteq \mathcal{F}_n^{\beta_1}$ for each $n \geq k_0$. For $i < k_0$ choose remote systems $\mathcal{F}_i^\beta$ on Y_i^β arbitrarily.

Assume we have constructed k_i and $\mathcal{F}_n^\beta$ for $n < k_i$. Choose $k_{i+1} > k_i$ such that $i + 1 \leq c(\mathcal{F}_n^{\beta_i})$ and $\mathcal{F}_n^{\beta_i} \sqsubseteq \mathcal{F}_n^{\beta_{i+1}}$ for each $n \geq k_{i+1}$ and use lemma 2.21 to construct $\mathcal{F}_n^\beta \sqsupseteq \mathcal{F}_n^{\beta_i}$ with $c(\mathcal{F}_n^\beta) \geq i$.

It is clear that in the end we will have constructed $\mathcal{F}_n^\beta$ satisfying (i)-(ii). This finishes the proof. □

2.2.2 Embedding Projective Covers

We now turn to another technique which uses K. Kunen's method for constructing O.K.-points to embed various spaces into ω^* in a very special way. First we will state a generalization of proposition 2.11 due to van Mill:

2.23 Definition. A closed set $F \subseteq X$ is κ-O.K if for each sequence $\{U_n : n < \omega\}$ of neighbourhoods of A there are neighbourhoods $\{V_\alpha : \alpha < \kappa\}$ of A such that for each finite $A \subseteq \kappa$.

$$\bigcap_{\alpha \in A} V_\alpha \subseteq U_{|A|}$$

2.24 Proposition ([vMill82a],2.1). *Suppose X is locally compact and σ-compact and $A \subseteq X^*$ is ω_1-O.K. Then the closure of any ccc subset of X^* disjoint from A is also disjoint from A.* □

The following is the main embedding theorem.

2.25 Theorem ([vMill82a],2.6). *If X is of the form $\omega \times Z$ where Z is a compact space of weight at most $\mathfrak{c}$, then the projective cover (see 1.15) of any continuous ccc image of ω^* can be embedded into X^* as a $\mathfrak{c}$-O.K. set.*

2.26 Corollary. *The projective cover of any continuous ccc image of ω^* can be embedded into ω^* as a $\mathfrak{c}$-O.K. set.* □

The proof splits into two parts. First one has to prove the proposition:

2.27 Proposition ([vMill82a],2.4). *If X is as above, $\mathfrak{F}$ is a closed filter on X such that for each $F \in \mathfrak{F}$ the set $\{n : F \cap \{n\} \times Z = \emptyset\}$ is finite and if Y is a continuous image of ω^* then there is a continuous surjection $g : X^* \to Y$ and a closed $\mathfrak{c}$-O.K. set $A \subseteq X^*$ such that $A \subseteq \bigcap_{F \in \mathfrak{F}} F^*$ and $g \restriction A$ is irreducible.* □

The proof of this proposition is a straightforward topological adaptation of the proof of theorem 2.9 and we do not include it. (During the induction one uses the fact that there are only $\mathfrak{c}$-many closed G_δ sets in X and one also has to make sure that g is irreducible.)

proof of theorem 2.25. By the previous proposition, there is a closed $\mathfrak{c}$-O.K. set $A \subseteq X^*$ which admits an irreducible map onto Y. Since X^* is an F-space (see theorem 1.13) and since A is ccc (by irreducibility and the fact that Y is ccc), A is extremally disconnected (see proposition 1.14). Hence $A \simeq \mathrm{E}Y$. □

Another embedding theorem, which we shall use in section 2.3, is due to P. Simon:

2.28 Theorem ([Sim85]). *The Čech-Stone compactification of any T_3 ED space of weight $\leq \mathfrak{c}$ can be embedded into ω^* as a closed weak P-set.* □

2.2.3 The 16 types

In this section we will use the techniques developed above to construct sixteen types of points in ω^*.

2.29 Definition (van Mill).

(T_1) $x \in T_0$ if it is a limit point of a countable discrete subset.

(T_2) $x \in T_1$ if it is a limit point of a countable crowded π-homogeneous set of countable π-weight.

(T_3) $x \in T_2$ if it is a limit point of a countable crowded π-homogeneous set of π-weight ω_1.

(T_4) $x \in T_3$ if it is a limit point of a locally compact ccc nowhere separable set.

2.30 Theorem (van Mill). *Given $A \subseteq \{1, 2, 3, 4\}$ there is $x \in \omega^*$ which is of type T_i for each $i \in A$ but not of type T_j for $j \notin A$.*

Proof.
1 Case $A = \emptyset$
Use theorem 2.9 to construct a $\mathfrak{c}$-O.K. point p in ω^* and note that, by proposition 2.24, $p \notin T_4$.
2 Case $A = \{1\}$
Notice that $\beta\omega$ is a continuous image of ω^* and, since $\beta\omega$ is ED, $\mathrm{E}\beta\omega \simeq \beta\omega$. To construct the required point, take a $\mathfrak{c}$-O.K. point in $X = \omega^*$. As $X \subseteq \beta\omega = Y$ we can use corollary 2.26 to embed Y in ω^* as a $\mathfrak{c}$-O.K. set. Then $p \in X \subseteq Y \subseteq \omega^*$ will be the required point.
3 Case $A = \{2\}$
Since the Cantor space is a continuous image of ω^*, its projective cover $\mathrm{E}2^\omega$ embeds into ω^* as a $\mathfrak{c}$-O.K. subset $X \subseteq \omega^*$. Let $\{X_n : n < \omega\}$ be a sequence of pairwise disjoint nonempty clopen subsets of X whose union is dense in X while $X_n \simeq X$. Since X is E.D., $\beta\big(\bigcup_{n<\omega} X_n\big) \simeq X$. By proposition 2.27 and theorem 2.18 there is a remote $\mathfrak{c}$-O.K. point $p \in X \setminus \bigcup_{n<\omega} X_n$. This point is as required.
4 Case $A = \{3\}$
Take $X = \mathrm{E}2^{\omega_1}$ and continue as in the previous case.
5 Case $A = \{4\}$
By [Bell80, 2.1] and [vMill79, 5.1] there is a ccc nowhere separable continuous image X of ω^*. The one-point compactification $\alpha(\omega \times X)$ is also a continuous image of ω^* so $\mathrm{E}\alpha(\omega \times X)$ embeds as a $\mathfrak{c}$-O.K. set $Y \subseteq \omega^*$. Notice that $Y \simeq \beta(\omega \times \mathrm{E}X)$. Let $\pi : \omega \times \mathrm{E}X \to \mathrm{E}X$ be the projection and for each countable $A \subseteq \omega \times \mathrm{E}X$ let $\{U_n(A) : n < \omega\}$ be a maximal pairwise disjoint collection of nonempty clopen subsets of $\mathrm{E}X$ disjoint from $\overline{\pi[A]}$. Since $\mathrm{E}X$ is nowhere separable, the set $D_A = \bigcup_{n<\omega} U_n(A)$ is dense for each such A. Let

$$F_A = \bigcup_{n<\omega} \{n\} \times D_A.$$

Then $F_A \cap \overline{A} = \emptyset$ and the closed filter generated by $\{F_A : A \in [\omega \times \mathrm{E}X]^\omega\}$ satisfies the requirements of theorem 2.27. Thus, by this theorem, there is $x \in \beta(\omega \times \mathrm{E}X) \simeq Y$ which is a $\mathfrak{c}$-O.K. point of $(\omega \times \mathrm{E}X)^*$ disjoint from the closure of any countable $A \subseteq \omega \times \mathrm{E}X$. As Y was a $\mathfrak{c}$-O.K. subset of ω^*, this finishes the proof.

6–8 Case $A = \{1,2\}, \{1,3\}, \{1,4\}$
Take $p \in T_i$ $(i = 2,3,4)$ and embed the ambient ω^* into $\omega^* \subseteq \beta\omega = X$ as a $\mathfrak{c}$-O.K. set. Then embed X into ω^* as a $\mathfrak{c}$-O.K. set.

9 Case $A = \{2,3\}$
Let $X = \mathrm{E}2^\omega$, $Y = \mathrm{E}2^{\omega_1}$. Assume that $\beta(\omega \times X) \subseteq \omega^*$ is $\mathfrak{c}$-O.K. Working in $\beta(\omega \times X)$ assume $\beta(\omega \times Y) \subseteq (\omega \times X)^*$ is $\mathfrak{c}$-O.K. and remote (by theorem 2.18). Finally choose a $\mathfrak{c}$-O.K. remote point $x \in (\omega \times Y)^*$.

10 Case $A = \{2,4\}$
As in case 9, replace Y by the projective cover of a ccc nowhere separable image of ω^* and use the point constructed in case 5.

11 Case $A = \{3,4\}$
As in case 10, replace X by $\mathrm{E}2^{\omega_1}$.

12–16 Case $A = \{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \{1,2,3,4\}$
Use the same techniques as before. □

2.3 Seventeenth Topological Type

The motivation for the seventeenth type comes from van Mill's second type:

2.31 Theorem (van Mill)**.** *There is a point $p \in \omega^*$ which is a limit point of a countable discrete set and the countable sets whose limit point it is form a filter.* □

This motivated P. Simon to define the following notion, which we have called a lonely point in [Ver08]. We want essentially the same type of point as in the above theorem only replacing the countable discrete set whose limit point it is by a crowded set:

2.32 Definition (Simon)**.** A point $p \in X$ is a *lonely point* provided:

(i) p is ω-discretely untouchable, i.e. not a limit point of a countable discrete set,

(ii) p is a limit point of a countable crowded (i.e. without isolated points) set and

(iii) The countable sets whose limit point p is form a filter.

In [Ver08] we were able to show that they exist in some open dense subspace of ω^* and we have later extended this result in [Ver11] to prove that they exist in ω^*. The aim of this section is to present this proof.

2.33 Theorem. *ω^* contains a lonely point.*

The following observation motivates our approach.

2.34 Observation. *If $F \subseteq X$ is a weak P-set of X and $x \in F$ is a lonely point of F then it is also a lonely point of X.* □

The idea is to construct an countable space X such that βX has a lonely point and then use the above observation together with the embedding theorem 2.28 of P. Simon. The space will be countable so as to make sure that the weight is at most $\mathfrak{c}$. From the definition of a lonely point, we immediately get that the space cannot have disjoint dense sets, i.e. it must be irresolvable. It would also be helpful, if no countable subset of the remainder had limit points in X. Such spaces are called $\aleph_0$-bounded and we will look at them shortly. First, however, we shall see what is known about irresolvable spaces.

2.3.1 Maximal Topologies & Irresolvable spaces

2.35 Definition (vanDouwen)**.** A crowded topological space is *resolvable* if it contains at least two disjoint dense subsets. It is *irresolvable* if it is not resolvable. It is *hereditarily irresolvable* (HI) if each subspace is irresolvable and it is *open hereditarily irresolvable* (OHI) if each open subspace is irresolvable.

The definition requires the space to have no isolated points, since any space with isolated points would be automatically irresolvable. These spaces have been first constructed by E. Hewitt ([Hew43]), who defined the notion of a hereditarily irresolvable space, and M. Katětov ([Kat47]) at the end of the forties. Van Douwen extended their work significantly in [vD93] where he used these spaces to construct a separable $\leq$ 2-to-one image of ω^*. Quite recently the notion of irresolvability (and especially resolvability) has become popular again, see e.g. [CW05], [JSS05] or [Pav05], which is a summary.

Irresolvable spaces are, in some sense, close to discrete spaces. This is suggested by the method Hewitt used to construct them — he considered maximal crowded topologies. These topologies turn out to be irresolvable. First we need a definition:

2.36 Definition. If P is a property of a topology (e.g. T_1, T_2, crowded, etc.), we say that τ is *maximal* P if it has P but cannot be refined to a strictly stronger topology having P.

2.37 Note. Originally Hewitt defined maximal topologies to be maximal crowded topologies. In this chapter we will use the modern terminology (as in the previous definition) not to confuse the reader.

2.38 Proposition (Hewitt)**.** *If τ is a maximal crowded topology on X, then any two τ-dense sets intersect.*

Proof. Suppose D_1, D_2 are disjoint dense. Then D_1 is not open and the topology generated by $\tau \cup \{D_1\}$ is a strictly finer topology which does not have any isolated point. □

We will need some of the results of van Douwen and we devote the rest of this section to their presentation.

2.39 Definition. A topological space is *perfectly disconnected* if no point is a limit point of two disjoint sets. It is *nodec* if every nowhere dense set is closed and is *ultradisconnected* if it is crowded and any two disjoint crowded subsets have disjoint closures.

Before continuing, notice that in a perfectly disconnected space every point is lonely. Thus, ideally, we would like to find a countable perfectly disconnected space with an $\aleph_0$-bounded remainder. The results from this section will allow us to do precisely that.

2.40 Theorem ([vD93, 2.2])**.** *For a crowded space X the following are equivalent*

(i) X is perfectly disconnected

(ii) a subset of X is open if and only if it is crowded

(iii) X is maximal crowded.

(iv) X is ultradisconnected and nodec

(v) X is extremally disconnected, OHI and nodec

□

Note, that an ultradisconnected space is hereditarily irresolvable (i.e. any crowded subspace is irresolvable) and extremally disconnected. See diagram 2.2 illustrating some relations between different irresolvability properties.

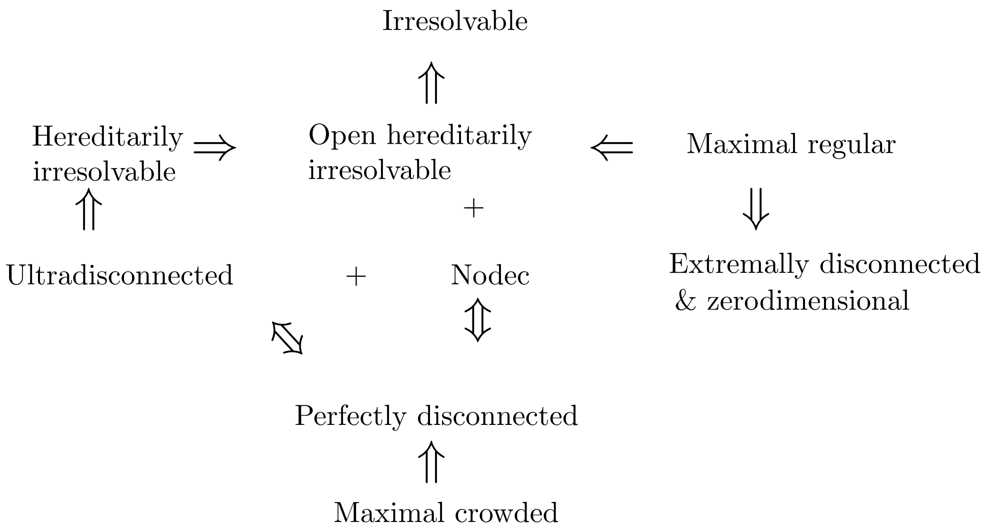


Figure 2.2: Relations between irresolvability properties

2.41 Theorem ([vD93],1.7,1.11). *Maximal regular spaces are zerodimensional, ED and OHI.* □

2.42 Theorem ([vD93],1.4,1.6). *If A, B are disjoint crowded subspaces of a maximal regular space, then $\overline{A}$ and $\overline{B}$ are disjoint.* □

2.43 Theorem ([vD93],2.2). *If X is ED and OHI and each nowhere dense subset of X is closed then X is perfectly disconnected.* □

The following theorem is not explicitly stated in van Douwen's paper, but its proof is essentially given in his Lemma 3.2 and Example 3.3.

2.44 Theorem (van Douwen). *Any countable maximal regular space X contains an open perfectly disconnected subspace.*

Proof. For each $Z \subseteq X$ let

$$A_Z = \{x \in Z : x \text{ is a limit point of a relatively discrete subset of Z}\}$$

Claim. $A_Z \neq Z$ for each nonempty open subset Z of X.

Proof of claim. Assume otherwise. Enumerate X as $\langle x_n : n < \omega\rangle$. By induction construct pairwise disjoint, relatively discrete sets $\langle D_n : n < \omega\rangle$ such that:

(i) $\bigcup_{i<n} D_i \subseteq \overline{D_n}$ for all $n < \omega$ and

(ii) $x_n \in \overline{D_n}$ for $n < \omega$.

This will lead to a contradiction with the irresolvability of Z (by theorem 2.41, X is OHI, so Z is irresolvable) since $\bigcup_{n<\omega} D_{2n}$ and $\bigcup_{n<\omega} D_{2n+1}$ would then be disjoint dense subsets of Z. To see that the construction can be carried out let $D_0 = \{x_0\}$ and assume we have constructed D_i for $i \leq n$. Let $Y = D_n \cup X \setminus \overline{D_n}$. Since D_n is relatively discrete, Y is open. Since Z is regular and D_n is countable and relatively discrete, there is a pairwise disjoint collection of open sets $\{U_x : x \in D_n\}$ such that $x \in U_x \subseteq Y$. Since we assumed $A_Z = Z$ we can choose for each $x \in D_n$ a relatively discrete set D_x such that $D_x \subseteq U_x$ and $x \in \overline{D_x} \setminus D_x$. Let $D'_{n+1} = \bigcup_{x\in D_n} D_x$. If x_{n+1} is a limit point of D'_{n+1} let $D_{n+1} = D'_{n+1}$ otherwise let $D_{n+1} = D'_{n+1} \cup \{x_{n+1}\}$. Then D_{n+1} is as required. ■

Claim. $\operatorname{int} A_X = \emptyset$.

Proof of claim. For any clopen U, $A_X \cap U = A_U$. Since X is regular and countable, it is zerodimensional. Suppose, aiming towards a contradiction, that U is nonempty clopen and $U \subseteq A_X$. By the previous claim $U \setminus A_U \neq \emptyset$. Take some $x \in U \setminus A_U$. This x is not a limit point of a relatively discrete subset of U so, since U is clopen, it is not a limit point of a relatively discrete subset of X so $x \notin A_X$ so $U \setminus A_X \neq \emptyset$ a contradiction. ■

Claim. A_X is nowhere dense.

Proof of claim. Take any open $U \subseteq X$. Then $U \setminus A_X$ is dense in U_X, since $\operatorname{int} A_X = \emptyset$. Since X is OHI (by theorem 2.41), U is irresolvable so A_X cannot be dense in U so $U \not\subseteq \overline{A_X}$. Thus $\operatorname{int} \overline{A_X} = \emptyset$. ■

Claim. If $A \subseteq X$ is nowhere dense then there is a discrete $D \subseteq A$ dense in A.

Proof of claim. Let $D = \{x \in A : x \text{ is isolated in } A\}$. Since X is regular and countable D is relatively discrete. Since A is nowhere dense, D is discrete. Let $E = A \setminus \overline{D}$. Then E has no isolated points. Also $X \setminus E$ has no isolated points. By theorem 2.42 E must be open which contradicts that A is nowhere dense. ■

Let

$$\vartheta = \{x \in X : x \text{ is not a limit point of a nowhere dense subset of } X\}$$

By the previous claim (and by the fact that each discrete subset of X is nowhere dense)

$$\vartheta = \{x \in X : x \text{ is not a limit point of a discrete set}\}$$

Then $X \setminus \vartheta \subseteq A_X$ so $X \setminus \vartheta$ is nowhere dense, so $\operatorname{int} \vartheta$ is nonempty. We finally show that $\operatorname{int} \vartheta$ is perfectly disconnected. By the definition of ϑ any nowhere dense subset of $\operatorname{int} \vartheta$ is closed. Now it remains to apply theorem 2.43 remembering that by theorem 2.41 $\operatorname{int} \vartheta$ is ED and OHI (any open subspace of a maximal regular space is maximal regular). □

2.3.2 Special spaces

We shall now turn our attention to spaces having an $\aleph_0$-bounded remainder.

2.45 Definition ([FlGuWe70]). A space X is $\aleph_0$-*bounded* provided every countable subset of X has compact closure in X.

The following definition and theorem is taken from [DGS88]:

2.46 Definition. Let $p \in \omega^*$ be a weak P-point. The space $\mathcal{G}_\omega$ is the space $\omega^{<\omega}$ of all finite sequences of natural numbers with $G \subseteq \omega^{<\omega}$ being open precisely when for each $\sigma \in G$ the set $\{n : \sigma^\frown n \in G\}$ is in p.

2.47 Theorem (Dow, Gubbi, Szymanski). *The remainder of $\mathcal{G}_\omega$ is $\aleph_0$-bounded. Moreover $\mathcal{G}_\omega$ is a T_2, zerodimensional, ED space.*

Proof. It is clear that the space is T_2: Given $\sigma, \tau \in \mathcal{G}_\omega$ if $\sigma \perp \tau$ then $G_\sigma = \{s \in \mathcal{G}_\omega : \sigma \subseteq s\}$ and $G_\tau = \{s \in \mathcal{G}_\omega : \tau \subseteq s\}$ are disjoint open sets separating σ from τ. If $\sigma \subseteq \tau$ let $G_\sigma = \{s \in \mathcal{G}_\omega : \tau \not\subseteq s\}$ and G_τ as before. Again we get two disjoint open sets separating σ from τ.

To see that the space is zerodimensional, notice that given $\tau \in U \subseteq \mathcal{G}_\omega$, the set

$$H_\tau = \{s \in \mathcal{G}_\omega : (\forall |\tau| \leq n \leq |s|)(s \restriction n \in U \ \&\ \tau \subseteq s)$$

is a clopen subset of U containing τ.

To see that it is ED consider an open set $U \subseteq \mathcal{G}_\omega$ with $t \in \overline{U}$. By recursion construct $\langle T_n : n < \omega\rangle$ such that $T_n \subseteq U$ and for each $s \in T_n$ the set $\{k : s^\frown k \in T_{n+1}\}$ is in p. Let $T_0 = \{t\}$. If we have constructed T_n and $s \in T_n$ then, $L_s = \{k : s^\frown k \in \overline{U}\} \in p$ (This is clear if $s \in U$ and if not, then for each $k \in \omega \setminus L_s$ there would be an open U_k containing $s^\frown k$ and disjoint from U. But then $\{s\} \cup \bigcup_{k \in \omega \setminus L_{n+1}} U_k$ would be a neighbourhood of s disjoint from U contradicting $s \in \overline{U}$.). Now let $T_{n+1} = \{s^\frown k : s \in T_n, k \in L_s\}$. This finishes the recursive definition and finally let $V = \bigcup_{n<\omega} T_n$. Then $V \subseteq \overline{U}$ is an open neighbourhood of t showing that $\overline{U}$ is open.

Finally we show $\mathcal{G}_\omega$ is $\aleph_0$-bounded. First notice that, since $\mathcal{G}$ is zerodimensional, $\beta\mathcal{G}_\omega \approx Ult(Clop(\mathcal{G}_\omega))$. We introduce some notation. For $s \in \mathcal{G}_\omega$ let $\mathcal{G}_\omega(s) = \{t \in \mathcal{G}_\omega : s \subseteq t\}$, $L_s(n) = \{t \in \mathcal{G}_\omega(s) : |t| = n + |s|\}$, $succ(s) = \{t \in L_s(1)\}$ and, given an open $U \subseteq \mathcal{G}_\omega$ let $\hat{U} = \{q \in \beta\mathcal{G}_\omega : U \in q\}$. In the following, closure will always be taken in $\beta\mathcal{G}_\omega$ unless otherwise stated.

2.48 Observation. *Each $\widehat{\mathcal{G}_\omega(s)}$ is a clopen subset of $\beta\mathcal{G}_\omega$ disjoint from $\overline{(\beta\mathcal{G}_\omega \setminus \mathcal{G}_\omega(s))}$.* ■

Note that $\overline{succ(s)}$ is isomorphic to $\beta\omega$ with s being taken to p by the isomorphism. Since p was a weak P-point, together with the above, we have:

2.49 Observation. *Each $s \in \mathcal{G}_\omega$ is a weak P-point in $\overline{L_\emptyset(|s|+1)}$.* ■

Let $D = \{p_n : n < \omega\}$ be a countable subset of $\beta\mathcal{G}_\omega \setminus \mathcal{G}_\omega$ and $t \in \mathcal{G}_\omega$. We will find a neighbourhood U of t disjoint from D. Let $X_n = \overline{L_t(n)}$ and $D_n = D \cap X_n$. We shall recursively build a neighbourhood T of t in $\mathcal{G}_\omega$ and in the end let $U = \hat{T} \cap \overline{\mathcal{G}_\omega(t)}$. We let $T_0 = \{t\}$ and suppose we have constructed T_n such that $D_n \cap \overline{T_n} = \emptyset$. Since each $s \in T_n$ is a weak P-point of $\overline{L_s(n+1)}$ and since X_{n+1} is countable and disjoint from $\mathcal{G}_\omega$, we may find an open set (in $\beta\mathcal{G}_\omega$) U_s such that $s \in U_s$ and $U_s \cap D_{n+1} = \emptyset$. Since $\beta\mathcal{G}_\omega \approx Ult(Clop(\mathcal{G}_\omega))$, we may find a clopen $U'_s \subseteq \mathcal{G}_\omega$ such that $\hat{U}'_s \subseteq U_s$. Let $A_s = \{U' \cap succ(s)\}$ and let $T_{n+1} = T_n \cup \bigcup_{s \in T_n} A_s$. At the end of the recursion let $T = \bigcup_{n<\omega} T_n$ and $U = \hat{T} \cap \widehat{\mathcal{G}_\omega(t)}$. Then U is an open neighbourhood of t disjoint from $\bigcup_{n<\omega} D_n$. Finally we let $D' = D \setminus \bigcup_{n<\omega} D_n$ and it remains to find a neighbourhood of t disjoint from D'. Enumerate D' as $\{q_n : n < \omega\}$ and pick U_n a clopen neighbourhood of $\{q_i : i \leq n\}$ disjoint from $\overline{L_n(t)}$ and let $U = \bigcup_{n<\omega} U_n$. We claim that $V = \mathcal{G}_\omega(t) \setminus U$ is an open neighbourhood of t (in $\mathcal{G}_\omega$). Pick $s \in \mathcal{G}_\omega(t) \setminus U$ and suppose to the contrary that $\{n : s^\frown n \notin V\} \in p$. Then $s \in \overline{\{s^\frown n : s^\frown n \notin V\}} \subseteq \overline{U}$ and, by the definition of U, $s \in \overline{\bigcup_{i \leq |s|-|t|} U_i}$ but this is impossible, since $\bigcup_{i \leq |s|-|t|} U_i$ is clopen so then it would follow that $s \in \bigcup_{i \leq |s|-|t|} U_i \subseteq U$ a contradiction with the choice of s. □

2.3.3 Putting it all together

Now we have all that we need to construct lonely points in ω^*. We will use the space from the previous section and refine the topology to a maximal regular topology. Then we will find an open perfectly disconnected subspace using 2.44. This will give us a countable perfectly disconnected space with an $\aleph_0$-bounded remainder because of the following proposition (and the fact that the space is zerodimensional).

2.50 Proposition. *If $(X,\tau)^*$ is a zerodimensional $\aleph_0$-bounded space and $\sigma \supseteq \tau$ is also zerodimensional, then $(X,\sigma)^*$ is $\aleph_0$ bounded.*

Proof. Note that any $p \in (X,\tau)^*$ corresponds to a closed subset of $(X,\sigma)^*$ (denote it $[p]$). Now given $\{q_n : n < \omega\} \subseteq (X,\sigma)^*$ we can find $\{p_n : n < \omega\} \subseteq (X,\tau)^*$ such that $\{q_n : n < \omega\} \subseteq \bigcup\{[p_n] : n < \omega\}$. Since $(X,\tau)^*$ is $\aleph_0$-bounded, $\overline{\{p_n : n < \omega\}}^{\beta(X,\tau)} \cap X = \emptyset$ so also $\overline{\{q_n : n < \omega\}}^{\beta(X,\sigma)} \cap X = \emptyset$ which implies that $(X,\sigma)^*$ is $\aleph_0$-bounded. □

Summarizing we have the following theorem.

2.51 Theorem. *There is a countable, ED, perfectly disconnected space X with an $\aleph_0$-bounded remainder.*

Proof. Take the space $\mathcal{G}_\omega$ from theorem 2.47, and refine the topology to a maximal regular topology. Then, by the previous proposition, this space still has an $\aleph_0$-bounded remainder and so does its open perfectly disconnected subspace given by theorem 2.44. Let X be this subspace. □

Notice that this space will have a dense set of lonely points. We now finish by embedding its Čech-Stone compactification into ω^* to get lonely points in ω^*.

2.52 Theorem. *ω^* contains a lonely point.*

Proof. Let X be the space from the previous theorem. Since it is crowded perfectly disconnected, each of its points is a lonely point of X. Since its remainder is $\aleph_0$-bounded, each of its points is also a lonely point of βX. Since it is ED, βX is also ED and since it is countable, βX has weight at most $\mathfrak{c}$. Hence, by theorem 2.28, βX can be embedded as a weak P-set into ω^* and each point of X will be a lonely point of ω^* (by observation 2.34). □

Albrecht Dürer: **Grass**

Chapter 3

Consistency Results

This chapter will give a sampling of the many constructions of ultrafilters which go beyond ZFC. Once one is allowed to use additional axioms, there is a wide range of constructions one can use, or invent. This chapter will present some of them, but the majority of the possibilities will not even be mentioned. We will also omit all of the “non-results” which construct models of ZFC where certain ultrafilters do *not* exist. Probably the first result in this direction was S. Shelah's construction of a model with no P-points ([Wim82]). Many other results have followed but intriguing open questions still remain. One of the most interesting representatives of these questions is probably the following:

3.1 Question. *Is there a model of ZFC with no P-points and no Q-points?*

This has been open for a long time and seems to be a very hard question. A positive solution requires the continuum to be higher than $\aleph_2$[1]. This rules out the obvious approach of forcing with a countable support iteration of proper forcings.

We will first start with two sections presenting several constructions based on additional combinatorial assumptions beyond ZFC. The other two sections, based on results from [BlHrVe11] and [HrVer11], will present two methods which use forcing to construct various ultrafilters.

3.1 Ketonen's construction of a P-point

In this section we will show how one can, assuming additional axioms, construct P-points. We start with W. Rudin's proof under CH, then we extract the essence of the proof and show that MA is sufficient. Finally we present J. Ketonen's construction of P-points under $\mathfrak{d} = \mathfrak{c}$.

3.2 Theorem ([Rud56]). *Assume $2^\omega = \omega_1$. Then there is a P-point p in ω^*.*

Proof. Let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of $\mathcal{P}(\omega)$ and let $\langle C_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of ${}^\omega\mathcal{P}(\omega)$ with each sequence listed cofinally often. By induction construct filters $\mathcal{F}_\alpha$ for $\alpha < \omega_1$ satisfying:

(i) For each $\alpha < \omega_1$ either $A_\alpha \in \mathcal{F}_{\alpha+1}$ or $(\omega \setminus A_\alpha) \in \mathcal{F}_{\alpha+1}$.

(ii) For each $\alpha < \omega_1$ if all elements of the sequence C_α are in $\mathcal{F}_\alpha$ then there is a $B \in \mathcal{F}_{\alpha+1}$ such that $|B \setminus C_\alpha(n)| < \omega$ for all $n \in \omega$.

(iii) For each $\alpha < \omega_1$ the filter $\mathcal{F}_\alpha$ has a countable basis.

Let $\mathcal{F}_0$ be the Fréchet filter on ω. If $\alpha < \omega_1$ is limit, let $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$ and both (i), (ii) and (iii) are satisfied. So suppose $\alpha < \omega_1$ is not limit. If there is an $F \in \mathcal{F}_\alpha$ such that $F \cap A_\alpha = \emptyset$ then let $\mathcal{F}'_\alpha = \mathcal{F}_\alpha \cup \{(\omega \setminus A_\alpha)\}$ otherwise let $\mathcal{F}'_\alpha$ be the filter generated by $\mathcal{F}_\alpha \cup \{A_\alpha\}$. $\mathcal{F}'_\alpha$ is a filter satisfying (i),(iii). Suppose the sequence C_α consists of elements of $\mathcal{F}'_\alpha$. Let $\{F_n : n < \omega\}$ be an enumeration of the basis of $\mathcal{F}'_\alpha$. Inductively for each $k < \omega$ choose $n_k \in (\bigcap_{i<k} F_i) \setminus \{n_0, \dots, n_{k-1}\}$. This is possible since $\mathcal{F}'_\alpha$ is centered. Now let $\mathcal{F}_{\alpha+1} = \langle \mathcal{F}_\alpha \cup \{\{n_k : k < \omega\}\}\rangle$. This is a centered system and the set $|\{n_k : k < \omega\} \setminus C_\alpha(n)| < \omega$ (in fact, if $C_\alpha(n) = F_i$, then the cardinality is at most i).

Now let $\mathcal{F} = \bigcup_{\alpha<\omega_1} F_\alpha$. (i) Guarantees that $\mathcal{F}$ is an ultrafilter and (ii) guarantees that it is a P-point. □

[1] Compare this with the $\mathfrak{p} = \mathfrak{t}$ problem

The continuum hypothesis turns out to be very strong and simplifies the study of ω^* considerably. But what if the continuum is larger and hence ω^* is richer? Booth noticed ([Boo70]), that Martin's axiom (see definition 1.34) is enough to prove the existence of P-points. The proof proceeds similarly as in 3.2 where condition (ii) is guaranteed using the following standard lemma:

3.3 Lemma. *Assume MA. If $\lambda < \mathfrak{c}$ and $\{A_\alpha : \alpha < \lambda\}$ is a centered system of subsets of ω then they have an infinite pseudointersection, that is there is an infinite A which is almost contained in all A_α's.*

Proof. Define $P = \{(F, I) : F \in [\omega]^{<\omega}, I \in [\lambda]^{<\omega}\}$, where $(G, J) \leq (F, I)$ if and only if $G \supseteq F$, $J \supseteq I$ and $G \setminus F \subseteq \bigcap_{\alpha\in I} A_\alpha$. Then $(P, \leq)$ is a ccc poset of size λ. Now consider the sets

$$\mathcal{D}_\alpha = \{(F, I) \in P : \alpha \in I\},\ \mathcal{D}'_n = \{(F, I) \in P : |F| \geq n\}.$$

Each of them is dense in $(P, \leq)$. So there is a filter $\mathcal{F}$ on $(P, \leq)$ which meets each of them. If we let $A = \bigcup\{F : (\exists I \in [\lambda]^{<\omega})(F, I) \in \mathcal{F}\}$ then A is infinite because $\mathcal{F}$ meets each $\mathcal{D}'_n$ and is almost contained in each A_α because some (F, I) is contained in $\mathcal{D}_\alpha \cap \mathcal{F}$. Then $A \setminus A_\alpha \subseteq F$. □

3.4 Note. The previous construction in fact constructs ultrafilters with much stronger properties. They are selective ultrafilters generated by a tower. They are also $P_{<\mathfrak{c}}$-points, i.e. the intersection of less than $\mathfrak{c}$ neighbourhoods is again a neighbourhood.

In [Ket76] it is shown that if we only want a P-point, much less is needed. Recall the following definition (see 1.35).

3.5 Definition. A family $\mathcal{F}$ of functions from ω to ω is *dominating* if and only if for any $g \in {}^\omega\omega$ there is an $f \in \mathcal{F}$ with $g \leq^* f$ (i.e. $g(n) > f(n)$ for only finitely many n's). The *dominating number* $\mathfrak{d}$ is defined to be the least cardinality of a dominating family.

It is easy to see, that $\omega < \mathfrak{d} \leq \mathfrak{c}$ so CH implies $\mathfrak{d} = \mathfrak{c}$.

3.6 Fact (MA). $\mathfrak{d} = \mathfrak{c}$. □

3.7 Lemma ([Ket76],1.3). *Assume $\mathfrak{d} = \mathfrak{c}$. If $\mathcal{F}$ is a (uniformly) centered system of size $< \mathfrak{c}$, and $\langle F_n : n < \omega\rangle$ is a descending sequence of sets which are $\mathcal{F}$-positive, i.e. which hit each $F \in \mathcal{F}$ in an infinite set, then there is an A which is almost contained in each F_n and such that $\mathcal{F} \cup \{A\}$ is centered.*

Proof. We may assume that the sequence is descending mod $\subseteq$. Define for $F \in \mathcal{F}$ a function f_F as follows: $f_F(n) = \min F_n \cap F$. Then the family $\{f_F : F \in \mathcal{F}\}$ has size $< \mathfrak{c}$ so it is not dominating by our assumption, so there is a $g \in {}^\omega\omega$ with $\{n : g(n) > f_F(n)\}$ infinite for each $F \in \mathcal{F}$. Then if we let $A = \bigcup_{i<\omega} F_i \cap g(i)$ we are done. □

Looking at the proof of 3.2, the following theorem immediately follows:

3.8 Theorem (Ketonen). *Assume $\mathfrak{d} = \mathfrak{c}$. Then there is a P-point in ω^*.* □

3.9 Note. Ketonen's theorem in fact characterizes $\mathfrak{d} = \mathfrak{c}$ in the sense that this equality is equivalent to the fact that every filter of character $< \mathfrak{c}$ can be extended to a P-point.

3.10 Definition. A filter $\mathcal{F}$ is a *P^+-filter* if any descending sequence of $\mathcal{F}$-positive sets has an $\mathcal{F}$-positive pseudointersection. It is a *P-filter* if for any descending sequence of sets from $\mathcal{F}$ has a pseudointersection in $\mathcal{F}$.

3.11 Note. For ultrafilters, the notion of a P-filter and P$^+$-filter coincide.

We have an easy corollary of lemma 3.7

3.12 Corollary. *Assume $\mathfrak{d} = \mathfrak{c}$. Then each filter with a basis of size $< \mathfrak{c}$ is a P^+-filter.* □

It is easy to see that an ultrafilter is a P-point if and only if it is a P-filter. Since the Fréchet filter is a P-filter, one is tempted to ask how close to an ultrafilter we can get with the P-point property in ZFC alone ([Kan78]). To quantify the "how close" A. R. D. Mathias introduced the notion of a feeble filter[2]:

3.13 Definition ([Mat78]). A filter $\mathcal{F}$ is *feeble* if it is RB-above the Fréchet filter.

It turns out that being non-feeble can be very close to being an ultrafilter

3.14 Theorem ([BlaLaf89]). *The following* Filter dichotomy *is consistent with ZFC. For each filter $\mathcal{F}$ on ω precisely one of the following happens:*

[2] Kanamori called P-filters which are not feeble *coherent*

(i) $\mathcal{F}$ is RB-above the Fréchet filter

(ii) $\mathcal{F}$ is RB-above some ultrafilter.

□

The above dichotomy follows from, e.g. the cardinal inequality $\mathfrak{u} < \mathfrak{g}$ (see [Blass90]), and in turn implies the *Near coherence of filters* (see [Blass86]). It is still an open question whether one of the implications

$$\mathfrak{u} < \mathfrak{g} \quad \Rightarrow \quad \text{Filter dichotomy} \quad \Rightarrow \quad \text{Near coherence of filters}$$

can be reversed.

S. A. Jalaili-Naini and, independently, M. Talagrand have shown that a filter $\mathcal{F}$ is *feeble* if and only if it is meager when considered as a subset of $\mathcal{P}(\omega)$ with the Cantor topology.

3.15 Theorem ([Jal76],[Tal80]). *A filter is* feeble *if it is a meager subset of ω with the Cantor topology.* □

A useful corollary of the previous theorem and 3.13 is the following characterization of non-meagerness for ideals.

3.16 Theorem (Talagrand,Jalaili-Naini). *An ideal $\mathcal{I}$ is non-meager if and only if for each interval partition $\langle I_n : n < \omega\rangle$ there is an infinite set $A \in [\omega]^{<\omega}$ such that $\bigcup_{n\in A} I_n \in \mathcal{I}$.* □

One may now ask:

3.17 Question ([Kan78]). *Is there a non-meager P-filter?*

Non-meager P-filters have been constructed from varying assumptions, e.g. $\mathfrak{c} \leq \aleph_{\omega+1}$ is sufficient, but in ZFC the question still remains open. It is known, however, that a negative answer implies the existence of large cardinals:

3.18 Theorem ([Mat78]). *If there are no non-meager P-filters then $0^{\#}$ exists.* □

A survey article summarizing current knowledge about the question is [JuMaPrSi90]. Quite recently N. Dobrinen and S. Todorcević found a curious characterization of non-meager P-ideals.

3.19 Definition ([SoTo04]). A separable metric space X together with a partial ordering $\leq$ is called *basic* if

(i) Each pair of elements has a least upper bound and the least upper bound, when considered as a map from X^2 to X, is continuous,

(ii) each bounded sequence has a converging subsequence and

(iii) each converging sequence has a bounded subsequence.

3.20 Example. $(\mathbb{R}, \leq)$ is basic. □

3.21 Theorem ([DoTo11]). *An ideal $\mathcal{I}$ extending Fin is a non-meager P-ideal if and only if it is a basic space with the inclusion ordering and the metric inherited from the Cantor space.*

Proof. Assume that $\mathcal{I}$ is basic. We shall first use theorem 3.16 to show that $\mathcal{I}$ is non-meager. So let $\langle I_n : n < \omega\rangle$ be an interval partition of ω. The I_n's converge to $\emptyset$ so they must have a bounded subsequence $\langle I_{n_k} : k < \omega\rangle$. Then $\bigcup_{k<\omega} I_{n_k} \in \mathcal{I}$ so $\mathcal{I}$ is non-meager. We shall now show that $\mathcal{I}$ is a P-ideal. Suppose $\langle A_n : n < \omega\rangle$ is a sequence of sets in $\mathcal{I}$ and that $A_n \subseteq A_{n+1}$. Let $A'_n = A_n \setminus n$. Then $\langle A'_n : n < \omega\rangle$ is a sequence in $\mathcal{I}$ converging to $\emptyset$, so it must have a bounded subsequence $\langle A'_{n_k} : k < \omega\rangle$. The bound $A = \bigcup_{k<\omega} A'_{n_k}$ must be in $\mathcal{I}$ and for each $n < \omega$, $A_n \subseteq^* A$.

Assume on the other hand that $\mathcal{I}$ is a non-meager P-ideal. Condition (i) is satisfied since $\mathcal{I}$ is closed under finite unions and (ii) follows from compactness of 2^ω and the fact that $\mathcal{I}$ is downwards closed. We need to prove (iii). Let $\langle A_n : n < \omega\rangle$ be sequence in $\mathcal{I}$ converging to A. Since $\mathcal{I}$ is a P-ideal we can choose a $B \in \mathcal{I}$ such that $A_n \subseteq^* B$ for each $n < \omega$. Recursively construct a strictly increasing sequence $\langle m_k : k < \omega\rangle$ such that

(i) $A_{m_{k+1}} \cap m_k = A \cap m_k$ for each $k < \omega$ and

(ii) $A_{m_{k+1}} \setminus m_{k+2} \subseteq B$ for each $k < \omega$ (equivalently $A_{m_k} \setminus m_{k+1} \subseteq B$ for each $0 < k < \omega$)

Since $\mathcal{I}$ is non-meager, there is an infinite $K \in [\omega]^\omega$ such that $C = \bigcup_{k\in K}[m_{3k}, m_{3k+3}) \in \mathcal{I}$. Then the subsequence $\langle A_{m_{3k+1}} : k \in K\rangle$ is bounded in $\mathcal{I}$. This follows since $A_{m_{3k+1}} \cap m_{3k} \subseteq A$ by (i), $A_{m_{3k+1}} \cap [m_{3k}, m_{3k+3}) \subseteq C$ by the choice of C and K and $A_{m_{3k+1}} \setminus m_{3k+3} \subseteq B$ by (ii), so $\bigcup_{k\in K} A_{m_{3k+1}} \subseteq A \cup B \cup C \in \mathcal{I}$. □

3.2 Selective ultrafilters, Q-points

As far as I know, the following concept first appeared in [Boo70] and/or in [Cho68]:

3.22 Definition. An ultrafilter $\mathcal{U}$ is *Selective*, or *Ramsey*, if for each partition $\{A_n : n < \omega\}$ of ω, either there is some $n < \omega$ such that $A_n \in \mathcal{U}$ or there is a set $A \in \mathcal{U}$ such that $|A \cap A_n| \leq 1$ for each $n < \omega$.

It is not hard to see that every selective ultrafilter is a P-point. The converse is not true, however. As we shall see in the next section, each strong P-point (see definition 3.34) is not selective. The following property extracts what is needed to get selectivity if the filter already is a P-point.

3.23 Definition ([Cho68],[Mat78])**.** A filter $\mathcal{F}$ is a *Q*-filter (*rare* in the terminology of Choquet), if for every (interval) partition $\{I_n : n < \omega\}$ of ω into finite sets there is a set $A \in \mathcal{F}$ such that $|A \cap I_n| \leq 1$. $\mathcal{F}$ is called *rapid* ([Mok67]) if for each such partition there is a set $A \in \mathcal{F}$ such that $|A \cap I_n| \leq n$. An ultrafilter that is Q is called a *Q-point*.

3.24 Observation. *An ultrafilter $\mathcal{U}$ is selective if and only if it is a P-point and a Q-point.* □

As we have already mentioned the existence of P-points, and hence also selective ultrafilters, is unprovable in ZFC. One may ask whether the existence of Q-points is provable. This question was answered by A. Miller in the negative.

3.25 Theorem ([Miller80])**.** *There are no Q-points in the Laver Model for Borel Conjecture.* □

As in the case of P-points selective ultrafilters can be easily constructed under CH (or MA). However $\mathfrak{d} = \mathfrak{c}$ is not enough to get selective ultrafilters. How about Q-points? Somewhat surprisingly A. R. D. Mathias was able to construct Q-points provided $\mathfrak{d} = \omega_1$.

3.26 Theorem ([Mat78])**.** *Assuming $\mathfrak{d} = \omega_1$ there is a Q-point.*

3.27 Lemma. *Suppose $M \subseteq V$ is a model of ZFC such that $\omega^\omega \cap M$ is a dominating family. If $\mathcal{F} \in M$ is a Q-filter in the sense of M, then any ultrafilter extending $\mathcal{F}$ is a Q-point in V.*

Proof. Let $\mathcal{U}$ extend $\mathcal{F}$. Suppose $\langle I_n : n < \omega\rangle$ is an interval partition of ω. Let $f : \omega \to \omega$ be the strictly increasing function such that $I_n = [f(n), f(n+1))$. Since $\omega^\omega \cap M$ is dominating, there is a $g \in M$ dominating f. We may assume g is strictly increasing and that $g(n) > n$. Define $g' : \omega \to \omega$, by recursion as follows: $g'(0) = g(0)$ and $g'(n+1) = g(g'(n))$. Clearly $g' \in M$. By our assumption on $\mathcal{F}$ we can choose $F \in \mathcal{F}$ such that $|F \cap [g'(n), g'(n+1))| \leq 1$. Then $|F \cap I_n| \leq 2$ for each $n < \omega$. Now split F into two sets F_0, F_1 such that $|F_0 \cap I_n|, |F_1 \cap I_n| \leq 1$ for each $n < \omega$. Since $\mathcal{U}$ is an ultrafilter extending $\mathcal{F}$ it must contain either F_0 or F_1. □

proof of theorem 3.26. Since $\mathfrak{d} = \omega_1$ we can choose a set of ordinals $A \subseteq \omega_1$ coding a dominating family in $(\omega^\omega, \leq^*)$. Let $M = L[A]$. Then $M \cap \omega^\omega$ is a dominating family since $A \in M$. Since $M \models \mathfrak{c} = \omega_1$, $M \models$ "There is a selective ultrafilter $\mathcal{F}$". Then $\mathcal{F}$ is a filter in V and, by the previous lemma, any extension of $\mathcal{F}$ to an ultrafilter is a Q-point in V. □

Miller's and Mathias's results of course motivated the question 3.1 mentioned in the introductory part of this chapter. Later Canjar and, independently, Bartoszyński and Judah proved a theorem for Q-points, which is reminiscent of Ketonen's characterization of $\mathfrak{d} = \mathfrak{c}$. This was then extended by Fremlin to rapid filters.

3.28 Theorem ([Can90],[BarJu88],Fremlin)**.** *The following are equivalent*

(i) $cov(\mathcal{M}) = \mathfrak{c}$

(ii) $\mathfrak{d} = \mathfrak{c}$ *and every filter of character* $< \mathfrak{c}$ *can be extended to a Q-point.*

(iii) $\mathfrak{d} = \mathfrak{c}$ *and every filter of character* $< \mathfrak{c}$ *can be extended to a rapid filter.*

□

The proof of the previous theorem may be found e.g. in [SRL, Lemmas 4.5.6 and 4.6.5].

3.3 Canjar Ultrafilters

This section will show how Canjar ultrafilters, which turn out to be a stronger version of P-points, can be constructed via forcing. Their definition is motivated by considering Mathias forcing.

3.29 Definition. *Mathias forcing* $\mathbb{M}$ consists of conditions which are pairs (s, A) such that $s \in [\omega]^{<\omega}$ and $A \in [\omega]^{\omega}$ ordered as follows: $(s, A) \leq (t, B)$ if

(i) $s \sqsupseteq t$ and $A \subseteq B$ and

(ii) $s \setminus t \subseteq B$

Given a family of subsets $\mathcal{A} \subseteq [\omega]^{\omega}$ we may define $\mathbb{M}_{\mathcal{A}}$ to be the restriction of Mathias forcing to conditions whose second coordinate is in $\mathcal{A}$.

It is not hard to see that $\mathbb{M}$ factors into two parts

$$\mathbb{M} = \mathcal{P}(\omega)/Fin * \mathbb{M}_{\dot{G}},$$

where the first forcing adds an ultrafilter[3] while the second forcing shoots a pseudointersection through this ultrafilter. It is not hard to see that the second forcing will add a dominating real, since $\dot{G}$ is forced to be rapid (even selective). M. Canjar in [Can88] was probably the first to ask whether this is always the case with Mathias-type forcings or whether one can have an ultrafilter $\mathcal{U}$ such that $\mathbb{M}_{\mathcal{U}}$ does not add a dominating real.

3.30 Question. *Is there an ultrafilter $\mathcal{U}$ such that $\mathbb{M}_{\mathcal{U}}$, the Mathias forcing relativized to the ultrafilter $\mathcal{U}$, does not add a dominating real?*

We shall call these ultrafilters *Canjar ultrafilters*, i.e.

3.31 Definition. A *Canjar ultrafilter* is an ultrafilter on ω such that $\mathbb{M}_{\mathcal{U}}$ does not add dominating reals.

M. Canjar established the following necessary condition for an ultrafilter to be Canjar:

3.32 Theorem (Canjar). *A* Canjar ultrafilter *must be a P-point with no rapid Rudin-Keisler predecessor.* □

From this theorem it is clear that we can only hope for a *consistent* positive answer to the question 3.30, since, e.g. in the model where there are no P-points, $\mathbb{M}_{\mathcal{U}}$ always adds a dominating real. Later in this section we will present M. Canjar's proof of a consistent positive answer, but first we shall look at Canjar ultrafilters in more detail.

In [Laf89] C. Laflamme considered, amongst other notions, what we call Canjar ultrafilters. He also introduced the notion of a strong P-point, which is motivated by the following observation.

3.33 Observation. *An ultrafilter $\mathcal{U}$ is a P-point if and only if for any descending sequence of sets $\langle X_n : n < \omega \rangle$ from $\mathcal{U}$ there is an interval partition $\langle I_n : n < \omega \rangle$ of ω such that*

$$X = \bigcup_{n<\omega} (I_n \cap X_n) \in \mathcal{U}.$$

□

Note that X will always be a pseudointersection of the X_n's, and the larger the intervals are, the larger it will be.

3.34 Definition ([Laf89]). An ultrafilter is a *strong P-point* if for any sequence $\langle \mathcal{C}_n : n < \omega \rangle$ of compact subsets of $\mathcal{U}$ (considering $\mathcal{U}$ as a subset of 2^{ω} with the product topology), there is an interval partition $\langle I_n : n < \omega \rangle$ such that for each choice of $X_n \in \mathcal{C}_n$ we have

$$X = \bigcup_{n<\omega} (I_n \cap X_n) \in \mathcal{U}.$$

It is easy to see that a strong P-point cannot be rapid (for example consider $\mathcal{C}_n = \{X : |\omega \setminus X| \leq n\}$) and in [Laf89, Lemma 6.8] it is proved that strong P-points are preserved when passing to RK-predecessors. Summarizing we have the following fact.

3.35 Fact (Laflamme). *A strong P-point is a P-point and it cannot have rapid RK-predecessors.* □

In the cited paper C. Laflamme noted without proof that every Canjar ultrafilter must be a strong P-point and conjectured that the two notions coincide. The topic was recently revisited by M. Hrušák and H. Minami in [HrMi∞] who invented a combinatorial characterization of Canjar ultrafilters. Before we can present their characterization we need the following notion which was probably first considered

[3] See the next section for a more detailed consideration of similar forcings

implicitly by S. M. Sirota ([Sir69]) and explicitly by A. Louveau ([Lou72]) in the construction of an extremally disconnected topological group:

3.36 Notation. *Given a filter $\mathcal{F}$ on ω we define $\mathcal{F}^{<\omega}$ to be the filter on $[\omega]^{<\omega}\setminus\{\emptyset\}$ generated by $\{[F]^{<\omega}\setminus\{\emptyset\} : F\in\mathcal{F}\}$.*

Note that $[\mathcal{F}]^{<\omega}$ is a filter on $[\omega]^{<\omega}\setminus\{\emptyset\}$ and it is easy to see that $\mathcal{U}^{<\omega}$ is never an ultrafilter, e.g. neither of the sets $\{a\in[\omega]^{<\omega} : |a|=2n\}$ and $\{a\in[\omega]^{<\omega} : |a|=2n+1\}$ can be in $\mathcal{U}^{<\omega}$. It is clear from the definition, that a set $A\subseteq[\omega]^{<\omega}$ is $\mathcal{U}^{<\omega}$-positive if it hits each $[U]^{<\omega}$ for $U\in\mathcal{U}$. The next lemma gives an alternative characterization.

3.37 Lemma. *If $\mathcal{U}$ is an ultrafilter on ω then $A\subseteq[\omega]^{<\omega}$ is $\mathcal{U}^{<\omega}$-positive if and only if each set $X\subseteq\omega$ such that every element $a\in A$ has nonempty intersection with X is in $\mathcal{U}$.*

Proof. Suppose A is positive and X hits each element of A. We will show that X intersects each $Y\in\mathcal{U}$: take $Y\in\mathcal{U}$, then $[Y]^{<\omega}\cap A\neq\emptyset$ so $Y\cap X\neq\emptyset$. Since $\mathcal{U}$ is an ultrafilter, $X\in\mathcal{U}$. On the other hand if A is not positive there is some $Y\in\mathcal{U}$ with $[Y]^{<\omega}\cap A=\emptyset$. Then $X=\omega\setminus Y$ hits every element of A. □

We are now ready to prove M. Hrušák and H. Minami's characterization of Canjar ultrafilters (for P^+-filters see definition 3.10).

3.38 Theorem ([HrMi∞])**.** *An ultrafilter $\mathcal{U}$ is Canjar if and only if $\mathcal{U}^{<\omega}$ is a P^+-filter.*

Proof. $\Leftarrow$: Assume $\mathcal{U}^{<\omega}$ is a P^+-filter and suppose, aiming towards a contradiction, that $\mathbb{M}_{\mathcal{U}}$ adds a dominating real. Let $\dot g$ be a name for it. For each $f\in\omega^\omega$ there is an $n_f<\omega$ and $(t_f,F_f)\in\mathbb{M}_{\mathcal{U}}$ such that

$$(t_f,F_f)\Vdash(\forall k\geq n_f)(f(k)\leq\dot g(k)).$$

Since $\mathfrak{b}>\omega$, we can fix $n<\omega$ and $t\in[\omega]^{<\omega}$ such that the family of functions $\mathcal{F}=\{f\in\omega^\omega : n_f=n\ \&\ t_f=t\}$ is a dominating family. For $k<\omega$ let

$$X'_k=\{s\in[\omega\setminus t]^{<\omega} : (\exists F\in\mathcal{U}, m\geq k, i<\omega)((t\cup s,F)\Vdash\dot g(m)=i)\}.$$

Clearly X'_k is $\mathcal{U}^{<\omega}$-positive and the sets decrease as k increases. Define $Y=\bigcap_{k<\omega}X'_k$ and let $X_k=X'_k\setminus Y$. Notice that the sets X_k are still decreasing and, if we can show that Y is not $\mathcal{U}^{<\omega}$-positive, then they will also be positive.

Claim. $Y\notin(\mathcal{U}^{<\omega})^+$.

Proof of claim. Suppose otherwise and for $s\in Y$ let $f_s : A_s\to\omega$ be a maximal (w.r.t. inclusion) function such that for each $m\in A_s$ there is $F^s_m\in\mathcal{U}$ such that $(t\cup s,F^s_m)\Vdash\dot g(m)=f_s(m)$. Note that each A_s is infinite. Choose $f\in\mathcal{F}$ eventually dominating $\{f_s : s\in Y\}$. Pick $F\in\mathcal{U}$ such that $(t,F)\Vdash(\forall m>n)(f(m)\leq\dot g(m))$. Since Y is positive, there must be some $s\in Y\cap[F]^{<\omega}$. Finally pick $m>n$ such that $m\in A_s$ and $f_s(m)<f(m)$ (this is possible since A_s is infinite and f eventually dominates f_s). But then $(t\cup s,F\cap F^s_m)\Vdash\dot g(m)=f_s(m)<f(m)\leq\dot g(m)$ — a contradiction. This completes the verification of the claim. ■

Since $\mathcal{U}^{<\omega}$ is a P^+-filter by assumption, there must be a $\mathcal{U}^{<\omega}$-positive set $X\subseteq X_0$ which is a pseudointersection of the X_k's. Define

$$f(k)=\max\{i+1 : (\exists s\in X\setminus X_{k+1},F\in\mathcal{U})((t\cup s,F)\Vdash\dot g(k)=i)\}\cup\{0\}.$$

Since the family $\mathcal{F}$ was a dominating family, choose $h\in\mathcal{F}$ dominating f above some $k_0<\omega$ with $n<k_0$. Since X is $\mathcal{U}^{<\omega}$-positive and $X\subseteq^* X_{k_0}$, we may find $s\in X\cap X_{k_0}\cap[F_h]^{<\omega}$. Let k be maximal such that $s\in X_k$. Then $k\geq k_0$. By the definition of the X_k's and f, there is $F\in\mathcal{U}$ and $i<f(k)$ such that $(t\cup s,F)\Vdash\dot g(k)=i$. But this contradicts the fact that $(t\cup s,F\cap F_h)\leq(t,F_h)$ forces $f(k)\leq\dot g(k)$.

$\Rightarrow$: Assume $\mathcal{U}^{<\omega}$ is not a P^+-filter. We shall show that $\mathcal{U}$ is not Canjar. Let $\langle X_n : n<\omega\rangle$ be a descending sequence of $\mathcal{U}^{<\omega}$-positive sets with no positive pseudointersection. Work in the extension by $\mathbb{M}_{\mathcal{U}}$ and let $F_g\subseteq\omega$ be the generic real. Notice that $[F_g\setminus n]^{<\omega}\cap X_n\neq\emptyset$. Otherwise there would be some condition (s,A) forcing $[F_g\setminus n]^{<\omega}\cap X_n=\emptyset$. However, since X_n positive with respect to $\mathcal{U}^{<\omega}$, there would be $t\in[A\setminus n]^{<\omega}\cap X_n$, and $(s\cup t,A)\Vdash t\in[F_g\setminus n]^{<\omega}\cap X_n$. This would contradict the choice of (s,A). So $[F_g\setminus n]^{<\omega}\cap X_n\neq\emptyset$ and we can recursively pick $x_n\in[F_g\setminus n]^{<\omega}\cap X_n$. Let $f(n)=\max x_n+1$ and notice that $x_n\in[f(n)]^{<\omega}\cap X_n$. Suppose some strictly increasing h is not dominated by f and let $X=\bigcup_{n<\omega}[h(n)]^{<\omega}\cap X_n$. Clearly X is a pseudointersection of the X_n's. Since h is not dominated by f, X contains infinitely many x_i's and it follows that it is positive: Suppose $F\in\mathcal{U}$. We will show $[F]^{<\omega}\cap X\neq\emptyset$. Find $n<\omega$ such that $F_g\setminus n\subseteq F$. Then we can pick $m>n$ such that $x_m\in X$ and $x_m\subseteq F_g\setminus m\subseteq F_g\setminus n\subseteq F$. This shows that X is positive. So h cannot be in the ground model. □

In [BlHrVe11] we were able to extend this result by proving Laflamme's conjecture.

3.39 Theorem ([BlHrVe11]). *An ultrafilter is Canjar if and only if it is a strong P-point.*

Proof. We actually rely on the previous theorem and show that $\mathcal{U}$ is a strong P-point if and only if $\mathcal{U}^{<\omega}$ is a P^+-filter.

$\Leftarrow$: Suppose $\mathcal{U}^{<\omega}$ is a P^+-filter but $\mathcal{U}$ is not a strong P-point. Let $\mathcal{C}_n$ witness the latter. We may assume that $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$ and that $\mathcal{C}_n$ is closed under intersections of up to $n+1$ elements, i.e. $(\forall \mathcal{C} \in [\mathcal{C}_n]^{\leq n+1})(\bigcap \mathcal{C} \in \mathcal{C}_n)$. Let

$$A_n = \{a \in [\omega]^{<\omega} : (\forall X \in \mathcal{C}_n)(a \cap X \neq \emptyset)\}.$$

Notice that $A_{n+1} \subseteq A_n$. In addition $A_n \in (\mathcal{U}^{<\omega})^+$. To show this, choose $F \in \mathcal{U}$ and check that $\{X \cap F : X \in \mathcal{C}_n\}$ is a compact set not containing $\emptyset$. In particular there is an $a \in [F]^{<\omega}$ such that $a \cap X \neq \emptyset$ for each $X \in \mathcal{C}_n$, so $a \in A_n \cap [F]^{<\omega}$. Now let A be a $\mathcal{U}^{<\omega}$-positive pseudointersection of the A_n's. We will deduce a contradiction. Let

$$g(n) = \min\{k : a \in A \setminus A_n \to a \subseteq k\}.$$

Enlarging $g(n)$, if necessary, we may assume it is increasing. By our assumption on the $\mathcal{C}_n$'s, there are X_n's with $X_n \in \mathcal{C}_n$ such that

$$\bigcup_{n<\omega} \big(X_n \cap [g(n), g(n+1))\big) \notin \mathcal{U}.$$

Define $Y_n = \bigcap_{i\leq n} X_n$ and notice that $Y_n \in \mathcal{C}_n$ since the sequence of $\mathcal{C}_n$'s is increasing and $\mathcal{C}_n$ is closed under intersections of at most $n+1$ elements. Moreover we have

$$Y = \bigcup_{n<\omega} \big(Y_n \cap [0, g(n+1))\big) \subseteq \bigcup_{n<\omega} \big(X_n \cap [g(n), g(n+1))\big) \notin \mathcal{U}.$$

Since A is positive Lemma 3.37 will give the desired contradiction if we show that Y hits each $a \in A$. Pick $a \in A$ and let

$$k = \max\{n : a \cap [g(n), g(n+1)) \neq \emptyset\}$$

Notice that $a \subseteq g(k+1)$ and, by the definition of g, $a \in A_k$. Hence $a \cap Y_k \neq \emptyset$ so $a \cap [0, g(k+1)) \cap Y_k \neq \emptyset$ so $a \cap Y \neq \emptyset$ and we are done.

$\Rightarrow$: Suppose on the other hand that $\mathcal{U}$ is a strong P-point and that $\langle A_n : n < \omega\rangle$ is a descending sequence of $\mathcal{U}^{<\omega}$-positive sets. We shall find a $\mathcal{U}^{<\omega}$-positive pseudointersection. Let

$$\mathcal{C}_n = \{X : (\forall a \in A_n)(a \cap X \neq \emptyset)\}.$$

Then $\mathcal{C}_n \subseteq \mathcal{U}$ by Lemma 3.37. Moreover $\mathcal{C}_n$ is closed (it is an intersection of clopen sets). Since $\mathcal{U}$ is a strong P-point, there is an interval partition $\langle I_n : n < \omega\rangle$ of ω satisfying the condition in the definition of a strong P-point. Let

$$A = \bigcup_{n<\omega} \big(A_n \cap \mathcal{P}(I_n)\big).$$

Since the A_n's were decreasing, A will be a pseudointersection of them. We have to show that it is positive. Pick $F \in \mathcal{U}$. We need to show that there is $n < \omega$ such that $[F]^{<\omega} \cap A_n \cap \mathcal{P}(I_n) \neq \emptyset$. Suppose this is not so. Then let $X_n = (\omega \setminus I_n) \cup (I_n \setminus F)$ and notice that $\bigcup_{n<\omega}(X_n \cap I_n) = \omega \setminus F \notin \mathcal{U}$. We will show that each $X_n \in \mathcal{C}_n$ which will contradict the choice of the interval partition, thus finishing the proof. But given some $a \in A_n$ either $a \notin \mathcal{P}(I_n)$ and then $a \cap X_n \neq \emptyset$ trivially, or $a \in \mathcal{P}(I_n)$ but then $a \notin [F]^{<\omega}$ so $a \cap X_n \neq \emptyset$ also. □

The paper [BlHrVe11] also contains a counterexample to C. Laflamme's conjecture that the implication in M. Canjar's theorem 3.32 can be reversed.

3.40 Theorem ([BlHrVe11]). *It is consistent that there is a P-point with no rapid RK-predecessors which is not a strong P-point.*

The proof of this theorem is a classical MA-style construction and we will not include it here but we shall present a different counterexample in the next section (see example 3.61).

We now return to the original question of M. Canjar and show a consistent positive answer.

3.41 Theorem ([Can88]). *It is consistent that there is a Canjar ultrafilter.*

Proof. Consider the forcing notion $\mathbb{P}_{F_\sigma}$ consisting of (proper) F_σ-ideals ordered by inclusion.

Claim. $\mathbb{P}_{F_\sigma}$ is σ-closed and hence it does not add new reals.

Proof of claim. Suppose $\langle I_n : n < \omega\rangle$ is a descending sequence of conditions and let $I = \bigcup_{n<\omega} I_n$. Clearly I is F_σ, since it is a countable union of F_σ sets, and $\omega \notin I$. ■

Claim. If G is $\mathbb{P}_{F_\sigma}$-generic, then $\mathcal{I}_G = \bigcup G$ is a maximal ideal.

Proof of claim. Since $\mathbb{P}_{F_\sigma}$ does not add new reals by the previous claim, we only need to consider subsets of ω in the ground model. Suppose $A \subseteq \omega$ and I is a condition. If $A \in I^*$ then $I \Vdash \omega \setminus A \in \dot{G}$. Otherwise $\{A\} \cup I$ generates a proper ideal J which is easily seen to be F_σ and $J \Vdash A \in \dot{G}$. ■

Claim. If G is $\mathbb{P}_{F_\sigma}$-generic then $\mathcal{I}_G^*$ is a strong P-point.

Proof of claim. Since $\mathbb{P}_{F_\sigma}$ does not add new reals (and since compact subsets of 2^ω are coded by reals) we need only consider sequences $\langle \mathcal{C}_n : n < \omega\rangle$ of compact sets from the ground model. Fix such a sequence. Since $\cup : 2^\omega \times 2^\omega \to 2^\omega$ is continuous, we may, without loss of generality, assume that $\mathcal{C}_n$ is closed under intersections of up to n elements. Let I be a condition such that $I \Vdash (\forall n < \omega)(\mathcal{C}_n \subseteq \mathcal{I}^*_{\dot{G}})$. By theorem 1.30 fix a lower semicontinuous submeasure μ such that $I = Fin(\mu)$. By recursion construct $\langle k_n : n < \omega\rangle$ such that for each $X \in \mathcal{C}_n$ we will have $\mu(X \cap [k_n, k_{n+1})) \geq n$. Let $k_0 = 0$ and assume we have constructed k_n. Consider the function $f : \mathcal{C}_n \to \omega$ defined as follows: $f(X) = \min\{k : \mu(X \cap [k_n, k)) \geq n\}$. This function is well defined (each X has infinite measure) and is a continuous function by monotonicity of μ. Since $\mathcal{C}_n$ is compact $f[\mathcal{C}_n]$ is bounded and has a maximum. Let k_{n+1} be this maximum. This clearly works. Let J be the ideal generated by

$$\mathcal{A} = \{Y \subseteq \omega : (\forall n < \omega)(\exists X_n \in \mathcal{C}_n)(Y \cap X_n \cap [k_n, k_{n+1}) = \emptyset)\}.$$

Since the generating set is clearly closed, J is an F_σ-ideal. We shall show that the ideal generated by $I \cup J$ is proper. It is enough to show that for each $Y \in J$ the measure $\mu(\omega \setminus Y)$ is infinite. Pick $Y \in J$ and write it as $\omega = Y_0 \cup \cdots \cup Y_m$ where each $Y_i \in \mathcal{A}$ as witnessed by $\langle X^i_n : n < \omega\rangle$. Let $X_n = \bigcap_{i=0}^m X^i_n$. By our assumption on $\mathcal{C}_n$ we have $X_n \in \mathcal{C}_n$ for $n > m+1$. Moreover $X_n \cap Y \cap [k_n, k_{n+1}) = \emptyset$. Let $X = \bigcup_{n>m+1} X_n \cap [k_n, k_{n+1})$. Then $X \cap Y = \emptyset$ and $\mu(X) = \infty$ (since $\mu(X \cap [k_n, k_{n+1})) \geq n$). So $I \cup J$ generates a condition $J' \in \mathbb{P}_{F_\sigma}$. It is easy to see that

$$J' \Vdash (\forall \langle X_n : n < \omega\rangle \in \prod_{n<\omega} \mathcal{C}_n)(\bigcup_{n<\omega} (X_n \cap [k_n, k_{n+1})) \in \mathcal{I}^*_G)$$

and this finishes the proof of the claim. ■

By the preceding claims it follows that $\mathbb{P}_{F_\sigma} \Vdash$ "$\mathcal{I}^*_G$ is a strong P-point" and it now remains to apply theorem 3.39. □

3.4 Adding Ultrafilters by Definable Quotients

We now turn to constructing ultrafilters via forcings of a special kind. In the previous section we presented an ultrafilter construction with a forcing where conditions were F_σ-ideals. In the present section we will look at ultrafilter constructions based on forcing notions of the form $\mathcal{P}(\omega)/\mathcal{I}$ for some definable $\mathcal{I}$. We will restrict ourselves to $\mathcal{I}$ such that $\mathcal{P}(\omega)/\mathcal{I}$ does not add new reals. There are two ways to look at this approach: (1) depending on the choice of $\mathcal{I}$ various types of ultrafilters may be added and (2) the definable ideals may be classified according to the ultrafilters which arise when forcing with $\mathcal{P}(\omega)/\mathcal{I}$. We will first take the second view and present characterizations of ideals based on the type of ultrafilters (P-points, selective, rapid ...) added by the forcings. Then we will take the first view and present constructions of interesting ultrafilters via these forcings.

This section is based on [HrVer11].

Instead of dealing with the quotient algebra $\mathcal{P}(\omega)/\mathcal{I}$ we will implicitly use the equivalent forcing notion $(\mathcal{I}^+, \subseteq)$. We introduce the following definition to simplify the statement of our theorems.

3.42 Definition. $\mathcal{I}$ (or $\mathcal{P}(\omega)/\mathcal{I}$) *adds an ultrafilter with property P* if every generic ultrafilter on $(\mathcal{I}^+, \subseteq)$ has property P.

Beware that this usage of the phrase "$\mathcal{P}(\omega)/\mathcal{I}$ adds an ultrafilter" differs from the usual meaning of "a forcing P adds an object Q"! We have chosen to use it in this modified sense since, e.g. under CH, the usual sense trivializes: under CH all forcings of this form are isomorphic and thus an ultrafilter added by one of the forcings is added by any other.

3.4.1 P-points

The following theorem may be found in [Zap09]

3.43 Theorem (Zapletal). *An ultrafilter $\mathcal{U}$ is a P-point if all analytic ideals disjoint from $\mathcal{U}$ can be separated from $\mathcal{U}$ by an F_σ ideal (i.e. are contained in an F_σ ideal disjoint from $\mathcal{U}$).*

We shall prove the forward implication which will be used in the proof of theorem 3.49.

Proof. We will need the following dichotomy due to Kechris-Louveaou-Woodin.

3.44 Theorem ([KeLoWo87]). *Suppose $\mathcal{U}, \mathcal{J}$ are two disjoint subsets of 2^ω such that $\mathcal{J}$ is analytic and cannot be separated from $\mathcal{U}$ by an F_σ-set. Then there is a perfect set $C \subseteq \mathcal{U} \cup \mathcal{J}$ such that $C \cap \mathcal{U}$ is countable dense in C.* □

Also recall that $\mathcal{U}$ is a P-point provided player I has no winning strategy in the game $G(\mathcal{U})$ where he chooses sets $A_n \in \mathcal{U}$ while player II chooses finite sets $a_n \in [A_n]^{<\omega}$ and II wins if at the end $\bigcup_{n<\omega} a_n \in \mathcal{U}$.

To prove the theorem, suppose $\mathcal{U}$ cannot be separated by an F_σ-set from an analytic ideal $\mathcal{J}$. We will show that player I has a winning strategy in $G(\mathcal{U})$. By the Kechris-Louveau-Woodin dichotomy there is a perfect set $\mathcal{C} \subseteq \mathcal{U} \cup \mathcal{J}$ with $\mathcal{C} \cap \mathcal{U}$ countable dense in $\mathcal{C}$. Enumerate $\mathcal{C} \cap \mathcal{U}$ as $\{C_n : n < \omega\}$. The strategy for player I will be as follows. He will play sets $A_n \in \mathcal{U} \cap \mathcal{C}$ and, on the side, he will write down finite initial segments $b_n \sqsubseteq A_n$ of A_n such that

(i) $\bigcup_{i<n} b_n \sqsubseteq A_n$ and

(ii) $A_n \neq C_n$ which will already be witnessed by b_n, i.e. $b_n \not\sqsubseteq C_n$.

Since $\mathcal{C} \cap \mathcal{U}$ is dense in $\mathcal{C}$ it is always possible to play according to this strategy. In the end, let $B = \bigcup_{n<\omega} b_n$. Then B is a limit of the A_n's by (i), so $B \in C$. By (ii), $B \neq C_n$ for any n, so $B \notin \mathcal{C} \cap \mathcal{U}$, so $B \in \mathcal{J}$. Since B contains all the moves of player II, it follows that player II lost. □

Compare the above theorem with a result of M. Hrušák and H. Minami ([HrMi∞]):

3.45 Theorem (Hrušák-Minami). *A Borel ideal $\mathcal{I}$ can be extended to an F_σ ideal if and only if it can be extended to a P^+-ideal.* □

Using Mazur's theorem (1.30) it is easy to prove the following which was first observed in [JuKr84]:

3.46 Observation (Just-Krawczyk). *If $\mathcal{I}$ is an F_σ ideal then $\mathcal{P}(\omega)/\mathcal{I}$ is σ-closed, in fact, $\mathcal{I}$ is a P^+-ideal.*

Proof. By Mazur's theorem 1.30 we may find a lscsm μ such that $\mathcal{I} = Fin(\mu)$. Assume $\langle A_n : n < \omega \rangle \subseteq \mathcal{I}^+$ is a descending sequence of conditions. Without loss of generality we may assume that $A_{n+1} \subseteq A_n$ (since $A_{n+1} \setminus A_n \in \mathcal{I}$). Using lower semicontinuity and the fact that $\mu(A_n) = \infty$ we recursively construct finite sets $a_n \in [A_n]^{<\omega}$ such that $\mu(a_n) \geq n$. Finally let $A = \bigcup_{n<\omega} a_n$. By monotonicity $\mu(A) = \infty$ so $A \in I^+$ is a condition. Since $\mu(A \setminus A_n) \leq \mu(\bigcup_{i<n} a_i) \leq \sum_{i<n} \mu(a_i) < \infty$, we have that $A \setminus A_n \in \mathcal{I}$ for each $n < \omega$. Hence A is a condition stronger than each A_n. □

3.47 Question. *Suppose I is Borel and $\mathcal{P}(\omega)/\mathcal{I}$ does not add new reals. Is $\mathcal{P}(\omega)/\mathcal{I}$ σ-closed?*

3.48 Observation (Folklore). *If $\mathcal{I}$ is F_σ then $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point.*

Proof. To prove that the generic ultrafilter G is a P-point suppose $A \in \mathcal{I}^+$ forces $\langle \dot{A}_n : n < \omega \rangle \subseteq \dot{G}$. Since $\mathcal{P}(\omega)/\mathcal{I}$ is σ-closed we may assume $\langle A_n : n < \omega \rangle \in V$. As above in the proof of observation 3.46 find $B \subseteq A$, $\mu(B) = \infty$ with $\mu(B \setminus A_n) < \infty$. Then $B \in \mathcal{I}^+$ is stronger than A and forces that $B \in \dot{G}$ is a pseudointersection of $\langle A_n : n < \omega \rangle$. □

We shall show that this is essentially the only case when a definable $\mathcal{I}$ not adding reals adds a P-point:

3.49 Theorem. *. Suppose $\mathcal{I}$ is analytic and $\mathcal{P}(\omega)/\mathcal{I}$ adds no new reals. Then $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point if and only if $\mathcal{I}$ is locally F_σ.*

Proof. If $\mathcal{I}$ is locally F_σ then $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point by observation 3.48.

Suppose on the other hand that $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point and that $A \in \mathcal{I}^+$. Work in the extension by some generic filter G containing A. Clearly $G \cap \mathcal{I} = \emptyset$ so, by Zapletal's theorem 3.43 there is an F_σ ideal J extending $\mathcal{I}$. Since $\mathcal{J}$ is given by a lscsm, which is essentially given by a real, and $\mathcal{P}(\omega)/\mathcal{I}$ adds no new reals, this J is already in the ground model and we may assume that $A \Vdash$ "$\dot{G} \cap \check{J} = \emptyset$". Since $\mathcal{J} \restriction A$ is an F_σ ideal it is sufficient to show $\mathcal{I} \restriction A = \mathcal{J} \restriction A$. The inclusion from left to right is clear. So suppose there was some $C \subseteq A$, $C \in \mathcal{J} \setminus \mathcal{I}$. Then $C \Vdash$ "$\check{C} \in \dot{G} \cap \check{J}$" which would be a contradiction. □

Note that being locally F_σ is not the same as being F_σ even in the class of Borel tall ideals:

3.50 Example. There are tall Borel ideals of arbitrarily high complexity which are locally F_σ.

Proof. Given a set $A \subseteq \omega^\omega$ let $\mathcal{I}_A$ be the ideal on $\omega^{<\omega}$ generated by sets of the form

(i) $\{f \restriction n : n < \omega\}$ for $f \in A$,

(ii) $\{f \restriction n : n \in X\}$ for $f \notin A$, $X \in \mathcal{I}_{1/n}$ (see definition 1.28) and

(iii) antichains in $\omega^{<\omega}$.

This is clearly a tall ideal.

The complexity of $\mathcal{I}_A$ is at least the complexity of A: Consider $\Phi : \omega^\omega \to \mathcal{P}(\omega^{<\omega})$ defined as follows $\Phi(f) = \{f \restriction n : n < \omega\}$. This is a continuous function and $\Phi^{-1}[\mathcal{I}_A] = A$.

Note that $\mathcal{I}_{\omega^\omega}$ is F_σ, as is $\mathcal{I}_A \restriction \{f \restriction n : n < \omega\}$ for $f \notin A$. Suppose $X \in \mathcal{I}_A^+$. Then either $\mathcal{I}_A \restriction X = \mathcal{I}_{\omega^\omega} \restriction X$ and then $\mathcal{I}_A \restriction X$ is F_σ or not, and then there is $f \notin A$ such that $Y = X \cap \{f \restriction n : n < \omega\} \in \mathcal{I}_A^+$. Then $\mathcal{I}_A \restriction Y$ is F_σ.

□

3.4.2 Selectivity

Recall the following classical theorem of A. R. D. Mathias ([Mat77])

3.51 Theorem (Mathias). *An ultrafilter $\mathcal{U}$ is selective if and only if $\mathcal{U}$ is disjoint from all tall analytic ideals.* □

The following fact is folklore:

3.52 Fact. *$\mathcal{P}(\omega)/Fin$ adds a selective ultrafilter.* □

We shall show that, in the class of analytic ideals not adding reals, Fin is in a sense the only ideal adding a selective ultrafilter:

3.53 Theorem. *Suppose $\mathcal{I}$ is analytic and $\mathcal{P}(\omega)/\mathcal{I}$ does not add reals. Then $\mathcal{P}(\omega)/\mathcal{I}$ adds a selective ultrafilter if and only if $\mathcal{I}$ is locally Fin.*

Proof. Suppose first that $\mathcal{I}$ is locally Fin. Given $A \in \mathcal{I}^+$ there is an $\mathcal{I}$-positive $B \subseteq A$ such that $\mathcal{I} \restriction B \simeq \mathcal{P}(\omega)/Fin$. Now use fact 3.52.

The other direction is a direct corollary of theorem 3.51: Suppose $A \in \mathcal{I}^+$. By assumption $A \Vdash$ "$\dot{G}$ is selective". We need to find a $B \in [A]^\omega$ such that $\mathcal{I} \restriction B = Fin$. Since $\mathcal{I} \cap \dot{G} = \emptyset$ and $\mathcal{I}$ is analytic, we may apply theorem 3.51 (taking A instead of ω) to see that $\mathcal{I} \restriction A$ is not tall. So there is an infinite $B \subseteq A$ such that $\mathcal{I} \restriction B = Fin$. □

3.54 Remark. This, of course, fails badly in the non-definable case, e.g. $\mathcal{P}(\omega)/\mathcal{I}(\mathcal{A})$ adds a selective ultrafilter for every MAD family $\mathcal{A}$ (see [Mat77]).

3.4.3 Q-points and rapid ultrafilters

Now we turn our attention to other properties of ultrafilters and prove two more characterizations.

3.55 Definition. Let $\Delta = \{(x, y) : x \leq y\}$. The ideal $\mathcal{ED}_{fin}$ on Δ consists of those sets which can be covered by finitely many functions.

3.56 Proposition. *Suppose $\mathcal{P}(\omega)/\mathcal{I}$ does not add new reals. Then the forcing $\mathcal{I}$ adds a Q-point if and only if it is locally not KB-above $\mathcal{ED}_{fin}$.*

Proof. Suppose that $\mathcal{I}$ adds a Q-point. We must show that $\mathcal{I}$ is locally not KB-above $\mathcal{ED}_{fin}$. Pick some $\mathcal{I}$-positive set A and a finite-to-one function $f : A \to \Delta$. Aiming towards a contradiction suppose this function witnesses $\mathcal{I} \restriction A \geq_{KB} \mathcal{ED}_{fin}$. Let $A_n = f^{-1}[\{(n, y) : y \leq n\}]$. Then A_n is a partition of A into finite sets with no positive selector so $A \Vdash$ "$\dot{G}$ is not a Q-point" a contradiction.

On the other hand suppose $A \in \mathcal{I}^+$ and $A \Vdash$ "$\dot{G}$ is not a Q-point". Since $\mathcal{P}(\omega)/\mathcal{I}$ does not add new reals, we can assume that there is an interval partition $\langle I_n : n < \omega\rangle$ such that A forces each each selector to be outside of $\dot{G}$. Fix an increasing sequence k_n of natural numbers such that $|I_n| \leq k_n$ and also fix bijections $\varphi_n : I_n \to k_n$. Finally define $f : A \to \Delta$ as follows. For $x \in A$ find $n < \omega$ such that $x \in I_n$ and let $f(x) = \varphi_n(x)$. It is easy to see that this function witnesses $\mathcal{I} \restriction A$ is KB-above $\mathcal{ED}_{fin}$. □

For dealing with rapid ultrafilters we use the following theorem of P. Vojtáš ([Voj94]).

3.57 Theorem (Vojtáš). *An ultrafilter is rapid if and only if it meets every tall summable ideal (see definition 1.28).* □

The following proposition, which can be found in [HrHe07], shows that we can replace tall summable ideals with tall analytic P-ideals in the above theorem.

3.58 Proposition (Hrušák-Hernandez). *Suppose $\mathcal{I}$ is a tall analytic P-ideal. Then there is a tall summable ideal contained in $\mathcal{I}$.*

Proof. By theorem 1.31 there is a lscsm μ such that $\mathcal{I} = Exh(\mu)$. We shall show that $\mu(\{n\}) \to 0$: Suppose otherwise. Then there is an $\varepsilon > 0$ and an infinite $A \subseteq \omega$ such that $\mu(\{a\}) \geq \varepsilon$ for each $a \in A$. Then any infinite subset of A has submeasure $\geq \varepsilon$ so it is not in $\mathcal{I}$ contradicting the tallness of $\mathcal{I}$. Now let $g(n) = \mu(\{n\})$. Since g converges to zero $\mathcal{I}_g$ is a tall summable ideal. We claim that $\mathcal{I}_g \subseteq \mathcal{I}$. To see this, let $A \subseteq \omega$ with $\sum_{a\in A} g(a) < \infty$. Since the sum converges, necessarily $\sum_{a\in A\setminus n} g(a) \to 0$. Moreover $\mu(A \setminus n) \leq \sum_{a\in A\setminus n} g(a)$ so also $\mu(A \setminus n) \to 0$ so $A \in \mathcal{I}$. □

3.59 Proposition. *Suppose $\mathcal{P}(\omega)/\mathcal{I}$ does not add new reals. Then forcing with $\mathcal{I}$ adds a rapid ultrafilter if and only if it is locally not KB-above a tall summable ideal.*

Proof. Suppose $\mathcal{I}$ adds a rapid ultrafilter and $A \in \mathcal{I}^+$. We shall show that $\mathcal{I} \restriction A$ is not KB-above a tall summable ideal. Aiming towards a contradiction suppose that $\mathcal{I} \restriction A \geq_{KB} \mathcal{I}_g$ as witnessed by some $f : A \to \omega$. and define $\mu(n) = g(f(n))$. Then $Fin(\mu) \subseteq \mathcal{I} \restriction A$ so, in particular, $A \Vdash$ "$\dot{G} \cap Fin(\mu) = \emptyset$" contradicting Vojtáš's characterization 3.57 ($Fin(\mu)$ is tall since f was finite-to-one, so $\mu \to 0$).

Suppose on the other hand that $\mathcal{I}$ does not add a rapid ultrafilter. Using Vojtáš's characterization again, since $\mathcal{P}(\omega)/\mathcal{I}$ does not add any new reals, there must be a condition $A \in \mathcal{I}^+$ and a tall summable ideal $\mathcal{J}$ such that $A \Vdash$ "$\dot{G} \cap \mathcal{J} = \emptyset$". Then necessarily $\mathcal{J} \restriction A \subseteq \mathcal{I} \restriction A$ and the identity map shows that $\mathcal{I} \restriction A$ is KB-above a summable ideal. The same argument shows that the restriction of $\mathcal{I}$ to any $\mathcal{I}$-positive B below A is also Katětov-above a summable ideal. □

3.4.4 Canjar Ultrafilters

On the other hand forcing with tall a analytic P-ideal gives a special kind of ultrafilter. Recall (see definition 3.31) that an ultrafilter $\mathcal{U}$ is Canjar if $\mathbb{M}_{\mathcal{U}}$ does not add a dominating real.

We will now present a counterexample to Laflamme's conjecture:

3.60 Conjecture (Laflamme). *If $\mathcal{U}$ is a P-point which has no rapid RK-predecessors then $\mathcal{U}$ is Canjar.*

The following theorem shows that counterexamples to the above conjecture may be added by forcing with $\mathcal{P}(\omega)/\mathcal{I}$ for a tall F_σ P-ideal $\mathcal{I}$.

3.61 Theorem. *If $\mathcal{I}$ is a tall F_σ P-ideal, then $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point with no rapid RK-predecessors which is not Canjar.*

Proof. By observation 3.48 $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point and by observation 3.46 it is σ-closed. We first show that the generic has no rapid RB-predecessors. By proposition 3.58 and the characterization 3.57 of rapid ultrafilters, it will be sufficient to show that for each $f : \omega \to \omega$ finite-to-one, $f_*(\mathcal{I})$ is a tall analytic P-ideal. (Recall that in definition 1.22 we have defined $f_*(\mathcal{I}) = \{A \subseteq \omega : f^{-1}[A] \in \mathcal{I}\}$.) Let $\mathcal{I} = Fin(\mu) = Exh(\mu)$ by theorem 1.31. Define $\mu_*(A) = \mu(f^{-1}[A])$. Then μ_* is a submeasure on ω and since f is finite-to-one it is lower semicontinuous. It is easy to see that $f_*(\mathcal{I}) = Fin(\mu_*)$. It remains to verify that $Exh(\mu_*) = Fin(\mu_*)$. The $\subseteq$ inclusion is clear and for the other one we use the fact that, since f is finite-to-one, for each n there is $k \geq n$ such that $f^{-1}[A \setminus k] \subseteq f^{-1}[A] \setminus n$. Since RK-predecessors of P-points are its RB-predecessors it remains for us to show that the generic is not Canjar. To do this, we show that it fails the combinatorial condition of theorem 3.38.

Let $X_n = \{a \in [\omega]^{<\omega} : \mu(a) \geq (n+1)\}$. Clearly $\mathcal{P}(\omega)/\mathcal{I} \Vdash X_n \in (\dot{G}^{<\omega})^+$. Pick $A \in \mathcal{I}^+$ and let X be a pseudointersection of the X_n's. We shall find a stronger condition $B \subseteq A$, $B \in \mathcal{I}^+$ which will force X to be in $(\dot{G}^{<\omega})^*$. Let $g(n) = \min\{k : a \in X \setminus X_n \to a \subseteq k\}$. By increasing g we may assume $1 \leq \mu([g(n), g(n+1)) \cap A)$ and $\mu(\{x\}) \leq 1/8$ for each $x \in [g(n), g(n+1)) \cap A$. For n let $b_n \subseteq [g(n), g(n+1)) \cap A$ be minimal such that $1/4 \leq \mu(b_n)$. Then, by the minimality of b_n, $\mu(b_n) < 1/2$. Let $B = \bigcup_{n<\omega} b_n$. Then $1/4 \leq \mu(B \setminus n)$ for each n so $B \in \mathcal{I}^+$ by exhaustivity and clearly $B \subseteq A$. We will show that $[B]^{<\omega} \cap X = \emptyset$. Let $b \in [B]^{<\omega}$ and let $k = \max\{n : b \setminus g(n) \neq \emptyset\}$. If $b \in X$ then $b \in X_k$ by the definition of g. Then $\mu(b) \geq (n+1)$. However $b \subseteq \bigcup_{i\leq n} b_i$ so $\mu(b) \leq \sum_{i\leq n} \mu(b_i) < (n+1)/2 \leq n+1$ which is absurd. This finishes the proof. □

One way to construct Canjar ultrafilters is to force with F_σ ideals (ordered by reverse inclusion). This suggests the following question:

3.62 Question. *Is there a Borel ideal $\mathcal{I}$ on ω such that $\mathcal{P}(\omega)/\mathcal{I}$ adds a Canjar ultrafilter?*

By the previous results such an ideal would have to be locally F_σ, locally KB-above a tall analytic P-ideal and locally not P.

3.4.5 Examples

We conclude by presenting a few illustrative examples.

3.63 Example. The ideal $\mathcal{ED}_{fin}$ adds a semiselective ultrafilter with a selective ultrafilter RB-below.

Proof. First notice that $\mathcal{ED}_{fin} = Fin(\mu)$ where $\mu(A) = \min\{|K| : K \subseteq \omega^\omega \ \& \ A \subseteq \bigcup K\}$. This shows that $\mathcal{ED}_{fin}$ is F_σ so it is σ-closed and adds a P-point.

To show it adds a rapid ultrafilter, pick $\langle a_n : n < \omega\rangle$ a partition of Δ into finite sets and a condition $A \in \mathcal{ED}_{fin}{}^+$. We must find $B \subseteq A$, $B \in \mathcal{ED}_{fin}{}^+$ such that $|B \cap a_n| \leq n$. We shall actually find a B such that $|B \cap a_n| \leq n^2$ which is clearly sufficient. For each $n < \omega$ let $k_n = \min\{k : (\forall i < n)(a_i \subseteq k \times k \cap \Delta)\}$. By induction pick an increasing sequence $\langle l_n : n < \omega\rangle$ such that $k_n < l_n$ and $|A \cap \{l_n\} \times l_n| \geq n$. Then choose $b_n \in [A \cap \{l_n\} \times l_n]^n$ and let $B = \bigcup_{n<\omega} b_n$. Clearly $B \in \mathcal{ED}_{fin}{}^+$ and, moreover, $|B \cap a_n| \leq n^2$ by the definition of k_n. This finishes the proof that $\mathcal{ED}_{fin}$ adds a rapid ultrafilter.

To see that the generic filter always has a selective ultrafilter below, let $\pi : \Delta \to \omega$ be the projection on the first coordinate. Given $A \in \mathcal{ED}_{fin}$ and $\langle I_n : n < \omega\rangle$ an interval partition of ω, choose an increasing sequence $\langle k_n : n < \omega\rangle$ such that $|\{k_n\} \times k_n \cap A| \geq n$. Then pick some infinite $N \subseteq \omega$ such that $(\forall j < \omega)(|\{k_n : n \in N\} \cap I_j| \leq 1)$, and let $B = \pi^{-1}\{k_n : n \in N\} \cap A$. Then clearly $B \in (\mathcal{ED}_{fin})^+$ and B forces that $\pi_*(G)$ contains a selector for the partition $\langle I_n : n < \omega\rangle$. □

Recall that $Fin \times Fin = \{X \subseteq \omega \times \omega : (\forall^\infty k)(|X \cap \{k\} \times \omega| < \omega)\}$. Even though $Fin \times Fin$ is not an F_σ-ideal (not even locally F_σ), $\mathcal{P}(\omega \times \omega)/Fin \times Fin$ is σ-closed (see [Dow89] or [Szy83]).

3.64 Example. The ideal $Fin \times Fin$ adds a Q-point which is not a P-point.

Proof. To see that it does not add a P-point, notice that if we let $A_n = [n, \infty) \times \omega$ then $A_n \in (Fin \times Fin)^*$ so they will be in any generic. However any pseudointersection of the A_n's is in $Fin \times Fin$.

To see that the generic is a Q-point, fix $A \in Fin \times Fin$ positive and some partition $\langle a_n : n < \omega\rangle$ of $\omega \times \omega$ into finite sets. Enumerate $\{n : |\{n\} \times \omega \cap A| = \omega\}$ as $\langle n_k : k < \omega\rangle$ so that each number appears infinitely often. By induction choose $x_l \in A \cap \{n_l\} \times [l, \omega) \setminus \bigcup_{i<s(l)} a_i$ where $s(l) = \max\{i : (\exists j < l)(x_j \cap a_i \neq \emptyset)\}$. Then $B = \{x_l : l < \omega\}$ is a $Fin \times Fin$-positive subset of A which is a selector for the partition. This shows that the generic is a Q-point. □

3.65 Example. Let $\mathcal{G}_C = \{X \subseteq [\omega]^2 : (\forall A \in [\omega]^\omega)([A]^2 \not\subseteq X)\}$ (see [Mez09]). Then $\mathcal{G}_C$ adds a rapid ultrafilter which is neither a P-point nor a Q-point.

Notice that $\mathcal{P}([\omega]^2)/\mathcal{G}_C$ is not σ-closed, but does not add new reals, since it has a dense subset isomorphic to $\mathcal{P}(\omega)/Fin$ (take the embedding $A \mapsto [A]^2$). This shows that the question 3.47 can be answered in the negative for co-analytic ideals.

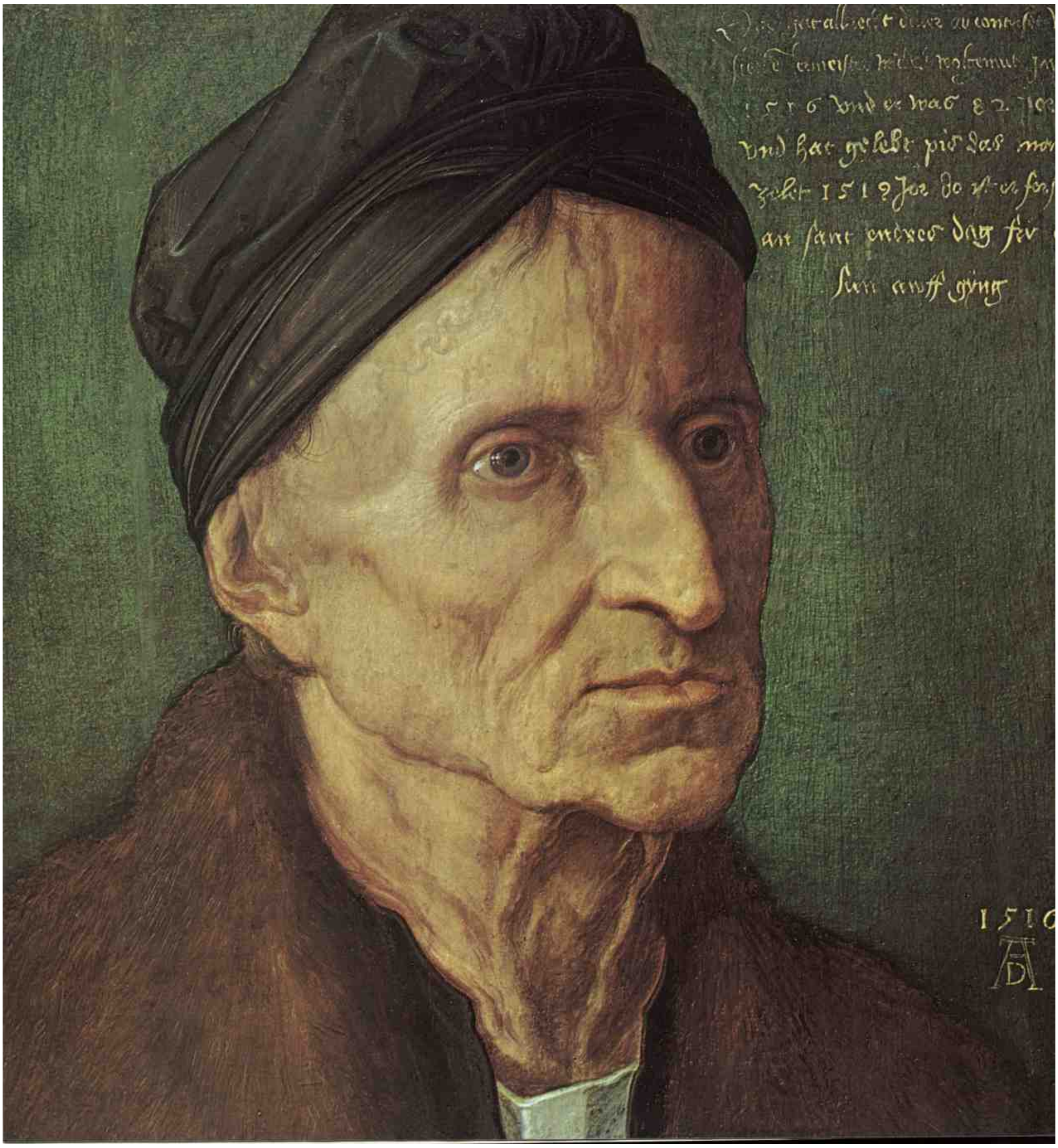
Proof of example 3.65. Notice that $\mathcal{ED}_{fin} \leq_{KB} \mathcal{G}_C$: let $F_n = \{\{m, n\} : m < n\}$ then $[\omega]^2 = \bigcup_{n<\omega} F_n$ and all selectors are in $\mathcal{G}_C$. Next $Fin \times Fin \leq_{KB} \mathcal{G}_C$: let $I_n = \{\{m, n\} : n < m\}$ then $[\omega]^2 = \bigcup_{n<\omega} I_n$ and each infinite subset A of ω has $|[A]^2 \cap I_n| = \omega$ for infinitely many n (every $n \in A$). The first implies that the generic is not a Q-point while the second shows that it cannot be a P-point via the same argument as for $Fin \times Fin$.

To show that it adds a rapid ultrafilter, by proposition 3.59 and homogeneity it suffices to show that $\mathcal{G}_C$ is not KB-above any tall summable ideal $\mathcal{I}_g$. So suppose $f : [\omega]^2 \to \omega$ is finite-to-one. Now construct a sequence $\langle n_i : i < \omega\rangle$ such that for each $i < j$, $g(f(\{n_i, n_j\})) < \frac{1}{j \cdot 2^j}$. This is easy to do and then $f''[\{n_i : i < \omega\}]^2 \in \mathcal{I}_g$, so $\mathcal{G}_C$ is not KB-above $\mathcal{I}_g$ via f. □

3.66 Definition ([HrZaRo∞])**.** If μ is a lscsm and there is a partition $\langle a_n : n < \omega\rangle$ of ω into finite sets and a sequence of submeasures $\langle \mu_n : n < \omega\rangle$, such that $\mu(A) = \sup\{\mu_n(A \cap a_n) : n < \omega\}$, then we say the submeasure μ is *fragmented*. The ideal $Fin(\mu)$ is then called a *fragmented ideal*.

3.67 Example. If $\mathcal{I}$ is a fragmented ideal, then $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point RB-above a selective ultrafilter.

Proof. The forcing adds a P-point by theorem 3.49. We show that the generic is RB-above a selective ultrafilter. Let $\langle a_n : n < \omega\rangle$ be the partition of ω witnessing the fragmentation of $\mathcal{I} = Fin(\mu)$. Fix some finite-to-one function such that $\mu(f^{-1}(n)) \geq n$ and each a_n is contained in some $f^{-1}(k)$. Suppose $A \in \mathcal{I}^+$ and $\langle X_n : n < \omega\rangle$ is a partition of ω. For $k < \omega$ choose n_k such that $\mu(f^{-1}(n_k) \cap A) \geq k$ (this is possible since $\mu(A) \leq \sup\{\mu(A \cap f^{-1}(n) : n < \omega\})$. Now either there is an infinite $X \subseteq \{n_k : k < \omega\}$ which is almost contained in some X_n or we can pick an infinite $X \subseteq \{n_k : k < \omega\}$ such that $|X \cap I_n| \leq 1$ for each $n < \omega$. Then $B = f^{-1}[X] \cap A \in \mathcal{I}^+$ and B forces that either some X_n is in the generic or there is a selector in the generic. □



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Ultrafilters and Independent Systems

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