Subset Synchronization of Transitive Automata

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Outline

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Finite Automata

- *DFA* is a triple $A = (Q, \Sigma, \delta)$
  - Q ... finite set of *states*
  - Σ ... finite set of *letters* (the *alphabet*)
  - δ ... total function $Q \times \Sigma \to Q$ (*transition function*)
- Extended transition function:

$$\delta : 2^Q \times \Sigma^\star \to 2^Q$$

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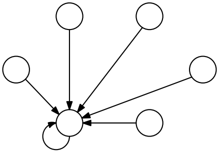
1. Synchronization of a DFA
   - DFA
   - Classical Synchronization
   - Subset Synchronization & The Result
2. Depth of Transformations
3. Proof Methods

Reset Words

- $w \in \Sigma^\star$ is a *reset word* of A if

$$|\delta(Q, w)| = 1,$$

i.e. if w acts like



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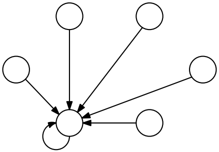

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Shortest Reset Words

- **Černý conjecture:**
  - If A is synchronizing, it has a reset word of length at most $(|Q|-1)^2$
- Known upper bounds:
  - $\frac{1}{3}|Q|^3 - n^2 + \frac{5}{3}n - 1$ (Kohavi, 1970)
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Subset Reset Words: an Example

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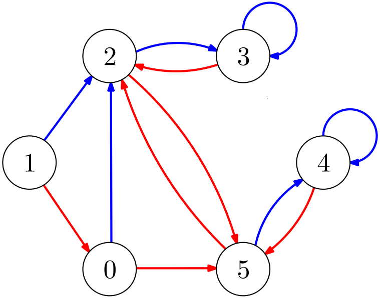

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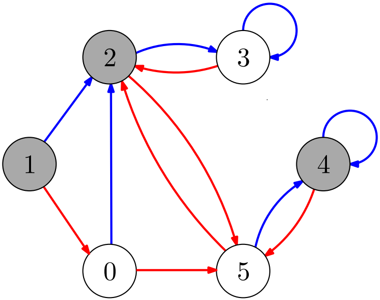

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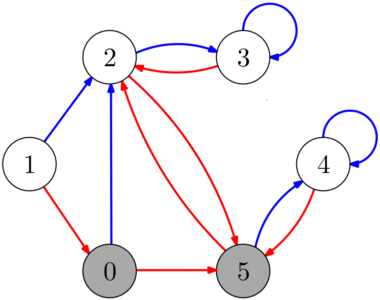

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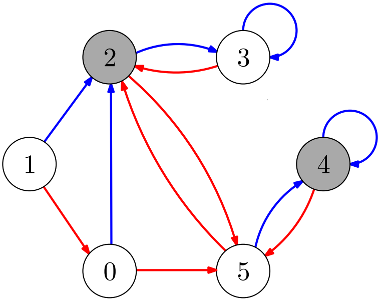

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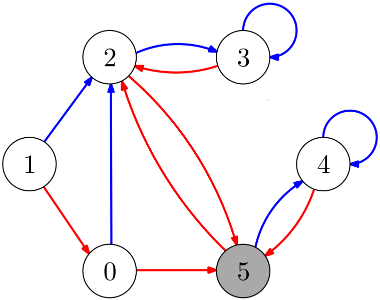

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Bounds for Shortest Reset Words

	Classical Synchronization
Upper Bounds	$\mathcal{O}\left(\lvert Q\rvert^3\right)$
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Lower Bound Construction

Infinite series of DFA satisfying:

- $|Q|$ grows, $|\Sigma| = 2$
- There is always a subset $S \subseteq Q$ with a shortest reset word of length $2^{\Omega(|Q|)}$.
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Depth of Transformations

- Full Transformation Monoid $\mathcal{T}_n$
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Infinite series of sets $\mathbf{G} \subseteq \mathcal{T}_n$ satisfying:

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Using bad cases of subset synchronization:

$$\text{DFA } A = ([n], \Sigma, \delta) \quad \longrightarrow \quad \mathbf{G} \subseteq \mathcal{T}_n$$

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A subset $S \subseteq Q$ is ***unstable*** if:

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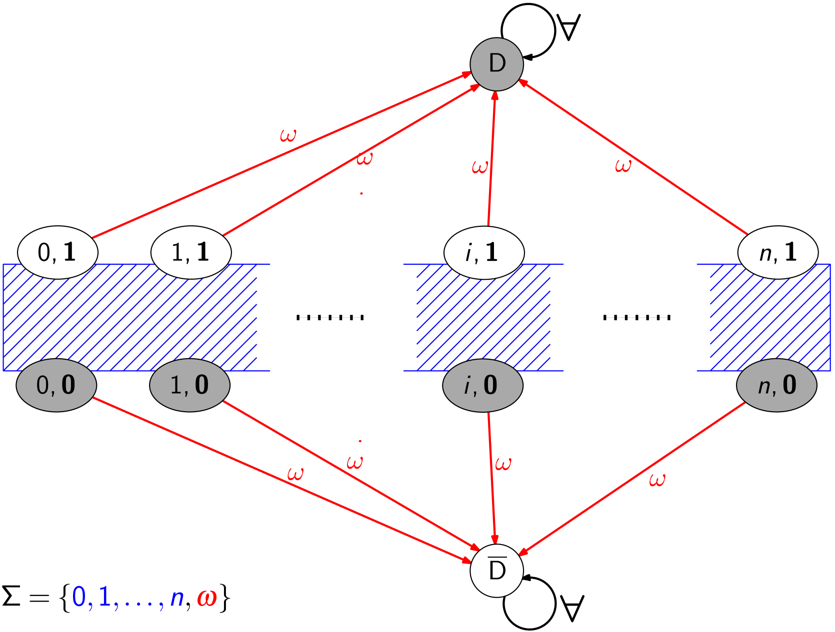

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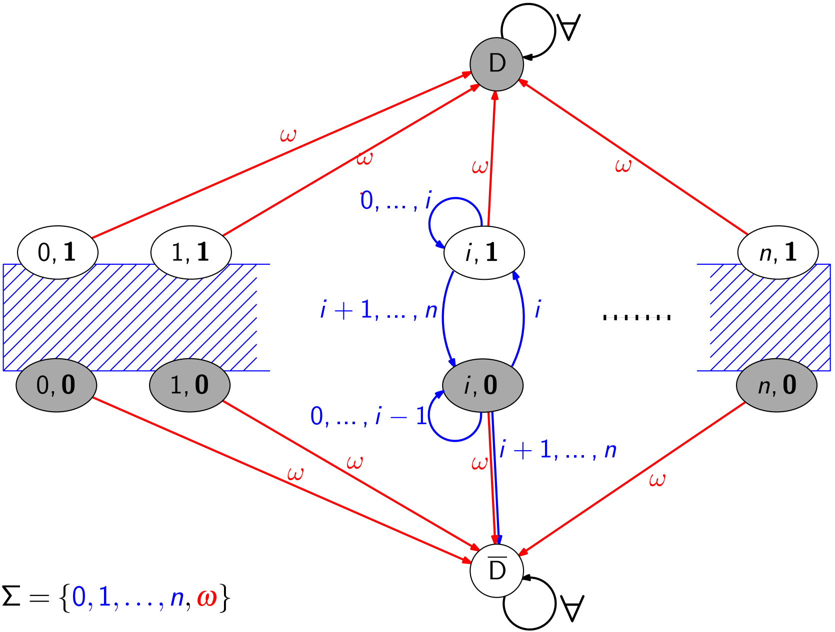

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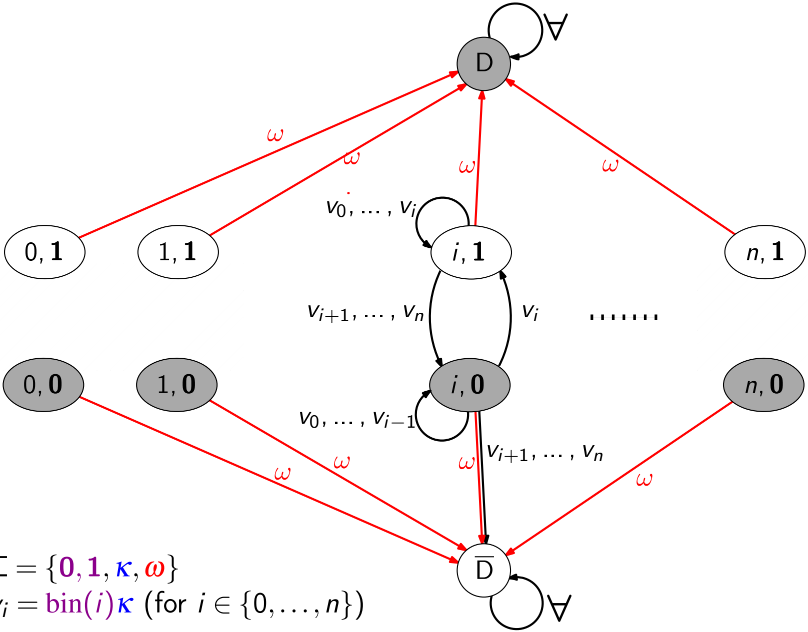

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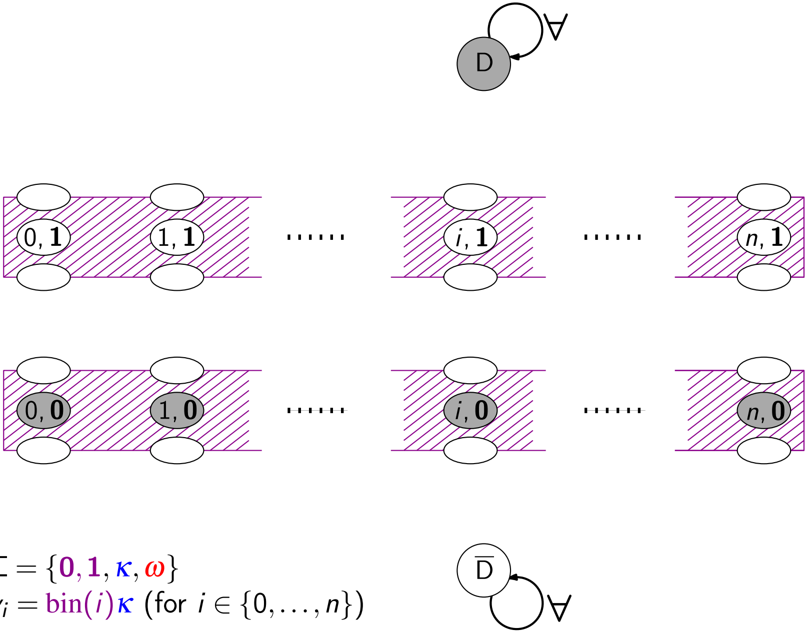

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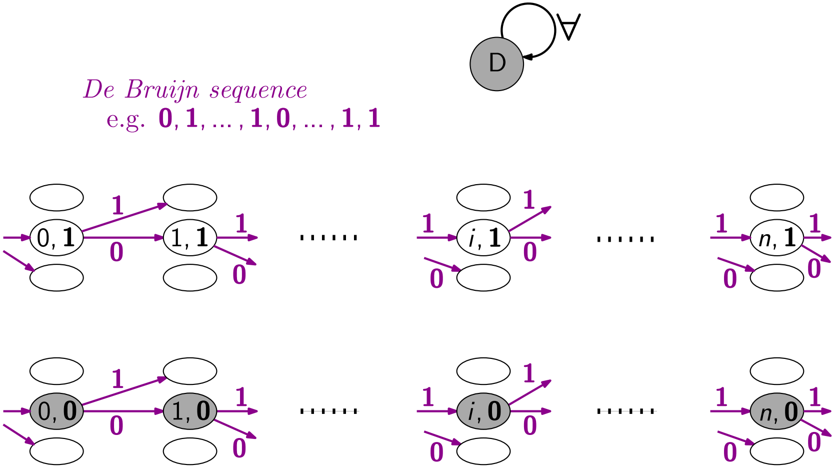

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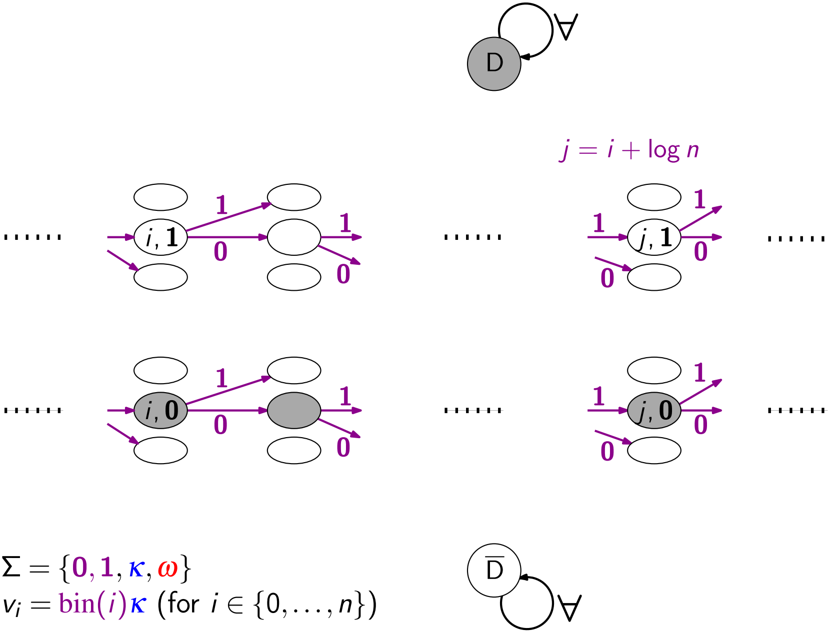

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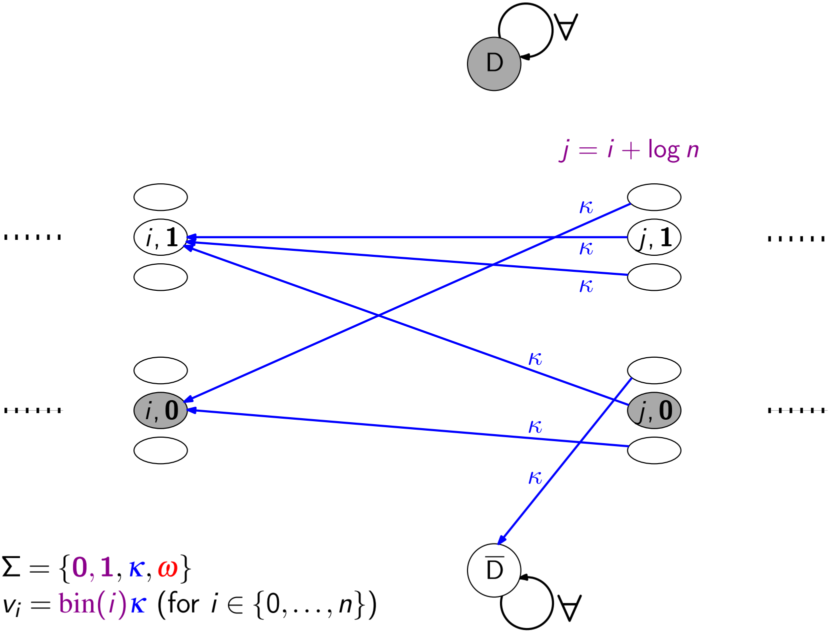

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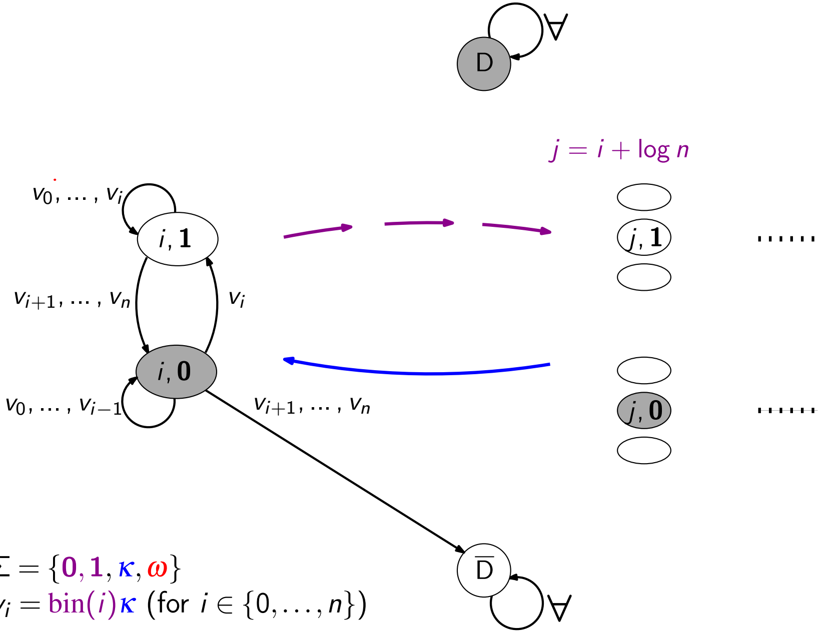

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Conclusion

- Some subsets require strongly exponential reset words even in transitive DFA with two-letter alphabets.
- Some transformations have strongly exponential depth even with respect to two generators.