Complexity of Road Coloring with Prescribed Reset Words

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Abstract

By the Road Coloring Theorem (Trahtman, 2008), the edges of any given aperiodic strongly connected directed multigraph with a constant out-degree can be colored such that the resulting automaton admits a reset word. There may also be a need for a particular reset word to be admitted. In this paper we consider the following problem: given a word w and digraph G, is it true that G has a coloring that is synchronized by w? We show that it is NP-complete for certain fixed words. For the binary alphabet we present a classification that separates such words from those that make the problem solvable in polynomial time. The classification differs if we consider only strongly connected multigraphs. In this restricted setting the classification remains incomplete.

Keywords: synchronizing word, reset word, Road Coloring Problem, synchronizing automata, Černý conjecture

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1. Introduction

Questions about synchronization of finite automata have been studied since the early times of automata theory. The basic concept is very natural: we want to find an input sequence that would bring a given machine to a unique state, no matter in which state the machine was before. Such sequence is called a *reset word*. If an automaton has a reset word, we call it a *synchronizing automaton*.

This area of research attracts much attention because of the famous Černý conjecture [1] posted in 1964 by Jan Černý. It says that an n-state synchronizing automaton has a reset word of length not longer than $(n-1)^2$. Since then the

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literature on the subject is constantly growing (see for example [2, 3, 4, 5, 6, 7, 8, 9]).

In the study of *road coloring*, synchronizing automata are created from directed multigraphs through edge coloring. A directed multigraph is said to be *admissible* if it has a constant out-degree and is aperiodic (that is, the greatest common divisor of the lengths of all cycles in the graph equals to 1). Sometimes we use the notion of admissible graph – in such case we always mean an admissible directed multigraph. A multigraph needs to be admissible in order to have a synchronizing coloring. Many papers in the area of synchronization focus on the road coloring type of problems, like algorithmic approach [10, 11], counting the colorings [12] or using various tools to investigate the road coloring problem itself [13, 14].

In this paper we deal with the complexity issues related to the road coloring problem. It is quite reasonable approach, as each synchronizing problem may be transformed to its road-coloring version in a very natural way:

- (synchronizing problem) Given an automaton $\mathcal{A}$ and some set of parameters and constraints Θ, check if some condition C holds.
- (road-coloring version) Given an admissible graph G and Θ, check if there exists a coloring δ such that condition C holds for the resulting automaton $\mathcal{A} = G(\delta)$.

Given an alphabet I and an admissible graph such that all out-degrees are equal to $|I|$, the following questions arise:

1. Is there a coloring such that the resulting automaton has a reset word?
2. Given a number $k \geq 1$, is there a coloring such that the resulting automaton has a reset word of length at most k?
3. Given a word $w \in I^\star$, is there a coloring such that w is a reset word of the resulting automaton?
4. Given a set of words $W \subseteq I^\star$, is there a coloring such that some $w \in W$ is a reset word of the resulting automaton?

For the first question it was conjectured in 1977 by Adler, Goodwyn, and Weiss [15] that for strongly connected graphs the answer is always *yes*. The conjecture was known as the *Road Coloring Problem* until Trahtman [16] in 2008, using some ideas from [13], found a proof, turning the claim into the *Road Coloring Theorem*.

The second question was initially studied in the paper [17] presented at LATA 2012, while the papers [18] and [19] give final results: the problem is NP-complete for any fixed $k \geq 4$ and any fixed $|I| \geq 2$. The instances with $k \leq 3$ or $|I| = 1$ can be solved by a polynomial-time algorithm. See [20, 19] for some recent results and a detailed discussion on the parameterized complexity for synchronizing automata.

The third question is the subject of the present paper. We show that the problem becomes NP-complete even if restricted to $|I| = 2$ and $w = abb$ or to $|I| = 2$ and $w = aba$, which may seem surprising. Moreover, we provide a complete classification of binary words: The NP-completeness holds for $|I| = 2$ and any $w \in \{a, b\}^\star$ that does not equal a^k, b^k, $a^k b$, nor $b^k a$ for any $k \geq 1$. On the other hand, for any w that matches some of these patterns, the restricted problem is solvable in polynomial time.

The fourth question was raised in [18] and it was emphasized that there are no results about the problem. Our results on the third problem provide an initial step for this direction of research.

It is an easy but important observation that any instance of the first question can be reduced to a strongly connected one by taking a suitable strongly connected component from the original digraph. So, with the Road Coloring Theorem in mind, it may seem that strong connectivity can be safely assumed even if dealing with other problems related to road coloring. Surprisingly, we show that this does not hold for complexity issues. If P is not equal to NP, the complexity of the third problem for strongly connected digraphs differs from the basic third problem in the case of $w = abb$. However, for the strongly connected case we are not able to provide a complete characterization as described above. We are able to give only partial results.

2. Preliminaries

Definition 2.1. A *deterministic finite automaton* (*DFA*) is a triple (Q, I, δ), where Q is a finite set of *states*, I is a finite *alphabet*, and $\delta : Q \times I \to Q$ is a total *transition function*. Slightly abusing the notation, we define $\delta(q, wx) = \delta(\delta(q, w), x)$ for $x \in I$, $w \in I^\star$ and $\delta(R, w) = \{\delta(r, w), r \in R\}$ for $R \subseteq Q$.

In order to work with the directed multigraphs, colorings, and the resulting DFAs, we introduce the following formalisms:

Definition 2.2.

- A *digraph* is a tuple $G = (Q, E, \mathrm{s}, \mathrm{t})$, where Q is a finite set of *vertices* and E is a finite set of *edges*, $\mathrm{s} : E \to Q$ defines *starts* of edges, and $\mathrm{t} : E \to Q$ defines *ends* of edges. We extend these notions for the sets of edges: $\mathrm{s}(F) = \{\mathrm{s}(f) : f \in F\}$ and $\mathrm{t}(F) = \{\mathrm{t}(f) : f \in F\}$, $F \subset E$. For the sake of simplicity we write (Q, E) instead of $(Q, E, \mathrm{s}, \mathrm{t})$. An edge $e \in E$ is *outgoing* from $\mathrm{s}(e)$ and *incoming* to $\mathrm{t}(e)$. Edges $e, e' \in E$ are *parallel* if $\mathrm{s}(e) = \mathrm{s}(e')$ and $\mathrm{t}(e) = \mathrm{t}(e')$.

- A *path* from $q \in Q$ to $r \in Q$ in G is a sequence $e_1, e_2, \ldots, e_d \in E$ such that $\mathrm{s}(e_1) = q$, $\mathrm{t}(e_d) = r$, and $\mathrm{s}(e_i) = \mathrm{t}(e_{i-1})$ for each $2 \leq i \leq d$. If $d \geq 1$ and $q = r$, the path is a *cycle*. An empty path leads from each $q \in Q$ to itself. The *length* of a path $e_1, e_2, \ldots, e_d$ is d, the number of its edges.

- We say that $q \in Q$ *reaches* $r \in Q$ and that r is *reachable* from q in G if there is a path from q to r in G. If there is a path of length at most d, then r is *d-reachable* from q.

- The term $\mathrm{d}_G(q,r)$ denotes the length of a shortest path from $q \in Q$ to $r \in Q$ or ∞ if there is no such path.
- If G is fixed, $r \in Q$ and $k \geq 0$, we denote

$$\begin{aligned} V_k(r) &= \{q \in Q \mid \mathrm{d}(q,r) = k\}\,, \\ V_{\leq k}(r) &= \{q \in Q \mid \mathrm{d}(q,r) \leq k\}\,, \\ V_{\geq k}(r) &= \{q \in Q \mid \mathrm{d}(q,r) \geq k\}\,. \end{aligned}$$

- If G is fixed, $r \in Q$ and $k \geq 0$, we denote

$$U_k(r) = \{q \in Q \mid \text{there is a path of length } k \text{ from } q \text{ to } r\}\,.$$

Definition 2.3.

- A digraph G has a *constant outdegree* $m \in \mathbb{N}$ if each $q \in Q$ has exactly m outgoing edges. For $m \geq 0$, the symbol $\mathbb{G}^m$ denotes the class of digraphs with constant outdegree m.
- A function $\delta[\,] : E \to I$ with a finite *alphabet* I is a *coloring* of a digraph G if:
  - G has constant outdegree $|I|$, and
  - for each $e_1, e_2 \in E$, $e_1 \neq e_2$ it holds that $\mathrm{s}(e_1) = \mathrm{s}(e_2) \Rightarrow \delta[e_1] \neq \delta[e_2]$.

Note that for each vertex $q \in Q$, a coloring $\delta[]$ of G must be a bijection from the outgoing edges of s onto the alphabet I. This allows us to define the transition function of the unique DFA based on G and $\delta[]$ (vertices become states). To emphasize this duality between colorings and transition functions, we use the symbol δ for both:

1. Using square brackets, $\delta[e] \in I$ assigns a letter to an edge.
2. Using parentheses, $\delta(q,x) \in Q$ assigns a target state to a start state and a letter.

More formally, for a coloring δ of G, we consider the DFA (Q, I, δ), where $\delta(q,x) = r$ if and only if there is $e \in E$ with $\mathrm{s}(e) = q$, $\mathrm{t}(e) = r$, and $\delta[e] = x$.

Definition 2.4. Let $w \in I^\star$ with $|I| = m$. A coloring δ is a *w-coloring* of G if $\delta(q,w) = \delta(r,w)$ for each $q, r \in Q$. By $\mathbb{G}^m_w$ we denote the class of all digraphs from $\mathbb{G}^m$ for which there exists a w-coloring.

We work with the following computational problem:

SRCW (*Synchronizing road coloring with prescribed reset words*)	
Input:	Alphabet I, digraph $G \in \mathbb{G}^{\lvert I\rvert}$, set $W \subseteq I^\star$
Output:	Is there a $w \in W$ such that $G \in \mathbb{G}^{\lvert I\rvert}_w$?

$k\backslash l$	0	1	2	3	4	≥ 5
1	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_4$	$\mathcal{P}_4$
2	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_5$	$\mathcal{P}_5$	$\mathcal{P}_5$	$\mathcal{P}_5$
3	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_5$	$\mathcal{P}_5$	$\mathcal{P}_5$	$\mathcal{P}_5$
4	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_5$	$\mathcal{P}_5$	$\mathcal{P}_5$	$\mathcal{P}_5$
≥ 5	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_5$	$\mathcal{P}_5$	$\mathcal{P}_5$	$\mathcal{P}_5$

Table 1: Classification of binary words of the form $w = a^k b^l$ with $k \geq 1$, $l \geq 0$

In this paper we study the restrictions of SRCW to various one-element sets W, which means that we consider the complexity of the sets $\mathbb{G}_w^{|I|}$ themselves.

Restrictions are denoted by subscripts and superscripts: $\mathrm{SRCW}_{k,w}^{\mathcal{M}}$ denotes SRCW restricted to inputs with $|I| = k$, $W = \{w\}$, and $G \in \mathcal{M}$, where $\mathcal{M}$ is a class of digraphs. If a digraph G has constant out-degree $|I|$, a vertex $v \in Q$ is called a *sink vertex* if there are $|I|$ loops on v. By $\mathcal{Z}$ and $\mathcal{SC}$ we denote the class of digraphs having a sink vertex and a family of strongly connected digraphs, respectively.

Binary words are divided to the following classes as follows. The first five classes cover words of the form $a^k b^l$, as visualized in Tab. 1.

$$\begin{aligned}
\mathcal{P}_1 &= \left\{a^k, b^k \mid k \geq 0\right\}, \\
\mathcal{P}_2 &= \left\{a^k b, b^k a \mid k \geq 1\right\}, \\
\mathcal{P}_3 &= \{abb, baa\}, \\
\mathcal{P}_4 &= \left\{ab^k, ba^k \mid k \geq 3\right\}, \\
\mathcal{P}_5 &= \left\{a^k b^l, b^k a^l \mid k, l \geq 2\right\}, \\
\mathcal{P}_6 &= \{a, b\}^\star \setminus (\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_5).
\end{aligned}$$

In our proofs we commonly assume that nonempty words start by a. The results of the present paper are summarized in Tab. 2.

The following lemmas are simple observations concerning distances in digraphs:

Lemma 2.5. *For each G, k and t we have*

$$V_k(t) \subseteq U_k(t) \subseteq V_{\leq k}(t).$$

Lemma 2.6. *Let $G = (Q, E) \in \mathbb{G}_{uv}^m$ with $u, v \in I^\star$ and let δ be a uv-coloring with $\delta(Q, uv) = \{q\}$. Then*

$$\delta(Q, u) \subseteq U_{|v|}(q).$$

In the following, the symbol $\preceq$ is used for the polynomial-time many-to-one reduction.

	general digraphs	strongly connected digraphs	digraphs with sink vertices
$w \in \mathcal{P}_1$	P (Cor. 4.2)	P	P
$w \in \mathcal{P}_2$	P (Th. 4.3)	P	P
$w \in \mathcal{P}_3$	NPC (Th. 4.4)	P (Th. 6.7)	P (Th. 5.1)
$w \in \mathcal{P}_4$	NPC	**?**	P (Th. 5.1)
$w \in \mathcal{P}_5$	NPC	NPC (Th. 6.8)	NPC (Cor. 5.5)
$w \in \mathcal{P}_6$	NPC	NPC (Th. 6.9)	NPC (Cor. 5.5)

Table 2: Complexity of $\mathrm{SRCW}_{2,w}$ according to the classification of binary words and special classes of digraphs. NPC means NP-complete.

3. Special Satisfiability Problems

In our reductions we use two particular variants of a generalized satisfiability problem. Let $R : \{\mathbf{0},\mathbf{1}\}^p \to \{\mathbf{0},\mathbf{1}\}$ be a p-ary Boolean function (or equivalently a Boolean relation). Having such R, we define the following computational task:

SAT(R)	
Input:	Finite set X of *variables*, finite set $\Phi \subseteq X^p$ of *clauses*.
Output:	Is there an assignment $\xi : X \to \{\mathbf{0},\mathbf{1}\}$ such that $R(\xi(z_1), \ldots, \xi(z_p)) = \mathbf{1}$ for each $(z_1, \ldots, z_p) \in \Phi$?

In [21], even more general notion is introduced, concerning the problem SAT(S) for any finite set S of boolean functions. For example, the classical problem 3-SAT is equivalent to SAT(S) with S containing four types of clauses - one for each possible number of negated literals. However, in this paper it is enough to consider variants with a single type of clauses.

Specifically, we work with the following two boolean functions:

$$\begin{aligned} R_1(x_1, x_2, x_3, x_4) &= (x_1 \vee x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2) \wedge (\neg x_3 \vee \neg x_4)\,, \\ R_2(x_1, x_2, x_3, x_4) &= (x_1 \oplus x_2) \vee (x_3 \oplus x_4)\,, \end{aligned}$$

where $\oplus$ denotes the exclusive disjunction. Let us prove, using a straightforward simplification of the Schaefer Dichotomy Theorem [21], that both SAT(R_1) and SAT(R_2) are NP-complete.

Definition 3.1. A function $R : \{\mathbf{0},\mathbf{1}\}^p \to \{\mathbf{0},\mathbf{1}\}$ is

- *weakly negative* if it is equivalent to a Horn formula,
- *weakly positive* if it is equivalent to a dual-Horn formula (i.e. a conjunction of disjunctions, each having at most one negated literal),
- *affine* if it is equivalent to a system of linear equations over the two-element field,
- *bijunctive* if it is equivalent to a formula in 2-CNF.

Theorem 3.2 (Schaefer Dichotomy Theorem)**.** *If $R = \bigwedge_i R_i$ satisfies at least one of the conditions below, then* SAT(R) *is polynomial-time decidable. Otherwise,* SAT(R) *is NP-complete.*

1. *for all i, $R_i(\mathbf{1},\ldots,\mathbf{1}) = \mathbf{1}$,*
2. *for all i, $R_i(\mathbf{0},\ldots,\mathbf{0}) = \mathbf{1}$,*
3. *all R_i are weakly negative,*
4. *all R_i are weakly positive,*
5. *all R_i are affine,*
6. *all R_i are bijunctive.*

Definition 3.3. Let $R(x) = \mathbf{1}$ for $x \in \{\mathbf{0},\mathbf{1}\}^p$. Then $C \subseteq \{1,\ldots,n\}$ is a *change set* for R, x if

$$R(x \oplus e_C) = \mathbf{1}$$

where $e_C \in \{\mathbf{0},\mathbf{1}\}^n$, $(e_C)_i = \mathbf{1}$ exactly for $i \in C$.

Theorem 3.4 ([21])**.** *A function $R : \{\mathbf{0},\mathbf{1}\}^p \to \{\mathbf{0},\mathbf{1}\}$ is*

- *Affine if and only if for each $x, y, z \in \{\mathbf{0},\mathbf{1}\}^p$ it holds that*

$$\textit{if } R(x) = \mathbf{1},\ R(y) = \mathbf{1},\ \textit{and } R(z) = \mathbf{1},\ \textit{then } R(x \oplus y \oplus z) = \mathbf{1}.$$

- *Bijunctive if and only if for any x with $R(x) = \mathbf{1}$ and any change sets C_1, C_2 for R, x it holds that $C_1 \cap C_2$ is also a change set for R, x.*

Corollary 3.5. SAT(R_1) *and* SAT(R_2) *are NP-complete.*

Proof. We just need to show that none of the conditions from Schaefer Dichotomy Theorem (Theorem 3.2) holds for R_1 or R_2. Indeed:

1. We have $R_1(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) = \mathbf{0}$ and $R_2(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) = \mathbf{0}$.
2. We have $R_1(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = \mathbf{0}$ and $R_2(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = \mathbf{0}$.
3. Any Horn formula is either satisfied by $\mathbf{0},\ldots,\mathbf{0}$ or contains a one-element clause x_j. The first case does not hold for R_1, R_2, and the second case implies that $x_j = \mathbf{1}$ in any satisfying assignment, which also does not hold for R_1, R_2.
4. Any Horn formula is either satisfied by $\mathbf{1},\ldots,\mathbf{1}$ or contains a one-element clause $\neg x_j$. The first case does not hold for R_1, R_2, and the second case implies that $x_j = \mathbf{0}$ in any satisfying assignment, which also does not hold for R_1, R_2.

5. For R_1 we apply Theorem 3.4 to the following assignments:

$$\begin{aligned} x &= (\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0}), \\ y &= (\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{0}), \\ z &= (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{1}) \end{aligned}$$

and we see that $R_1(x \oplus y \oplus z) = R_1(\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{1}) = 0$, so R_1 is not affine. For R_2 we take

$$\begin{aligned} x &= (\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0}), \\ y &= (\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{0}), \\ z &= (\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{1}) \end{aligned}$$

and we see that $R_2(x \oplus y \oplus z) = R_2(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) = 0$, so R_2 is not afine.

6. For R_1 we apply Theorem 3.4 to the following setting:

$$\begin{aligned} x &= (\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0}), \\ C_1 &= \{1,2\}, \\ C_2 &= \{1,3\}, \end{aligned}$$

hence

$$\begin{aligned} x \oplus e_{C_1} &= (\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{0}), \\ x \oplus e_{C_2} &= (\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{0}), \\ x \oplus e_{C_1 \cap C_2} &= (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}), \end{aligned}$$

so R_1 is not bijunctive. For R_2 we take:

$$\begin{aligned} x &= (\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{1}), \\ C_1 &= \{1,2,3\}, \\ C_2 &= \{2,3,4\}, \end{aligned}$$

hence

$$\begin{aligned} x \oplus e_{C_1} &= (\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{1}), \\ x \oplus e_{C_2} &= (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0}), \\ x \oplus e_{C_1 \cap C_2} &= (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}), \end{aligned}$$

which means that R_2 is not bijunctive as well.

$\square$

4. SRCW in General Graphs

Lemma 4.1. *Let $G = (Q, E) \in \mathbb{G}^m$ and $k \geq 0$. Then $G \in \mathbb{G}^m_{a^k}$ if and only if G has a sink vertex $q \in Q$ such that $\mathrm{d}_G(r, q) \leq k$ for each $r \in Q$.*

Proof. As for the forward implication, let δ be an a^k-coloring with $\delta(Q, a^k) = \{q\}$. As $\delta(q, a^k) = q$, the vertex q lies on a cycle of edges labeled by a. For each vertex r lying on this cycle, $\delta(r, a^k) = r$, hence the cycle must be a loop. Checking that $\mathrm{d}_G(r, q) \leq k$ for each $r \in Q$ is trivial.

As for the backward implication, we fix a tree $T \subseteq E$ of shortest paths from all the vertices to q, including one loop on q. A coloring δ with $\delta[e] = a$ if and only if $e \in T$ is a valid a^k-coloring. □

Corollary 4.2. *For each $w \in \mathcal{P}_1$ (i.e. $w = a^k$ with $k \geq 0$),* $\mathrm{SRCW}_{2,w} \in \mathrm{P}$.

Proof. For each $q \in Q$ we can easily check whether q is a sink vertex and if it is k-reachable from each $r \in Q$. According to Lemma 4.1, we answer *yes* if and only if some $q \in Q$ meets these conditions. □

Theorem 4.3. *For each $w \in \mathcal{P}_2$ (i.e. $w = a^k b$ with $k \geq 1$),* $\mathrm{SRCW}_{2,w} \in \mathrm{P}$.

Proof. For a fixed G and $q \in Q$ denote

$$Q_1 \quad = \quad \{s \in Q \mid s \text{ has an outgoing edge leading to } q\}\,.$$

For each $s \in Q_1$ fix an outgoing edge e_s leading to q. Let

$$\begin{aligned} T &= \{e \in E \mid \mathrm{s}(e) \in Q_1, \mathrm{t}(e) \in Q_1\} \setminus \{e_s \mid s \in Q_1\}\,, \\ Q_2 &= \{s \in Q_1 \mid s \text{ lies on a cycle of edges in } T\}\,, \\ Q_3 &= \{s \in Q_1 \mid \text{there is a path in } T \text{ from } s \text{ to some } r \in Q_2\}\,. \end{aligned}$$

As usually, we also consider empty paths, thus $Q_2 \subseteq Q_3$. Let us prove that for $q \in Q$ the coloring δ with $\delta(Q, a^k b) = \{q\}$ exists if and only if:

1. $\mathrm{d}_G(s, q) \leq k + 1$ for each $s \in Q$.
2. For each $s \in Q$ there exists $r \in Q_3$ such that $\mathrm{d}_G(s, r) \leq k$.

First, let us check the backward implication. For each $r \in Q_3$, set $\delta[e_r] = b$. Then we fix a forest $U \subseteq E$ of shortest paths from all the vertices of $Q \backslash Q_3$ into Q_3. Due to the second condition above, the paths within U have length at most k. Set $\delta[e] = a$ for each $e \in T$. We have completely specified δ. Now, for any $s \in Q$ there is a prefix a^j of $a^k b$ such that $\delta(s, a^j) \in Q_3$ due to the edges from U. Moreover, because each $e \in T$ with $\mathrm{s}(e) \in Q_3$ has $\delta[e] = a$, we have also $\delta(s, a^k) \in Q_3$ using the definitions of Q_2, Q_3. Together, $\delta(s, a^k b) = q$ using the edge e_r for $r = \delta(s, a^k)$.

As for the forward implication, the first condition is trivial. For the second one, take any $s \in Q$ and denote $s_j = \delta(s, a^j)$ for $j \geq 0$. Clearly, $s_k \in Q_1$, but we show also that $s_k \in Q_3$ and set $r = s_k$ in the second condition. Indeed, whenever $s_j \in Q_1$ for $j \geq k$, we note that $s_j \in \delta\big(s, a^{k-1}\big)$, thus $s_{j+1} \in \delta\big(s, a^k\big)$ and thus $s_{j+1} \in Q_1$ as well. Since j can grow infinitely, there must be a cycle within Q_1 reachable from s_k, all colored with a. It remains to deduce that for each $j \geq k$, the edge from s_j to s_{j+1} colored by a lies in T and thus $s_k \in Q_3$.

Indeed, if the edges outgoing from s_j are not parallel, the one leading to q must be e_{s_j} and it must be colored by b because $s_j \in \delta\big(Q, a^k\big)$. Thus, the other outgoing edge lies in T. If the edges outgoing from s_j are parallel, it makes no difference which one is chosen to be e_s, and thus we can just assume that e_s is colored by b and thus the other one lies in T.

$\square$

Next, we present a simple reduction method for proving NP-completeness of the problems SRCW_{2,ab^k}, $k \geq 2$, which covers the classes $\mathcal{P}_3$ and $\mathcal{P}_4$.

In a formula $\Phi = \{C_1, \ldots, C_m\}$, $C_i \in X^4, 1 \leq i \leq m$, we denote $C_j = (z_{j,1}, z_{j,2}, z_{j,3}, z_{j,4})$ for each $1 \leq j \leq m$ and $\Phi[j,p] = i$ if $z_{j,p} = x_i$ for each $1 \leq p \leq 4$. In the figures depicting particular colorings bold lines stand for a and dotted lines stand for b.

Theorem 4.4. *For each $w \in \mathcal{P}_3 \cup \mathcal{P}_4$ (i.e. $w = ab^k$ with $k \geq 2$),* $\mathrm{SRCW}_{2,w} \in$ NPC.

Proof. We perform a reduction from $\mathrm{SAT}(R_1)$. For a given (X, Φ) with $X = \{x_1, \ldots, x_n\}$, $\Phi = \{C_1, \ldots, C_m\}$, we construct a digraph $\mathbf{G}_1(w, X, \Phi) = (Q, E)$ with $|Q| = 3m + 2n + 3$, independently of the particular value of k, see Fig. 1. The vertices of $\mathbf{G}_1(w, X, \Phi)$ are: $\mathrm{C}_j, \mathrm{C}'_j, \mathrm{C}''_j$ for each $1 \leq j \leq m$, $\mathrm{V}_i, \mathrm{V}'_i$ for each $1 \leq i \leq n$, and $\mathrm{D}_0, \mathrm{D}_1, \mathrm{D}_2$.

1. First, we show that for each ab^k-coloring δ of $\mathbf{G}_1(w, X, \Phi)$ there is a satisfying assignment ξ_δ of Φ. The following are necessary key properties of the coloring δ:

   - $\mathrm{D}_0 \in \delta(Q, a)$. Indeed, if all the edges incoming to D_0 were colored by b, then we would observe that e.g. $\delta(\mathrm{V}_1, ab) = \mathrm{D}_0$ and $\delta(\mathrm{C}_1, abb) = \mathrm{D}_0$. But as there is no loop on $\mathrm{D}_0, \mathrm{D}_1, \mathrm{D}_2$, $\delta\big(\mathrm{D}_0, b^i\big) \neq \delta\big(\mathrm{D}_0, b^{i+1}\big)$ for each $j \geq 0$.

   - The edges outgoing from $\mathrm{D}_0, \mathrm{D}_1, \mathrm{D}_2$ are colored according to some of the variants in Fig 2. Indeed, suppose first that $\delta(\mathrm{D}_0, a) = \mathrm{D}_1$. Then $\delta(\mathrm{D}_1, b) = \mathrm{D}_2$ because otherwise both possible values of $\delta(\mathrm{D}_2, a)$ would imply that $\mathrm{D}_0, \mathrm{D}_1, \mathrm{D}_2 \in \delta(Q, a)$ and subsequently $\big|\delta\big(Q, ab^k\big)\big| \neq 1$ for any $k \geq 0$. We observe that the only remaining variants for $\delta(\mathrm{D}_0, a) = \mathrm{D}_1$ are (A) and (B). Symmetrically, supposing that $\delta(\mathrm{D}_0, a) = \mathrm{D}_2$ leads to variants (A'), (B').

   - $\mathrm{D}_0 \notin \delta(Q, ab)$. Indeed, otherwise $\delta\big(\mathrm{D}_0, b^k\big) = \delta\big(\mathrm{D}_0, b^{k-1}\big)$ which contradicts with each of the variants above.

   We fix the following assignment ξ_δ:

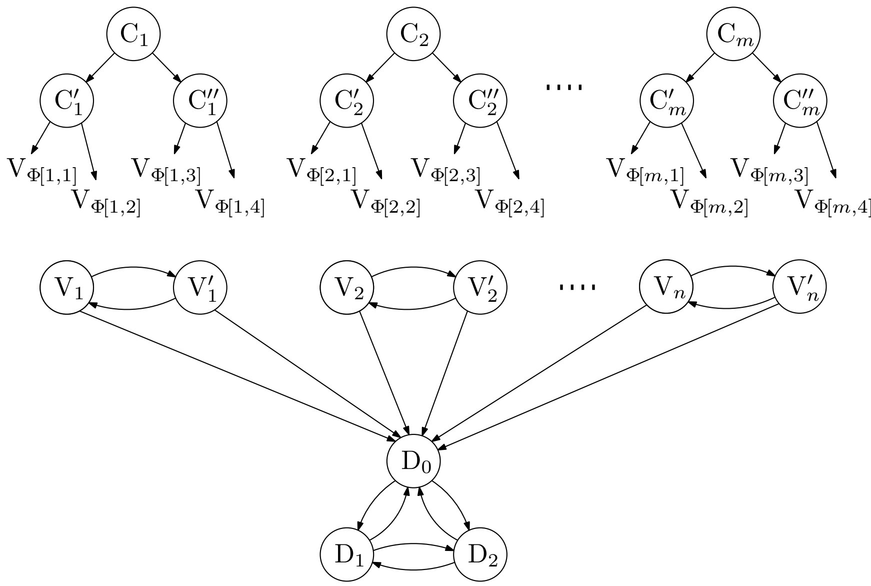

Figure 1: The digraph $\mathbf{G}_1(w, X, \Phi)$

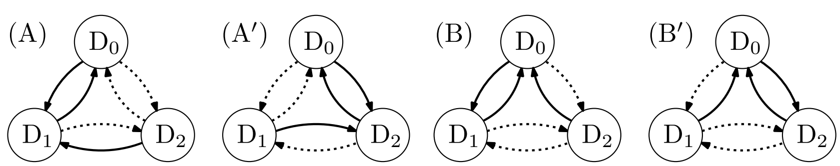

Figure 2: Possible colorings of a part of $\mathbf{G}_1(w, X, \Phi)$

$$\xi_\delta(x_i) = \begin{cases} \mathbf{0} & \text{if } \delta(\mathrm{V}_i, a) = \mathrm{D}_0, \\ \mathbf{1} & \text{if } \delta(\mathrm{V}_i, b) = \mathrm{D}_0. \end{cases} \tag{1}$$

In order to check that ξ_δ is a satisfying assignment, we choose an arbitrary clause C_j, $1 \le j \le m$ and apply the following reasoning.

- Obviously, there is $i \in \{\Phi[j,1], \ldots, \Phi[j,4]\}$ such that $\delta(\mathrm{C}_j, ab) = \mathrm{V}_i$. Notice also that there must be $\delta(\mathrm{V}_i, b) = \mathrm{D}_0$. If, for a contradiction, $\delta(\mathrm{V}_i, b) = \mathrm{V}'_i$, then necessarily $\delta(\mathrm{V}'_i, b) = \mathrm{D}_0$ and thus $\delta(\mathrm{V}'_i, abb) = \mathrm{D}_0$. We see that $k \ge 3$, $\mathrm{D}_0 \in \delta(Q, abb)$, and $\mathrm{D}_0 \in \delta\big(Q, ab^3\big)$. Thus $\delta\big(\mathrm{D}_0, b^{k-2}\big) = \delta\big(\mathrm{D}_0, b^{k-3}\big)$, but no loop colored by b is reachable from D_0 using edges labeled by b, which is a contradiction. Thus, $\delta(\mathrm{V}_i, b) = \mathrm{D}_0$ and $\xi_\delta(x_i) = \mathbf{1}$.
- There are also $i' \in \{\Phi[j,1], \Phi[j,2]\}$ and $i'' \in \{\Phi[j,3], \Phi[j,4]\}$ such that $\delta\big(\mathrm{C}'_j, a\big) = \mathrm{V}_{i'}$ and $\delta\big(\mathrm{C}''_j, a\big) = \mathrm{V}_{i''}$. As $\mathrm{D}_0 \notin \delta(Q, ab)$, it follows that $\delta(\mathrm{V}_{i'}, b) \ne \mathrm{D}_0$ and $\delta(\mathrm{V}_{i''}, b) \ne \mathrm{D}_0$. Thus, $\xi_\delta(x_{i'}) = \mathbf{0}$ and $\xi_\delta(x_{i''}) = \mathbf{0}$.

Together, we have checked that $R_1(\xi_\delta(z_{j,1}), \ldots, \xi_\delta(z_{j,4})) = \mathbf{1}$ and thus ξ_δ satisfies C_j.

2. Second, from a satisfying assignment ξ of Φ we infer an ab^k-coloring δ_ξ of $\mathbf{G}_1(w, X, \Phi)$:

   - The edges outgoing from $\mathrm{D}_0, \mathrm{D}_1, \mathrm{D}_2$ are colored according to the variant (A) above.
   - For each $1 \le i \le n$, we set

$$\delta_\xi(\mathrm{V}_i, b) = \begin{cases} \mathrm{D}_0 & \text{if } \xi(x_i) = \mathbf{1}, \\ \mathrm{V}'_i & \text{otherwise}, \end{cases} \tag{2}$$

$$\delta_\xi(\mathrm{V}'_i, b) = \begin{cases} \mathrm{D}_0 & \text{if } \xi(x_i) = \mathbf{0}, \\ \mathrm{V}_i & \text{otherwise}. \end{cases} \tag{3}$$

   - For each $1 \le j \le m$, the clause C_j is satisfied by ξ. Thus, there is $p \in \{1,2,3,4\}$ such that $\xi(z_{j,p}) = \mathbf{1}$, there is $p' \in \{1,2\}$ such that $\xi(z_{j,p'}) = \mathbf{0}$, and there is $p'' \in \{3,4\}$ such that $\xi(z_{j,p''}) = \mathbf{0}$. Let

$$\delta_\xi(\mathrm{C}_j, a) = \begin{cases} \mathrm{C}'_j & \text{if } i \in \{1,2\}, \\ \mathrm{C}''_j & \text{if } i \in \{3,4\}, \end{cases} \tag{4}$$

$$\delta_\xi\big(\mathrm{C}'_j, a\big) = \mathrm{V}_{\Phi[j,i']}, \tag{5}$$

$$\delta_\xi\big(\mathrm{C}''_j, a\big) = \mathrm{V}_{\Phi[j,i'']}. \tag{6}$$

We claim that for each $q \in Q$, $\delta_\xi\big(q, ab^k\big) = \mathrm{D}_0$ if k is even and $\delta_\xi\big(q, ab^k\big) = \mathrm{D}_1$ if k is odd. This implies that $\big|\delta\big(Q, ab^k\big)\big| = 1$:

- For $q \in \{\mathrm{D}_0, \mathrm{D}_1, \mathrm{D}_2\}$ we check the claim easily.

- If $1 \leq i \leq n$, for $q \in \{\mathrm{V}_i, \mathrm{V}'_i\}$ it holds that $\delta_\xi(q,a) = \mathrm{D}_0$ or $\delta_\xi(q,abb) = \mathrm{D}_0$, which implies the claim.
- For $1 \leq j \leq m$, we know that $\delta(\mathrm{C}_j, ab) = \mathrm{V}_i$ with $\xi(x_i) = \mathbf{1}$. Thus, $\delta(\mathrm{V}_i, b) = \mathrm{D}_0$ and $\delta(\mathrm{C}_j, abb) = \mathrm{D}_0$.
- For $q \in \left\{\mathrm{C}'_j, \mathrm{C}''_j\right\}$ with $1 \leq j \leq m$, we know that $\delta(q,a) = \mathrm{V}_{i'}$ with $\xi(x_{i'}) = \mathbf{0}$. Thus, $\delta(\mathrm{V}_{i'}, bb) = \mathrm{D}_0$ and $\delta(q, abb) = \mathrm{D}_0$.

□

5. SRCW in Graphs with Sink Vertices

Theorem 5.1. *For each $w \in \mathcal{P}_3 \cup \mathcal{P}_4$ (i.e. $w = ab^k$ with $k \geq 2$),* $\mathrm{SRCW}^{\mathcal{Z}}_{2,w} \in \mathrm{P}$.

Proof. Let $w = ab^k$ for $k \geq 2$. Let $G = (Q,E) \in \mathbb{G}^2$ be a given digraph with a sink vertex $q \in Q$. We suppose that $Q \subseteq V^G_{\leq k+1}(q)$ as otherwise the answer is trivially *no*. Let $H = \{s \in Q \mid s \text{ reaches some } r \in V_{k+1}(q)\}$. We show that $G \in \mathbb{G}^2_{ab^k}$ if and only if each $s \in H$ has an outgoing edge $e \in E$ with $\mathrm{t}(e) \notin H$.

1. For the forward implication, let δ be an ab^k-coloring of G. Now, suppose for a contradiction that both the edges e and e' outgoing from s_0 have $\mathrm{t}(e), \mathrm{t}(e') \in H$. Then $s_1 = \delta(s_0, a) \in H$ via some $e_0 \in \{e, e'\}$. Let $e_0, e_1, \ldots, e_d$ be a path from $s_0 \in H$ to $s_d \in V_{k+1}(q)$. Let $0 \leq c \leq d$ be the largest number with $\delta[e_c] = a$ (we know that at least 0 has this property). Then $\delta\big(\mathrm{s}(e_c), ab^i\big) \in V_{k+1}(q)$ for some $i \geq 0$, which contradicts the fact that $\delta\big(\mathrm{s}(e_c), ab^k\big) = q$.
2. For the backward implication, let $U \subseteq E$ be a set containing for each $s \in H$ exactly one edge with $\mathrm{s}(e) = s$ and $\mathrm{t}(e) \notin H$. Moreover, let $T \subseteq E$ be a tree of the shortest paths from the vertices of $Q \backslash H$ to the sink vertex q. Consider the following coloring:

$$\delta[e] = \begin{cases} a & \text{if } \mathrm{s}(e) \in H, e \in U \text{ or } \mathrm{s}(e) \notin H, e \notin T, \\ b & \text{if } \mathrm{s}(e) \notin H, e \in T \text{ or } \mathrm{s}(e) \in H, e \notin U. \end{cases}$$

   This is a valid construction as each $s \in H$ and each $s \notin H$ has exactly one outgoing edge in U or T respectively. From the definition of H it follows that no edge starting within $Q \backslash H$ ends in H. Thus, the construction of δ implies $\delta(Q, a) \subseteq Q \backslash H$.
   Because $V_{k+1}(q) \subseteq \mathrm{s}(T)$, T has height of at most k and thus $\delta\big(Q \backslash H, b^k\big) = \{q\}$ and $\delta\big(Q, ab^k\big) = \{q\}$.

□

Lemma 5.2. *Let $v \in I^\star$ be a factor of $w \in I^\star$, that is $w = u_1 v u_2$ for some $u_1, u_2 \in I^*$. Then* $\mathrm{SRCW}^{\mathcal{Z}}_{2,v} \preceq \mathrm{SRCW}^{\mathcal{Z}}_{2,w}$.

Proof. Suppose that $w = u_1 v u_2$. Then the reduction takes a digraph $G = (Q,E)$ with a sink vertex $q \in Q$ and produces $G' = (Q', E')$, which is obtained from G in a following way:

1. For each $s \in Q$, add a chain of $|u_1|$ vertices with consecutive pairs of parallel edges, ending in s.
2. Remove the two loops on q.
3. Add a chain of $|u_2|$ vertices with consecutive pairs of parallel edges, starting in q. The last vertex of the chain is the new sink vertex q'.

If δ is a v-coloring of G, it corresponds to a unique coloring δ' of G'. For each $s \in Q'$ and $t = \delta'(s, u_1)$ we have $t \in Q$ or t is some of the vertices added in the step (3) above. In both cases it follows that $\delta(t, vu_2) = q'$. On the other hand, if δ' is a w-coloring of G', it corresponds to a unique coloring δ of G. Each $s \in Q$ lies in $\delta'(Q', u_1)$, so for each $s \in Q$ we have $\delta'(s, vu_2) = q'$. As each path from s to q' contains q, we have $\delta(s, v') = q$ for some prefix v' of vu_2. As each path from q to q' is of length at least $|u_2|$, we have $|v'| \leq |v|$ and thus v' is a prefix of v. It follows that $\delta(s, v') = q$ and $\delta(s, v) = q$.

□

Theorem 5.3. $\mathrm{SRCW}^{\mathcal{Z}}_{2,aabb} \in \mathrm{NPC}$.

Proof. We perform a reduction from $\mathrm{SAT}(R_2)$. For a given (X, Φ) with $X = \{x_1, \ldots, x_n\}$, $\Phi = \{C_1, \ldots, C_m\}$, we construct a digraph $\mathbf{G}_2(X, \Phi) = (Q, E)$ with $|Q| = 8m + 5n + 1$. The vertices of $\mathbf{G}_2(X, \Phi)$ are $\mathrm{C}_{j,1}, \ldots, \mathrm{C}_{j,8}$ for each $1 \leq j \leq m$, $\mathrm{V}_1, \ldots, \mathrm{V}_n$, and $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3, \mathrm{D}_4$, where D_4 is a sink vertex, see Fig. 3.

1. First, we show that for each $aabb$-coloring δ of $\mathbf{G}_2(X, \Phi)$ there is a satisfying assignment ξ_δ of Φ. Let
$$\xi_\delta(x_i) = \begin{cases} \mathbf{0} & \text{if } \delta(\mathrm{V}_{i,1}, a) = \mathrm{D}_4, \\ \mathbf{1} & \text{if } \delta(\mathrm{V}_{i,1}, b) = \mathrm{D}_4. \end{cases}$$
In order to check that ξ_δ is a satisfying assignment, we choose an arbitrary clause C_j, $1 \leq j \leq m$ and continue with the following steps:
   - Observe that for each $1 \leq i \leq n$, $\mathrm{V}_i \notin \delta(Q, aa)$. Indeed, if $\mathrm{V}_i \in \delta(Q, aa) \subseteq \delta(Q, a)$, then surely either D_2 or D_3 lies in $\delta(Q, aabb)$, which is a contradiction.
   - From the shape of a clause gadget it is obvious that some $r \in \{\mathrm{C}_{j,3}, \mathrm{C}_{j,4}\}$ lies in $\delta(Q, aa)$. If $r = \mathrm{C}_{j,3}$, then for some $s, s' \in \{\mathrm{C}_{j,5}, \mathrm{C}_{j,6}\}$ we have $s \in \delta(Q, aaa) \subseteq \delta(Q, a)$ and $s' \in \delta(Q, aab)$. From the observation (a) it follows that $\delta(s, a) \notin \{\mathrm{V}_{i,1} \mid 1 \leq i \leq n\}$, and thus $\delta(s, a) = \mathrm{D}_4$. Moreover, it is obvious that $\delta(s', b) = \mathrm{D}_4$. Thus, $\delta(s, b) = \mathrm{V}_{i,1}$, $\mathrm{V}_{i,1} \in \delta(Q, aab)$ and $\delta(s', a) = \mathrm{V}_{i',1}$ for $x_i, x_{i'} \in \{z_{j,1}, z_{j,2}\}$. It follows that $\delta(\mathrm{V}_{i,1}, b) = \mathrm{D}_4$ and $\delta(V_{i',1}, a) = \mathrm{D}_4$ and thus $\xi(z_{j,1}) \neq \xi(z_{j,2})$. If $r = \mathrm{C}_{j,4}$, the situation is symmetric and we obtain $\xi(z_{j,3}) \neq \xi(z_{j,4})$. Together, $R_2(\xi_\delta(z_{j,1}), \ldots, \xi_\delta(z_{j,4})) = \mathbf{1}$ and ξ_δ satisfies C_j.
2. Second, from a satisfying assignment ξ of Φ we infer an $aabb$-coloring δ_ξ of $\mathbf{G}(X, \Phi)$.

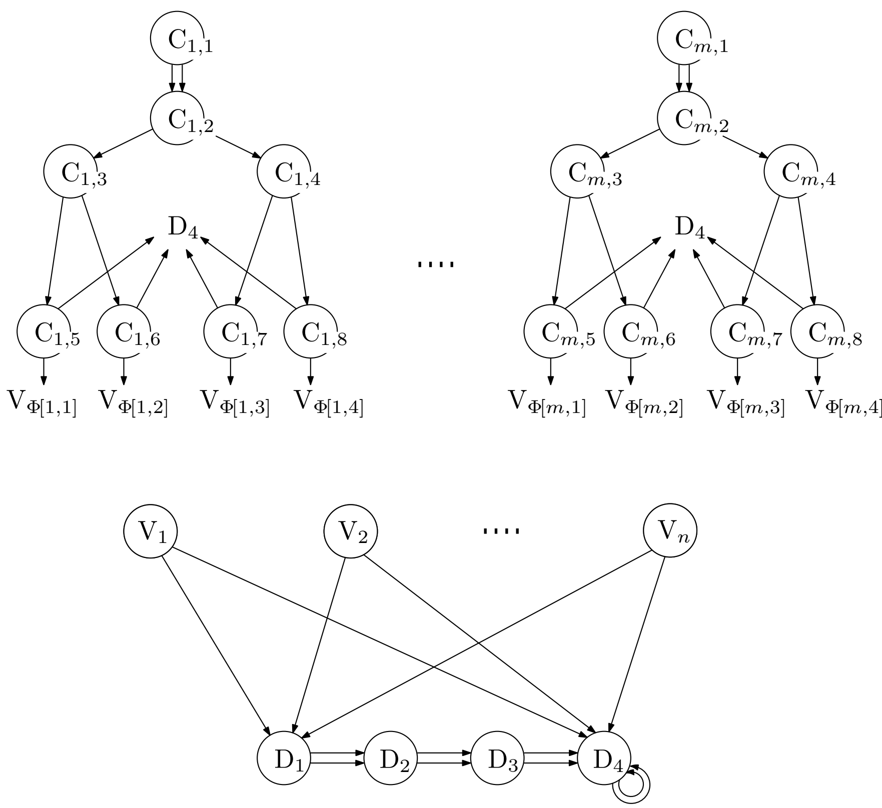

Figure 3: The digraph $\mathbf{G}_2(X, \Phi)$

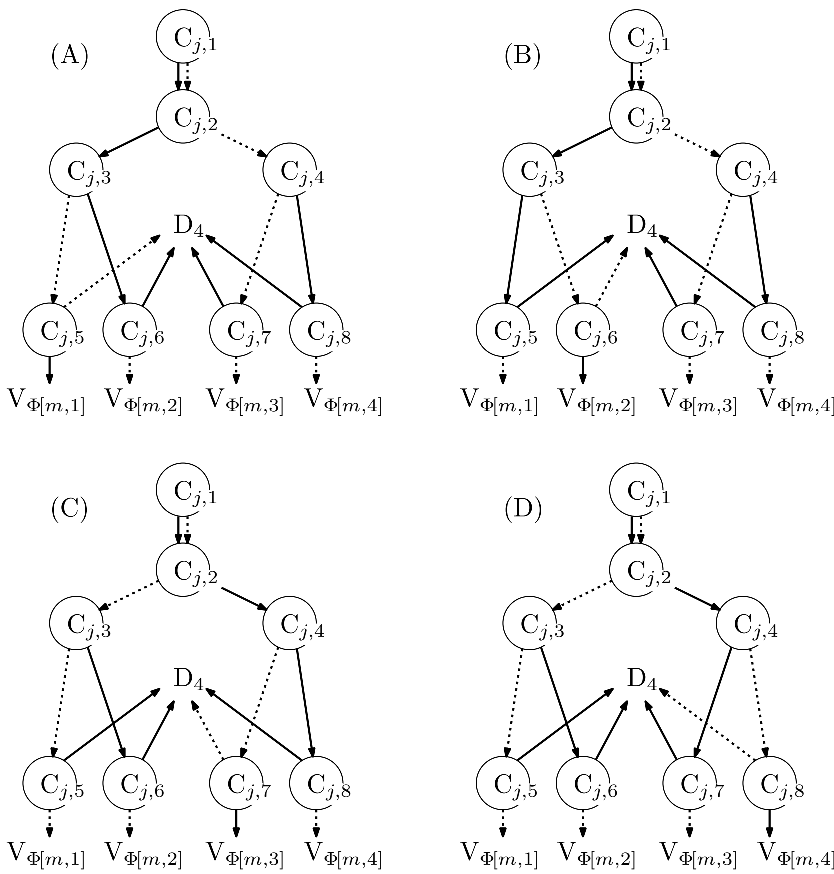

Figure 4: Possible colorings of parts of $\mathbf{G}_2(X, \Phi)$

- For each $1 \leq j \leq m$ some of the following possibilities must hold:
  (A) $\xi(z_{j,1}) = \mathbf{0}$ and $\xi(z_{j,2}) = \mathbf{1}$,
  (B) $\xi(z_{j,1}) = \mathbf{1}$ and $\xi(z_{j,2}) = \mathbf{0}$,
  (C) $\xi(z_{j,3}) = \mathbf{0}$ and $\xi(z_{j,4}) = \mathbf{1}$,
  (D) $\xi(z_{j,3}) = \mathbf{1}$ and $\xi(z_{j,4}) = \mathbf{0}$,

  Let δ_ξ be defined on edges outgoing from $\mathrm{C}_{j,1}, \ldots, \mathrm{C}_{j,8}$ according to a suitable variant depicted in Fig. 4.
- For each $1 \leq i \leq n$, let $\delta(\mathrm{V}_{i,1}, a) = \mathrm{D}_4$ if $\xi(x_i) = \mathbf{0}$, and $\delta(\mathrm{V}_{i,1}, b) = \mathrm{D}_4$ otherwise.

This defines δ_ξ on all edges that are not parallel. It remains to verify that δ_ξ is an $aabb$-coloring.

- Any path of length 4 from $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3, \mathrm{D}_4$ and $\mathrm{V}_1, \ldots, \mathrm{V}_n$ ends in D_4.
- Take $1 \leq j \leq m$ and check which variant was used for a coloring of $\mathrm{C}_{j,1}, \ldots, \mathrm{C}_{j,8}$. If (A) is the case, we have:
  - $\delta_\xi(r, aabb) = \mathrm{D}_4$ holds immediately for $r \in \{\mathrm{C}_{j,1}, \mathrm{C}_{j,3}, \mathrm{C}_{j,4}, \mathrm{C}_{j,6}, \mathrm{C}_{j,7}, \mathrm{C}_{j,8}\}$.
  - $\delta_\xi(\mathrm{C}_{j,2}, aab) = \mathrm{V}_{i,1}$, where $z_{j,2} = x_i$ and $\xi(x_i) = \mathbf{1}$. Thus $\delta_\xi(\mathrm{V}_{i,1}, b) = \mathrm{D}_4$ and $\delta_\xi(\mathrm{C}_{j,2}, aabb) = \mathrm{D}_4$.
  - $\delta_\xi(\mathrm{C}_{j,5}, a) = \mathrm{V}_{i',1}$, where $z_{j,1} = x_{i'}$ and $\xi(x_{i'}) = \mathbf{0}$. Thus $\delta_\xi(\mathrm{V}_{i,1}, a) = \mathrm{D}_4$ and $\delta_\xi(\mathrm{C}_{j,5}, aabb) = \mathrm{D}_4$.

  If (B), (C), or (D) is the case, the situation is symmetrical.

□

Theorem 5.4. *For each $k \geq 1$ $\mathrm{SRCW}^{\mathcal{Z}}_{2,ab^ka} \in \mathrm{NPC}$.*

Proof. We perform a reduction from $\mathrm{SAT}(R_1)$. For a given (X, Φ) with $X = \{x_1, \ldots, x_n\}$, $\Phi = \{C_1, \ldots, C_m\}$, we construct a digraph $\mathbf{G}_3(X, \Phi) = (Q, E)$ with $|Q| = 3m + (k+2)\,n + 1$, having a sink vertex D. The vertices of $\mathbf{G}_3(X, \Phi)$ are $\mathrm{C}_j, \mathrm{C}'_j, \mathrm{C}''_j$ for each $1 \leq j \leq m$, $\mathrm{V}_{i,j}$ for each $1 \leq i \leq n$, $1 \leq j \leq k+2$ and D which is a sink vertex, see Fig. 5.

1. First, we show that for each ab^ka-coloring δ of $\mathbf{G}_3(X, \Phi)$ there is a satisfying assignment ξ_δ of Φ. Let

$$\xi_\delta(x_i) = \begin{cases} \mathbf{0} & \text{if } \delta(\mathrm{V}_{i,k}, b) = \mathrm{D}, \\ \mathbf{1} & \text{if } \delta(\mathrm{V}_{i,k}, a) = \mathrm{D}. \end{cases}$$

   In order to check that ξ_δ is a satisfying assignment, we choose an arbitrary clause C_j, $1 \leq j \leq m$ and apply the following reasoning:

   - Obviously, there is $i \in \{\Phi[j,1], \ldots, \Phi[j,4]\}$ such that $\delta(\mathrm{C}_j, ab) = \mathrm{V}_{i,1}$ and thus also $\delta\big(\mathrm{C}_j, ab^k\big) = \mathrm{V}_{i,k}$ and $\delta(\mathrm{V}_{i,k}, a) = \mathrm{D}$. We get $\xi_\delta(x_i) = \mathbf{1}$.

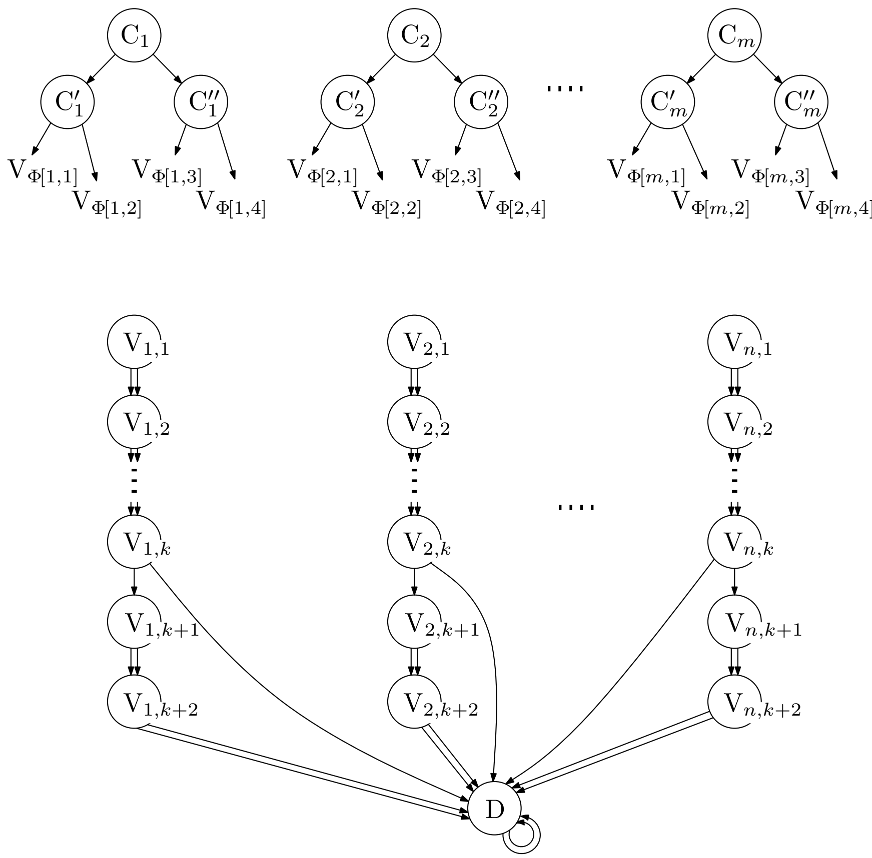

Figure 5: The digraph $\mathbf{G}_3(X, \Phi)$

- There are also $i' \in \{\Phi[j,1], \Phi[j,2]\}$ and $i'' \in \{\Phi[j,3], \Phi[j,4]\}$ such that $\delta\big(\mathrm{C}'_j, a\big) = \mathrm{V}_{i',1}$ and $\delta\big(\mathrm{C}''_j, a\big) = \mathrm{V}_{i'',1}$. Thus also $\delta\big(\mathrm{C}'_j, ab^{k-1}\big) = \mathrm{V}_{i',k}$ and $\delta\big(\mathrm{C}''_j, ab^{k-1}\big) = \mathrm{V}_{i'',k}$. The only paths of length 2 from $\mathrm{V}_{i',k}$ and $\mathrm{V}_{i'',k}$ to D use the direct edge to D. Thus, $\delta(\mathrm{V}_{i',k}, b) = \mathrm{D}$ and $\delta(\mathrm{V}_{i'',k}, b) = \mathrm{D}$. It follows that $\xi_\delta(x_{i'}) = \mathbf{0}$ and $\xi_\delta(x_{i''}) = \mathbf{0}$.

Together, $R_1(\xi_\delta(z_{j,1}), \ldots, \xi_\delta(z_{j,4})) = \mathbf{1}$ and ξ_δ satisfies C_j.

2. Second, from a satisfying assignment ξ of Φ we infer an ab^ka-coloring δ_ξ of $\mathbf{G}_3(X, \Phi)$:

- For each $1 \leq i \leq n$, we set

$$\delta_\xi(\mathrm{V}_{i,k}, a) = \begin{cases} \mathrm{D} & \text{if } \xi(x_i) = \mathbf{1}, \\ \mathrm{V}_{i,k+1} & \text{otherwise,} \end{cases}$$

- For each $1 \leq j \leq m$, the clause C_j is satisfied by ξ. Thus, there is $p \in \{1,2,3,4\}$ such that $\xi(z_{j,p}) = \mathbf{1}$, there is $p' \in \{1,2\}$ such that $\xi(z_{j,p'}) = \mathbf{0}$, and there is $p'' \in \{3,4\}$ such that $\xi(z_{j,p''}) = \mathbf{0}$. Let

$$\begin{aligned} \delta_\xi(\mathrm{C}_j, a) &= \begin{cases} \mathrm{C}'_j & \text{if } i \in \{1,2\}, \\ \mathrm{C}''_j & \text{if } i \in \{3,4\}, \end{cases} \\ \delta_\xi\big(\mathrm{C}'_j, a\big) &= \mathrm{V}_{\Phi[j,i']}, \\ \delta_\xi\big(\mathrm{C}''_j, a\big) &= \mathrm{V}_{\Phi[j,i'']}. \end{aligned}$$

We claim that for each $s \in Q$, $\delta_\xi\big(s, ab^ka\big) = \mathrm{D}$:

- For $s = \mathrm{D}$ and $s \in \{\mathrm{V}_{i,1}, \ldots, \mathrm{V}_{i,k+2}\}$, $1 \leq i \leq n$, each path of length $k+2$ starting in s ends in D.
- For $1 \leq j \leq m$ we know that $\delta(\mathrm{C}_j, ab) = \mathrm{V}_{i,1}$, $\delta\big(\mathrm{C}_j, ab^k\big) = \mathrm{V}_{i,k}$ with $\xi(x_i) = \mathbf{1}$. Thus, $\delta(\mathrm{V}_{i,k}, a) = \mathrm{D}$ and $\delta\big(\mathrm{C}_j, ab^ka\big) = \mathrm{D}$.
- For $s \in \big\{\mathrm{C}'_j, \mathrm{C}''_j\big\}$ with $1 \leq j \leq m$ we know that $\delta(s, a) = \mathrm{V}_{i',1}$, $\delta\big(s, ab^{k-1}\big) = \mathrm{V}_{i',k}$ with $\xi(x_{i'}) = \mathbf{0}$. Thus, $\delta(\mathrm{V}_{i',k}, b) = \mathrm{D}$ and $\delta\big(s, ab^ka\big) = \mathrm{D}$.

□

Corollary 5.5. *For each $w \in \mathcal{P}_5 \cup \mathcal{P}_6$, $\mathrm{SRCW}^{\mathcal{Z}}_{2,w} \in \mathrm{NPC}$.*

Proof. Observe that each $w \in \mathcal{P}_5$ has $aabb$ as a factor, and each $w \in \mathcal{P}_6$ has a factor of the form ab^ka with $k \geq 1$. Accordingly, we just apply Theorems 5.3 and 5.4. □

6. SRCW for Strongly Connected Graphs

Next, we deal with strong connectivity. Before presenting a polynomial-time algorithm for the case of $w = abb$ (i.e. $w \in \mathcal{P}_3$), we introduce a technical definition and five simple lemmas that will be also used later in the proofs of NP-completeness.

Definition 6.1. Let $w \in I^\star$ and $G \in \mathbb{G}^{|I|}$. A w-coloring δ of G is an *exact w-coloring* if $q \notin \delta(Q, v)$ for each proper non-empty prefix v of w, where $q = \delta(Q, w)$.

Lemma 6.2. *Let $w = ab^k$ and $G \in \mathbb{G}^2$. Then G has an exact w-coloring if and only if there is $q \in Q$ such that each $s \in Q$ has an outgoing edge ending in $V_k(q)$.*

Proof. First, fix an exact ab^k-coloring δ with $\delta\big(Q, ab^k\big) = \{q\}$. Clearly (due to Lemma 2.6), no $e \in E$ with $\delta[e] = a$ has $\mathrm{t}(e) \in V_{\geq k+1}(q)$. Let $0 \leq m \leq k$ be the least number such that some $e \in E$ with $\delta[a]$ has $\mathrm{t}(e) \in V_m(q)$. Each vertex $r \in V_i(q)$ for $1 \leq i \leq m$ has an outgoing edge ending in $V_{i-1}(q)$, and due to minimality of m all these edges are colored by b. It follows that $q \in \delta(Q, ab^m)$. As δ is exact, we get $m = k$. As each $s \in Q$ has an outgoing edge labeled by a, each $s \in Q$ has an outgoing edge leading to $V_k(q)$.

Second, let G and $q \in Q$ meet the key condition. For each $s \in Q$ we choose some $e \in E$ with $\mathrm{s}(e) = s$, $\mathrm{t}(e) \in V_k(q)$ and set $\delta[e] = a$. Then $\delta(Q, a) \subseteq V_k(q)$. For each $1 \leq i \leq k$, each $s \in V_m(q)$ has an outgoing edge ending in $V_{i-1}(q)$ and these edges are labeled by b. Thus, $\delta\big(Q, ab^k\big) = \{q\}$. □

Lemma 6.3. *Let $G \in \mathbb{G}^2_{ab^k} \cap \mathcal{SC}$ and let δ be a non-exact coloring with $\delta\big(Q, ab^k\big) = \{q\}$. Then for each $r \in Q$ there is $0 \leq d \leq k$ with $r \in \delta\big(Q, ab^d\big)$.*

Proof. Choose $r \in Q$ and fix a shortest path $e_1, \ldots, e_h$ from $\mathrm{s}(e_1) = q$ to $\mathrm{t}(e_h) = r$. Let $1 \leq i \leq h$ be the greatest number with $\delta[e_i] = a$.

- If there is no such i, for each $j > h$ denote by e_j the unique edge with $\mathrm{s}(e_j) = \mathrm{t}(e_{j-1})$ and $\delta[e_j] = b$. Let $\overline{h} \geq h$ be the least number such that $e_{\overline{h}+1} \in \big\{e_1, \ldots, e_{\overline{h}}\big\}$. Informally, we have elongated the path labeled by b and reached a cycle. As δ is non-exact, let ab^l with $l \geq 0$ be the nonempty proper prefix of w with $q \in \delta\big(Q, ab^l\big)$. It follows that $\delta\big(q, b^{k-l}\big) = q$, thus q lies on a cycle labeled by b and thus $\mathrm{t}\big(e_{\overline{h}}\big) = q$ and $\overline{h} \leq k - l$. We get $r \in \delta\big(Q, ab^{l+h}\big)$ and $d = l + h \leq k$.
- If $i < h - k$, we get $\delta\big(\mathrm{s}(e_i)\,, ab^k\big) = \mathrm{t}(e_{i+k})$ and thus $\mathrm{t}(e_{i+k}) = q$. This is a contradiction as q does not lay on a shortest path from q to r.
- If $i \geq h - k$, we get $\delta\big(\mathrm{s}(e_i)\,, ab^{h-i}\big) = \mathrm{t}(e_h) = r$ and $d = h - i \leq k$.

□

Lemma 6.4. *Let $G \in \mathbb{G}^2_{ab^k} \cap \mathcal{SC}$ and let δ be a non-exact coloring with $\delta\big(Q, ab^k\big) = \{q\}$. Then $V_{\geq k+1}(q) = \emptyset$.*

Proof. For a contradiction, let $r \in V_{\geq k+1}(q)$. According to Lemma 6.3, suppose that $r \in \delta\big(Q, ab^d\big)$ for $0 \leq d \leq k$. This easily contradicts Lemma 2.6 as there is no path of length at most k from r to q. $\square$

Lemma 6.5. *Let $G \in \mathbb{G}^2_{ab^k} \cap \mathcal{SC}$ and let δ be a non-exact coloring with $\delta\big(Q, ab^k\big) = \{q\}$. Let $m \leq k$ and $s \in V_m(q)$. Then $s \notin \delta\big(Q, b^{k-m+1}\big)$.*

Proof. For a contradiction, let $\delta\big(r, b^{k-m+1}\big) = s$ for $s \in V_m(q)$. According to Lemma 6.3, suppose that $r \in \delta\big(Q, ab^d\big)$ for $d \leq k$ and thus $s \in \delta\big(Q, ab^{d+k-m+1}\big)$. Lemma 2.6 implies that $s \in V_{\leq m-d-1}(q)$. However, $V_{\leq m-d-1}(q) \cap V_m(q) = \emptyset$, so we get a contradiction. $\square$

Lemma 6.6. *Let $G \in \mathbb{G}^2_{ab^k} \cap \mathcal{SC}$ and let δ be a non-exact coloring with $\delta\big(Q, ab^k\big) = \{q\}$. Then $V_k(q) \subseteq \delta(Q, a)$.*

Proof. Choose $r \in V_k(q)$. According to Lemma 6.3, suppose that $r \in \delta\big(Q, ab^d\big)$ for $d \leq k$. If $d > 0$, we get a contradiction with Lemma 2.6 (there is no path of length at most $k-1$ from r to q). Thus, $d = 0$ and $r \in \delta(Q, a)$. $\square$

Theorem 6.7. *For each $w \in \mathcal{P}_3$ (i.e. $w = aab$)* $\mathrm{SRCW}^{\mathcal{SC}}_{2,w} \in \mathrm{P}$.

Proof. For each $q \in Q$ we apply the following procedure. First, we use Lemma 6.2 to check whether $\delta(Q, abb) = \{q\}$ for some exact abb-coloring δ. If not, we will look for a non-exact one. If $V_{\geq 3}(q) \neq \emptyset$, according to Lemma 6.4 we can answer *no*. Otherwise, $Q \subseteq V_{\leq 2}(q)$. We check whether q has a loop. If so, Lemma 4.1 says that there is a coloring δ with $\delta(Q, bb) = \{q\}$ and thus $\delta(Q, abb) = \{q\}$, we answer *yes*. So, assume that q has no loop.

Any coloring δ that may remain to be found has $q \in \delta(Q, a)$. To see this, assume the opposite and choose an arbitrary $e \in E$ with $\delta[e] = a$, thus $\mathrm{t}(e) \neq q$. We have $\mathrm{t}(e) \notin V_1(q)$ because each $t \in V_1(q)$ has an outgoing edge leading to q colored by b. Hence $\delta(t, ab) = q$, but q has no loop. Thus, each e with $\delta[e] = a$ has $\mathrm{t}(e) \in V_2(q)$ and there is an exact coloring according to Lemma 6.2, which is a contradiction.

Using the observation above and Lemma 6.6 we obtain $\{q\} \cup V_2(q) \subseteq \delta(Q, a)$ for any possible remaining coloring δ. We encode the structure of G to a propositional formula Ψ with $|E|$ variables, $\mathbf{x}_e$ for each $e \in E$. Let Ψ be the conjuction of the following subformulas:

1)	$\mathbf{x}_e \oplus \mathbf{x}_f$	for each $e, f \in E$ with $\mathrm{s}(e) = \mathrm{s}(f)$
2)	$\mathbf{x}_e$	for each $e \in E$ with $\mathrm{t}(e) \in V_2(q)$
3)	$\mathbf{x}_e \rightarrow \mathbf{x}_f$	for each path $e, f \in E$ with $\mathrm{t}(f) = q$
4)	$\mathbf{x}_e \vee \mathbf{x}_f$	for each path $e, f \in E$ with $\mathrm{t}(e) \neq q$, $\mathrm{t}(f) \neq q$

This encoding can be done in a polynomial time, since the length of the formula is bounded by the polynomial of $|E|$ and $|Q|$: the number of formulas of type 1) is $O(|Q|)$, because each formula is related to some state. The number of formulas of type 2) is $O(|E|)$, as each of them is related to some edge. The number of formulas of type 3) and 4) is $O(|E|^2)$, since they represent different paths of length 2. So, in total, the number of subformulas is $O(|Q| + |E|^2)$.

We show that Ψ is satisfiable if and only if there exists an abb-coloring of G with the properties enforced so far. Specifically, assignments of Ψ correspond to the colorings of G in the sense that a variable $\mathbf{x}_e$ stands for the proposition $\delta[e] = a$.

First, let δ be an abb-coloring of G. Let ξ be the assignment with $\xi(\mathbf{x}_e) = \mathbf{1}$ if and only if $\delta[e] = a$.

1. The formulas of (1) are satisfied as δ is a coloring.
2. The formulas of (2) hold due to Lemma 6.5 with $m = k = 2$.
3. As for formulas of (3), let $\xi(\mathbf{x}_e) = \mathbf{1}$ and $\xi(\mathbf{x}_f) = \mathbf{0}$. Thus, $\delta[e] = a$, $\delta[f] = b$, and $q \in \delta(Q, ab)$, which contradicts the assumption of no loop on q.
4. As for the formulas of (4), let $\xi(\mathbf{x}_e) = \xi(\mathbf{x}_f) = \mathbf{0}$ and thus $\delta[e] = \delta[f] = b$. According to Lemma 6.3 $\mathrm{s}(e) \in \delta\big(ab^d\big)$ for $d \leq 2$. If $d \leq 1$, we get $\mathrm{t}(f) \in \delta(Q, abb)$ or $\mathrm{t}(e) \in \delta(Q, abb)$, which contradicts with the abb-coloring of δ. If $d = 2$ we get $\mathrm{s}(e) \in \delta(Q, abb)$ and thus $\mathrm{s}(e) = q$. But we have enforced that $q \in \delta(Q, a)$, thus $\mathrm{t}(f) \in \delta(Q, abb)$, which is again a contradiction.

Second, let ξ be a satisfying assignment of Ψ. Let $\delta[e] = a$ if and only if $\xi(\mathbf{x}_e) = \mathbf{1}$. Choose $s \in Q$ and denote $t = \delta(s, ab)$. Let e be the incoming edge of t with $\delta[e] = b$.

1. Suppose that $t \in V_1(q)$. If both outgoing edges of t lead to q , we have $\delta(t, b) = q$. If t has an outgoing edge f not leading to q, according to the fact that the formulas (4) are satisfied we get $\delta[f] = a$. As t has necessarily an outgoing edge leading to q, that edge is colored by b and we get $\delta(t, b) = q$.
2. If $t \in V_2(q)$, the fixed edge e with $\delta[e] = b$ incomes to $V_2(q)$, which contradicts the fact that the formulas (2) are satisfied.
3. If $t = q$, we have $\delta[e] = a$ and $\delta[f] = b$ for f with $\mathrm{t}[f] = q$, $\xi(\mathbf{x}_e) = \mathbf{1}$ *and* $\xi(\mathbf{x}_f) = \mathbf{0}$. This contradicts the assumption that the formulas (3) are satisfied.

□

Theorem 6.8. *For each* $w \in \mathcal{P}_5$ *(i.e.* $w = a^k b^l$ *with* $k, l \geq 2$*)* $\mathrm{SRCW}^{\mathcal{SC}}_{2,w} \in \mathrm{NPC}$.

Proof. We perform a reduction from $\mathrm{SAT}(R_1)$ by constructing a strongly connected digraph $\mathbf{G}_4(w, X, \Phi)$ that includes a structure similar to the digraph $\mathbf{G}_1(w, X, \Phi)$ from the proof of Theorem 4.4.

Specifically, vertices of $\mathbf{G}_4(w, X, \Phi) = (Q, E)$ are:

1. $\mathrm{C}_j, \mathrm{C}'_j, \mathrm{C}''_j$ for each $1 \leq j \leq m$ and $\mathrm{V}_i, \mathrm{V}'_i$ for each $1 \leq i \leq n$ (by $Q_1 \subseteq Q$ we denote the set of these vertices and we set $\alpha = 3m + 2n = |Q_1|$),
2. $\mathrm{D}_0, \ldots, \mathrm{D}_{l-2}$, $\mathrm{E}_0, \ldots, \mathrm{E}_{l-1}$, B,

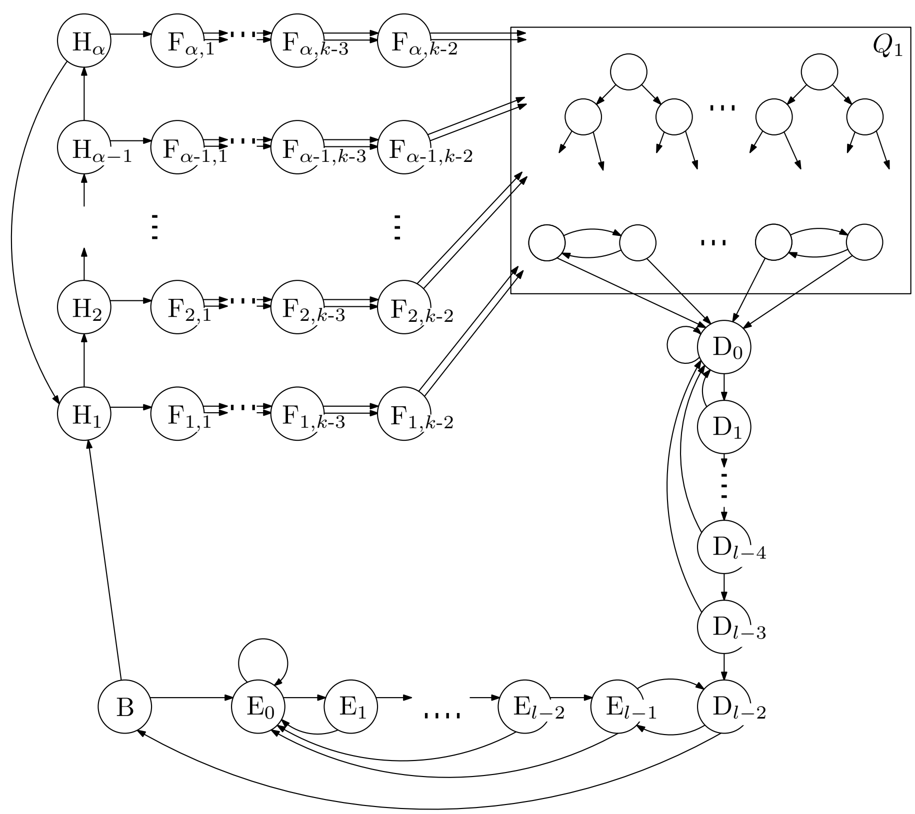

Figure 6: The digraph $\mathbf{G}_4(w, X, \Phi)$

3. H_i, $\mathrm{F}_{i,j}$ for each $1 \leq i \leq \alpha$ and $1 \leq j \leq k-2$. From each $\mathrm{F}_{i,k-2}$, both outgoing edges lead to t_i, where $t_1, \dots, t_\alpha$ is a fixed enumeration of all vertices from Q_1.

The edges are defined in Fig. 6. For small values of k, l, some of the depicted vertices and edges are not present: If $l = 2$, we have $\mathrm{D}_0 = \mathrm{D}_{l-2}$ and the outgoing edges of this vertex lead to $\mathrm{E}_{l-1}, \mathrm{B}$. If $k = 2$, no $\mathrm{F}_{i,j}$ exists and each H_i has an outgoing edge leading directly to t_i.

First, suppose that $\mathbf{G}_4(w, X, \Phi)$ has a w-coloring δ. The following are necessary properties of any such coloring:

1. It holds that $\delta(Q, w) = \{\mathrm{D}_{l-2}\}$. Indeed:
   - From $\mathrm{F}_{1,1}$, only vertices from Q_1, $\{\mathrm{D}_0, \dots, \mathrm{D}_{l-2}\}$and $\{\mathrm{E}_{l-1}\}$ are possibly reachable by paths of length $(k+l)$.
   - From E_0, no vertex from Q_1 nor $\{\mathrm{D}_0, \dots, \mathrm{D}_{l-3}\}$ is reachable by such paths.
   - From H_1, E_{l-1} is not reachable by such paths.
2. Each H_i for $1 \leq i \leq \alpha$ has $\delta(\mathrm{H}_i, a) = \mathrm{F}_{i,1}$. Indeed, in the opposite case, Lemma 2.6 requires $\mathrm{H}_{(i+1) \bmod \alpha} \in V_{\leq k+l-1}(q)$, which is false.
3. It follows that $Q_1 \subseteq \delta\big(Q, a^{k-1}\big)$ and thus $\delta\big(Q_1, ab^l\big) = \{\mathrm{D}_{l-2}\}$.

As in the proof of Theorem 4.4, there is no loop on D_{l-2}, thus $\delta\big(\mathrm{D}_0, b^{l-1}\big) \neq \mathrm{D}_{l-2}$ and $\mathrm{D}_0 \notin \delta(ab)$. The situation within Q_1 is similar to the one from the proof of Theorem 4.4 and we can use the construction (1) of an assignment ξ and employ literally the same arguments to show that ξ satisfies Φ.

Second, suppose that ξ is a satisfying assignment of Φ. Then, a w-coloring δ_ξ is constructed as follows:

1. The edges outgoing from Q_1 are colored according to (2)...(6) as in the proof of Theorem 4.4.
2. The other edges are colored according to Fig. 7.

Observe that $\delta\big(Q, a^{k-1}\big) \subseteq Q_1 \cup \{\mathrm{D}_0, \mathrm{B}, \mathrm{E}_0\}$, while obviously $\delta\big(\{\mathrm{D}_0, \mathrm{B}, \mathrm{E}_0\}, ab^l\big) = \{\mathrm{D}_{l-2}\}$ and for each $s \in Q_1$ we have $\delta(s, a) = \mathrm{D}_0$ or $\delta(s, abb) = \mathrm{D}_0$ according to the final part of proof of Theorem 4.4. It remains to observe that $\delta\big(\mathrm{D}_0, b^l\big) = \mathrm{D}_{l-2}$ and $\delta\big(\mathrm{D}_0, b^{l-2}\big) = \mathrm{D}_{l-2}$.

□

Theorem 6.9. *For each $w \in \mathcal{P}_6$, $\mathrm{SRCW}^{\mathcal{SC}}_{2,w} \in \mathrm{NPC}$.*

Proof. For each $w \in \mathcal{P}_6$ we perform a reduction from the NP-complete problem $\mathrm{SRCW}^{\mathcal{Z}}_{2,w}$ (see Corollary 5.5) to $\mathrm{SRCW}^{\mathcal{SC}}_{2,w}$. As usual, we suppose that w starts with a. In certain parts of the following general construction we distinguish two cases:

(A) w ends with a,
(B) w ends with b.

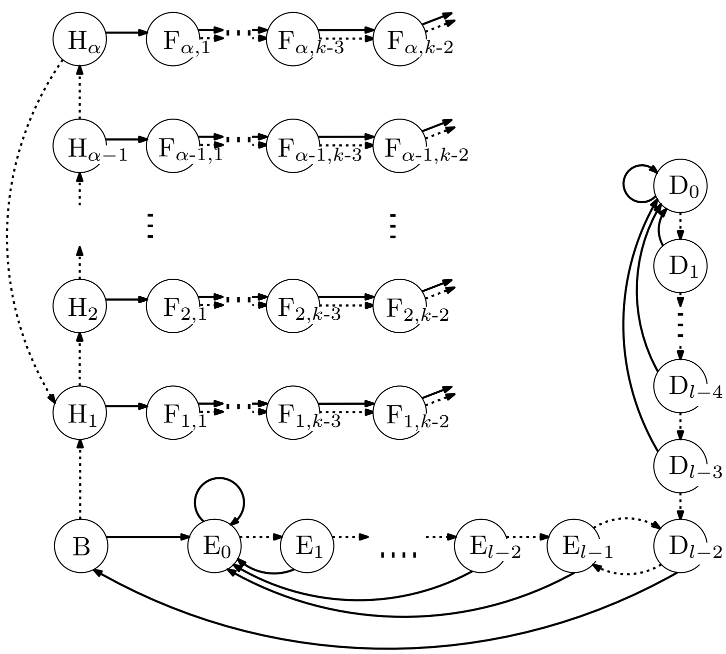

Figure 7: A part of a coloring of $\mathbf{G}_4(w, X, \Phi)$

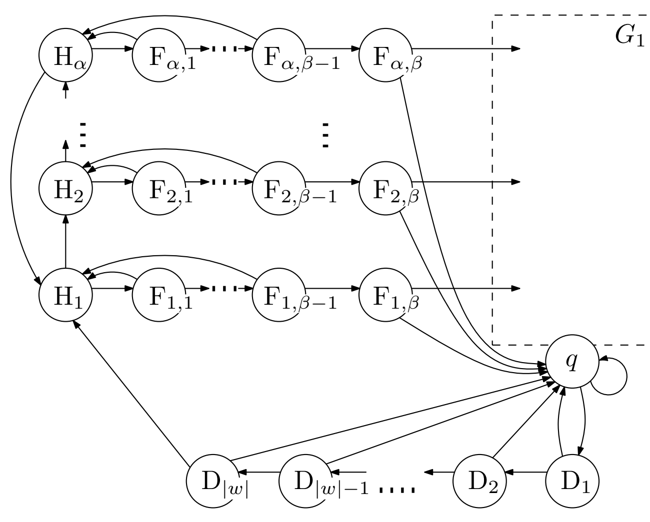

Figure 8: The digraph $\mathbf{G}_5(w, G_1)$, variant (A)

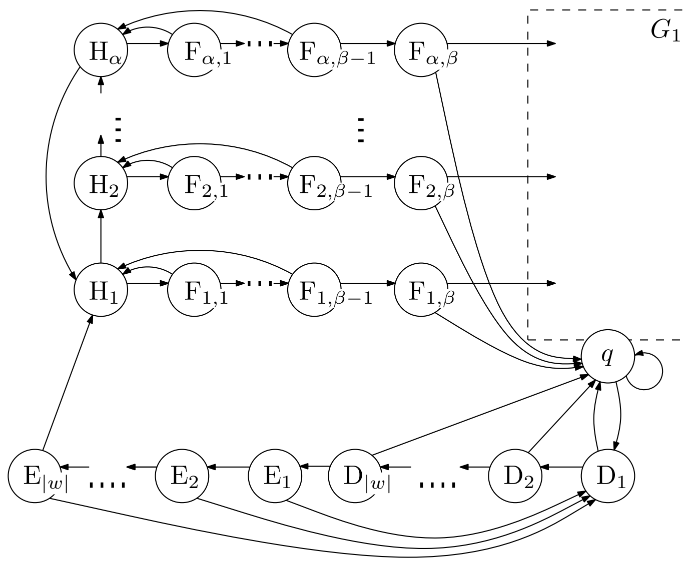

Figure 9: The digraph $\mathbf{G}_5(w, G_1)$, variant (B)

Let $G_1 = (Q_1, E_1)$ be an instance of $\mathrm{SRCW}^{\mathcal{Z}}_{2,w}$ with a sink vertex $q \in Q_1$. Denote $\alpha = |Q_1|$ and fix the unique $\gamma, \beta \geq 1$ such that $a^\gamma b^\beta a$ is a prefix of w. As $w \in \mathcal{P}_6$, the existence of γ and β is guaranteed. We build a digraph $\mathbf{G}_5(w, G_1) = (Q, E)$ by removing one of the loops on q in G_1 and adding certain vertices and edges to the resulting digraph. The resulting structure of $\mathbf{G}_5(w, G_1)$ depends on which of the cases (A), (B) is met by w.

1. In any case we add vertices H_i, $\mathrm{F}_{i,j}$ for each $1 \leq i \leq \alpha$ and $1 \leq j \leq \beta$.
2. In the case (A) we add vertices $\mathrm{D}_1, \ldots, \mathrm{D}_{|w|}$.
3. In the case (B) we add the vertices $\mathrm{D}_1, \ldots, \mathrm{D}_{|w|}$ and $\mathrm{E}_1, \ldots, \mathrm{E}_{|w|}$.

See Figures 8 and 9.

First, assume that $\mathbf{G}_5(w, G_1)$ has a w-coloring δ. Except for loops on q, each edge of E_1 is present in E, so δ induces a coloring δ_1 of G_1. In $\mathbf{G}_5(w, G_1)$, only the vertices q, $\mathrm{D}_1, \ldots, \mathrm{D}_{|w|}$ are $|w|$-reachable from q. Thus $\delta(Q, w) = \{r\}$, where r is one of these vertices. From the structure of $\mathbf{G}_5(w, G_1)$ it follows easily that for each $s \in Q_1$ there is a prefix w' of w such that $\delta(s, w') = q$. This implies that $\delta_1(Q, w) = \{q\}$, i.e. δ_1 is a w-coloring of G_1.

Second, assume that G_1 has a w-coloring δ_1. We extend δ_1 to a coloring δ of $\mathbf{G}_5(w, G_1)$ as follows:

- The edges outgoing from H_i, $\mathrm{F}_{i,j}$, $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$ are colored according to Fig. 10.
- The edges outgoing from $\mathrm{D}_1, \ldots, \mathrm{D}_{|w|}$ (and from $\mathrm{E}_1, \ldots, \mathrm{E}_{|w|}$ if used) are colored according to Figures 11 and 12.

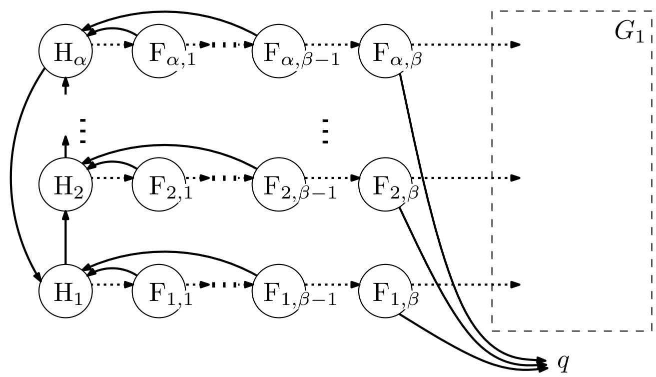

Figure 10: A coloring of a part of $\mathbf{G}_5(w, G_1)$

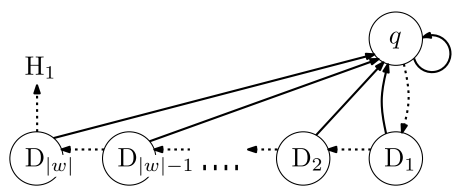

Figure 11: A coloring of a part of $\mathbf{G}_5(w, G_1)$ in the case (A)

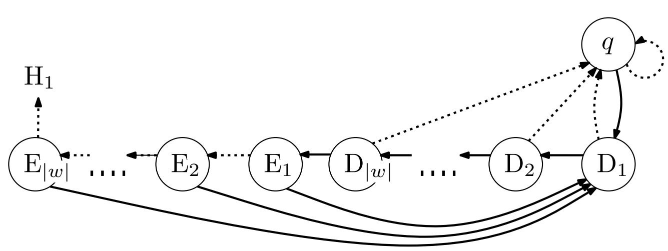

Figure 12: A coloring of a part of $\mathbf{G}_5(w, G_1)$ in the case (B)

Observe that in the case (A) $\delta(q, u) = q$ for any u ending with a and in the case (B) $\delta(q, u) = q$ for any uv ending with b. Together, for any suffix u of w we have $\delta(q, u) = q$.

Finally, choose any $s \in Q$.

- If $s \in Q_1$, there is the shortest prefix v of w with $\delta(s, v) = q$, so we can easily use the above observation.
- If $s = \mathrm{D}_i$ for $1 < i < |w|$ and (A) is the case, we get $\delta(s, a) = q$ and use the above observation.
- If $s = \mathrm{D}_i$ or $s = \mathrm{E}_i$ for $1 < i < |w|$ and (B) is the case, we get $\delta(s, a^\gamma b) \in \{q, \mathrm{E}_2\}$ and $\delta\big(s, a^\gamma b^\beta a\big) = \mathrm{D}_1$. As w ends by b, we easily see that $\delta(s, v) = \mathrm{D}_0$ for a prefix v of w, and conclude using the above observation.
- If $s = \mathrm{H}_i$ or $s = \mathrm{F}_{i,j}$ for $1 \le i \le \alpha$, $1 \le j \le \beta$, we see that $\delta\big(s, a^\gamma b^\beta a\big) = q$ and use the above observation.

□

Conclusion

In this paper we investigated the complexity issues for a road-coloring version of a synchronizing problem parameterized by a given word: "Given an admissible graph and a word $w \in I^\star$, is there a coloring such that w is a reset word of the resulting automaton?". Notice that the "classical" version of this problem: "Given an automaton $\mathcal{A}$ and a word $w \in I^*$ check if w is a synchronizing word for $\mathcal{A}$" is obviously in P, as we just need to find whether $\delta(Q, w)$ is a singleton.

In a road-coloring version of this problem things look different. For some words the problem remains in P, but for a broad class of words it become NP-complete. This means that the road-coloring versions are usually much harder than their original "automata" versions. The other interesting thing is that the complexity may depend on whether we deal with strongly connected digraphs or not (see Theorems 4.4 and 6.7).

We were not able to find the complexity in one case ($w \in \mathcal{P}_4$ for strongly connected digraphs), so the problem remains open. Nevertheless, the results presented in this paper show that the world of the road-coloring types of problems is much more complicated than in the case of classical synchronization of finite automata and full of unexpected, surprising results.

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