Complexity of Road Coloring with Prescribed Reset Words

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Deterministic Finite Automata

- *DFA:* a triple $A = (Q, \Sigma, \delta)$
  - Q ... finite set of *states*
  - Σ ... finite set of *letters* (the *alphabet*)
  - δ ... total function $Q \times \Sigma \to Q$ (*transition function*)

Reset Words

- $w \in \Sigma^\star$ is a *reset word* of a DFA if

$$\delta(s_1, w) = \delta(s_2, w)$$

  for each $s_1, s_2 \in Q$.

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Road Coloring

directed multigraph with constant out-degree k

$\downarrow$

DFA with $|\Sigma| = k$

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synchronizing
DFA with $|\Sigma| = k$

Road Coloring

road colorable

directed multigraph with constant out-degree k

$\downarrow$

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DFA with $|\Sigma| = k$

Admissible Graphs

FACT.
Each road colorable graph is aperiodic.

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k-admissible $\equiv$ aperiodic and
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Decision Problems

1. Given an $|\Sigma|$-admissible graph G,
   is there a synchronizing coloring δ of G?
2. Given an $|\Sigma|$-admissible graph G and **an integer** $k \geq 1$,
   is there a coloring δ of G with a reset word of length $\leq k$?
3. Given an $|\Sigma|$-admissible graph G and **a word** $w \in \Sigma^\star$,
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Problem No. 2 (SRCP)

2. Given an $|\Sigma|$-admissible graph G and **an integer** $k \geq 1$, is there a coloring δ of G with a reset word of length $\leq k$?

	$k = 2$	$k = 3$	$k = 4, 5, \ldots$
$\|\Sigma\| = 2$	P		
$\|\Sigma\| = 3$	P		NPC
$\|\Sigma\| = 4, 5, \ldots$	P		NPC

Adam Roman, LATA 2012

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Roman and Drewienkowski, 2014

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Vorel and Roman, 2014

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3. Given a **strongly connected** $|\Sigma|$-adm. graph G and $w \in \Sigma^\star$, is there a coloring δ of G with reset word w?

	$\|\Sigma\| = 2$	
	general	S. C.
$w = a^k$ $(k \geq 1)$	P	
$w = a^k b$ $(k \geq 1)$	P	
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$w = abb$	**NPC**	**P**
$w = ava$ $(v \notin \{a\}^\star)$	NPC	NPC
otherwise	NPC	?

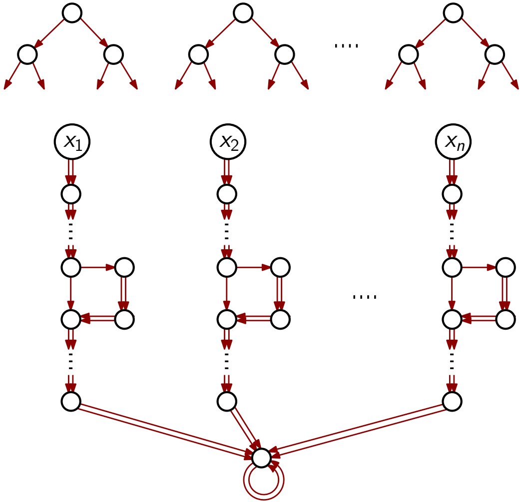
A Special Variant of SAT

Input: A formula in CNF, clauses in triples of the form

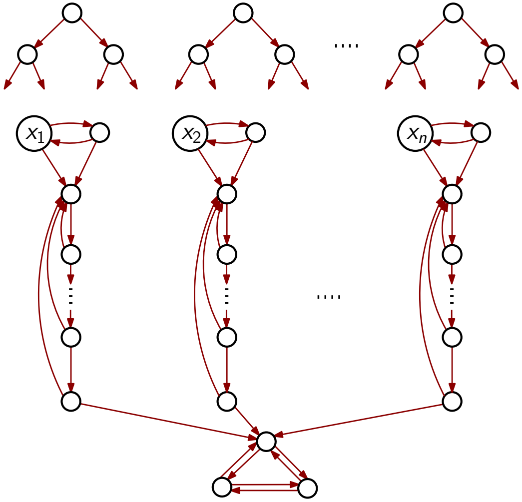
$$(x_1 \vee x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2) \wedge (\neg x_3 \vee \neg x_4).$$

Output: Satisfiable?

NP-Completeness of SRCW with $w \in a^{\geq 1} b^{\geq 1} a^{\geq 1}$



NP-Completeness of SRCW with $w \in a^{\geq 1} b^{\geq 2}$



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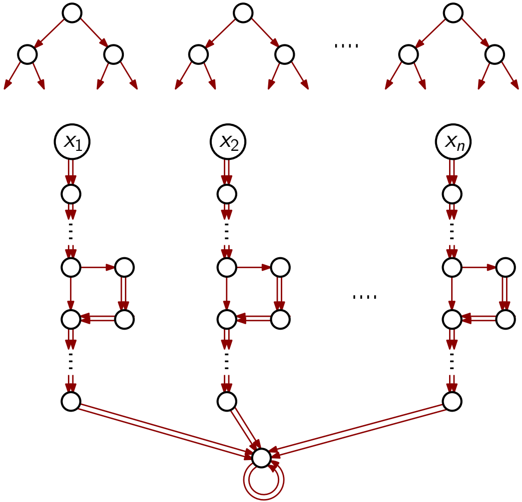

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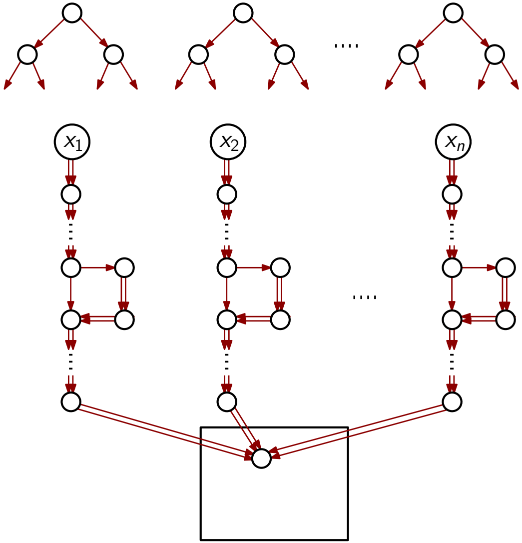

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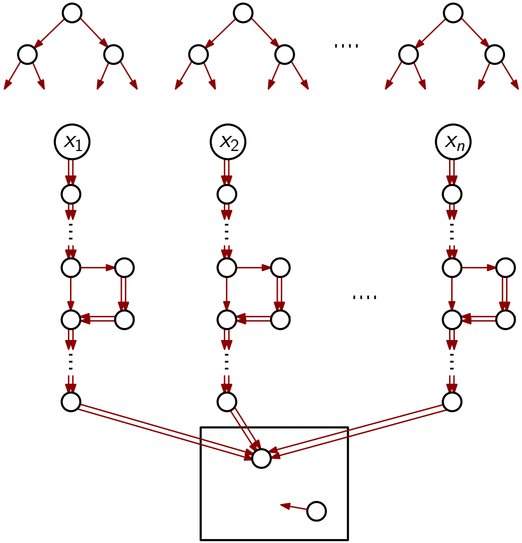

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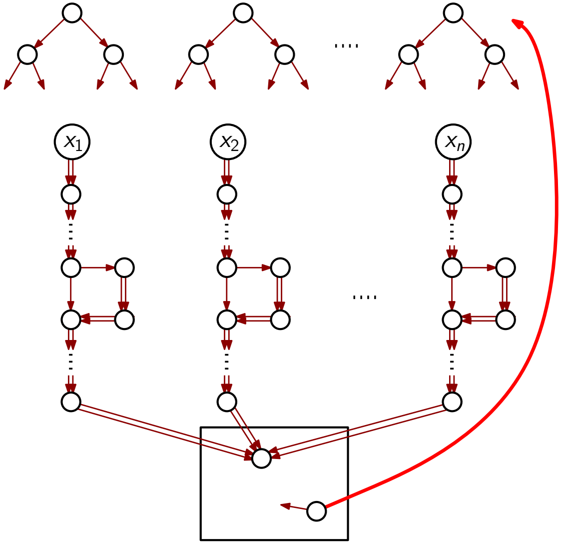

Conclusion

	$\lvert\Sigma\rvert = 2$		$\lvert\Sigma\rvert = 3, 4, \ldots$	
	general	S. C.	general	S.C.
$w = a^k$ $(k \geq 1)$	P	P	P	P
$w = a^k b$ $(k \geq 1)$	P	P	P	P
$w = abb$	NPC	P	?	?
$w = ava$ $(v \notin \{a\}^\star)$	NPC	NPC		
$w \neq a^k b^l$ with a S.C. incomplete sink device	NPC	NPC		
otherwise	NPC	?		

Which binary words have strongly connected incomplete sink devices?