

Implementation of algorithms and data structures

9. seminar

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- (V, E) is a directed graph on n vertices and m edges
- $c : E \rightarrow \mathbb{R}_0^+$ is capacity of edges
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- Capacity constraint: $0 \leq f(e) \leq c(e)$ for every edge e
- Kirchhoff law: $\sum_{u:uv \in E} f(uv) = \sum_{u:vu \in E} f(vu)$ for every vertex v except s, t

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Overflow of a vertex v is $f^\Delta(v) = \sum_{u:uv \in E} f(uv) - \sum_{u:vu \in E} f(vu)$

Maximum flow in a network

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Terminology

- Reserve (residual) of an edge uv is $r(uv) = c(uv) - f(uv) + f(vu)$
- Edge uv is saturated if $r(uv) = 0$

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- Improved version: From all vertices with positive overflow, choose the highest one.

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$$2 \quad f(uv) = \begin{cases} c(uv) & \text{for } u = s \\ 0 & \text{otherwise} \end{cases}$$

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4   if exists an edge  $uv$  satisfying  $r(uv) > 0$  and  $h(u) > h(v)$  then
5     | transfer overflow on edge  $uv$ 
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- A list P of vertices with positive overflow which can find a highest one

1st option: Pointers to opposite edges

- Structure for an edge
 - Destination vertex
 - Capacity
 - Flow
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- Every vertex has a list of edges
- Disadvantage: there are two instances of the edge structure for every edge

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2nd option: Shared edge structure for both directions

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3rd option: Hash table

- Every vertex has a list of incident edges
- Hash table: (vertex,vertex) \rightarrow edge data (i.e. capacity, flow)
- Disadvantage: We no longer have worst-case complexity but expected

List of non-saturated downhill edges $L(u)$

Representation

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Trivial operations in $O(1)$ -time

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Removing the opposite edge vu from $L(v)$ when $h(u)$ is increased

Problem: find the position of vu in the list $L(v)$

- Intrusive list can find and delete in $O(1)$ -time
- Lazy solution: Delete vu from $L(v)$ when the vertex v is processed

What we need?

- Find an arbitrary vertex of P with the largest height
- Remove the vertex from P after transferring whole overflow
- Increase height of a vertex of P by one
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Approach

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- Store every group in a special list
- Access groups using a main list/array indexed by the height
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2nd option: Array indexed by the height with index to the highest non-empty group

After removing the last vertex from the highest group, the index has to be updated which cannot be done in $O(1)$ -time

- Index is always increased by one
- Total number of increments is at most $2n^2$
- If the index is decremented by ones, total number of decrements is at most $2n^2$

Public functions creating instance and obtaining results

- Add a vertex
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- Run the algorithm
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How to identify an edge

- Edges can be indexed from 1 to m (impractical)
- Only iterate edges incident with a given vertex
- Internally use a hash table (vertex,vertex) \rightarrow edge

Example of libraries

- Networkx in Python
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API is only one function

- Argument: A graph with capacity as an edge property
- Return value: The size of a maximum flow
- The function sets a maximum flow as an edge property

Testing correctness of a solution

- Capacity constraint
- Kirchhoff law
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Testing graphs

How to choose a graph?

- Small graphs, examples from literature
- Complete graphs, path, cycles, ...
- Random graphs
- Adversary graphs
 - Disconnected graphs, isolated vertices and edges
 - Graphs having dead branches accessible from the source
 - Find graphs on which the algorithm is slowest

How to choose capacities

- Unit capacity
- Regular, e.g. from 1 to m
- Random, e.g. uniform or normal distribution

- Correct structure of a graph (depends on representation)
- Check overflows on all vertices
- Check that f is a wave
 - Capacity constraint: $0 \leq f(e) \leq c(e)$ for every edge e
 - Non-negative overflows: $f^\Delta(v) \geq 0$ for every vertex v except the source
- Every vertex v satisfies $0 \leq h(v) \leq 2n$
 - except $h(z) = n$ and $h(s) = 0$
- Lists P and $L(u)$ contains only the expected vertices
- Every edge uv with $r(uv) > 0$ satisfies $h(u) - h(v) \leq 1$
- There exists a non-saturated from every vertex with positive overflow to the source
- Calculate the number of operations and potentials

General approach

- Design data representation and API and implement them
- Implement tests
- Implement algorithm
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- 2nd step: Implement P
- 3rd step: Implement $L(u)$

Implement the algorithm step by step

General approach

- Design data representation and API and implement them
- Implement tests
- Implement algorithm
- Test and debug

Split implementation into peaces

- 1st step: Correct but slow version
 - Skip lists P and $L(U)$
 - Functions using P and $L(U)$ are implemented trivially
 - Debug graph representation, main parts of the algorithm and all tests
- 2nd step: Implement P
- 3rd step: Implement $L(u)$

Discussion

- Can the first step be simplified?
- Can the implementation be split into more testable steps?

Reason: Testing the tests

- Tests often contains bugs reporting non-existing bugs as well as missing bugs
- Running tests on incompletely (improperly) implemented functions verifies correctness of (some) tests
- No needs to write even more code, just run tests when implementing and check that tests fails as expected for the current code

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- Similarly for lists of non-saturated downhill edges
- Implement increasing heights before transferring overflows
⇒ The program should loop forever but tests checking the height upper bound should fail

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Implementation questions

- Develop unit and fuzz tests
- Based on our analysis, develop as many data consistency tests as possible
- Find data representation so that whole algorithm has complexity $O(n^2m)$
- How to find a highest vertex with positive overflow to improve the complexity to $O(n^2\sqrt{m})$?

The first assignment: Goldberg's algorithm

- Study and understand the algorithm including analysis and complexity
- Write data representation such that all operations has expected complexity
- Write tests
- Write API