Long cycles in hypercubes with optimal number of faulty vertices

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Abstract

Let $f(n)$ be the maximum integer such that for every set F of at most $f(n)$ vertices of the hypercube Q_n, there exists a cycle of length at least $2^n - 2|F|$ in $Q_n - F$. Castañeda and Gotchev conjectured that $f(n) = \binom{n}{2} - 2$. We prove this conjecture. We also prove that for every set F of at most $(n^2 + n - 4)/4$ vertices of Q_n, there exists a path of length at least $2^n - 2|F| - 2$ in $Q_n - F$ between any two vertices such that each of them has at most 3 neighbors in F. We introduce a new technique of potentials which could be of independent interest.

1 Introduction

The n-dimensional hypercube Q_n is the (bipartite) graph with all binary vectors of length n as vertices and edges joining every two vertices that differ in exactly one coordinate. The bipartite classes of Q_n consist of vertices with even, respectively odd, weight, where the *weight* $|u|$ of a vertex $u \in V(Q_n) = \{0,1\}^n$ is defined as the number of 1's in u. A set $F \subseteq V(Q_n)$ in which all vertices are from the same bipartite class, is called a *monopartite* set.

Applications of the hypercube in the theory of interconnection networks inspired many questions related to its robustness. In particular, if some faulty (or busy) vertices $F \subseteq V(Q_n)$ and all incident edges are removed from Q_n, is there a cycle in the remaining graph, denoted by $Q_n - F$, which covers 'almost' all vertices? And how many vertices in the worst-case can be removed?

Clearly, if F is monopartite, the length of any cycle in $Q_n - F$ cannot exceed $2^n - 2|F|$. This leads to the following definition. A cycle of length at least $2^n - 2|F|$ in $Q_n - F$ is called a *long F-free cycle in Q_n*. Let $f(n)$ be the maximum integer such that $Q_n - F$ has a long F-free cycle for every set F of at most $f(n)$ vertices in Q_n.

The study of this parameter has a numerous literature. Firstly, Chan and Lee [2] showed that $f(n) \geq (n-1)/2$. Then, Yang et al. [15] improved it to $f(n) \geq n-2$, and Tseng et al. [13] to $f(n) \geq n-1$. Next, Fu [7] significantly increased it to $f(n) \geq 2n-4$ for $n \geq 3$, and

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Castañeda and Gotchev [1] strengthened it further to $f(n) \geq 3n-7$ for $n \geq 5$. Recently, Fink and Gregor [6] obtained the first quadratic lower bound $f(n) \geq n^2/10 + n/2 + 1$ for $n \geq 15$.

On the other hand, Koubek [10] and independently Castañeda and Gotchev [1] noticed that for every $n \geq 4$ there is a set F of $\binom{n}{2} - 1$ vertices such that $Q_n - F$ contains no cycle of length at least $2^n - 2|F|$, so $f(n) \leq \binom{n}{2} - 2$. An example of a such set F consists of all but one vertex of weight 2. Indeed, since all vertices of F have even weight, any long F-free cycle in Q_n must visit all the remaining vertices of even weight. Namely, it has to visit the vertex $\mathbf{0} = (0, \ldots, 0)$ and some vertex of weight 4, which is clearly impossible as they are in different 2-connected components of $Q_n - F$.

From the previous results it follows that the above upper bound is sharp for $n = 4$ [7] and for $n = 5$ [1]. It was conjectured [1] that it is sharp for all $n \geq 4$, i.e. $f(n) = \binom{n}{2} - 2$ for $n \geq 4$. In this paper we prove this conjecture.

Theorem 1.1. *For every set F of at most $\binom{n}{2} - 2$ vertices in Q_n and $n \geq 4$, the graph $Q_n - F$ contains a cycle of length at least $2^n - 2|F|$.*

To prove Theorem 1.1, we need to consider a modification of this problem for long paths with prescribed endvertices. Similarly as above, a path in $Q_n - F$ between vertices u and v, and of length at least $2^n - 2|F| - 2$ is called a *long F-free uv-path in Q_n*. Note that in case u and v are from different bipartite classes, the length of any long F-free uv-path is at least $2^n - 2|F| - 1$. Also note that in the case when $F \cup \{u, v\}$ is monopartite, the length of any uv-path in $Q_n - F$ cannot exceed $2^n - 2|F| - 2$, and hence a long F-free uv-path has optimal length.

Fu [8] showed that $Q_n - F$ contains a long path between any two vertices if $|F| \leq n-2$ and $n \geq 3$. To improve this result for larger sets F, one needs to introduce additional conditions on the neighbors of prescribed endvertices. Kueng et al. [11] strengthened the number of tolerable faults to $|F| \leq 2n - 5$ under the condition that the minimal degree of $Q_n - F$ is at least 2. Recently, Fink and Gregor [6] showed that a much weaker condition is both necessary and sufficient for sets F with $|F| \leq 2n - 4$. Let $N(u)$ be the set of neighbors of a vertex u in Q_n.

Theorem 1.2 (Fink and Gregor [6])**.** *Let F be a set of at most $2n - 4$ faulty vertices of Q_n and $n \geq 5$. For every two vertices u and v of $Q_n - F$, there exists a long F-free uv-path in Q_n if and only if $N(u) \nsubseteq F \cup \{v\}$ and $N(v) \nsubseteq F \cup \{u\}$.*

Note that for $|F| \leq n - 2$, the right side of the equivalence in Theorem 1.2 is always satisfied. Hence, we obtain the following direct corollary.

Corollary 1.3 (Fu [8])**.** *For every set F of at most $n - 2$ vertices of Q_n and $n \geq 2$, there is a long F-free uv-path in Q_n between every two vertices u and v of $Q_n - F$.*

In this paper, we show that F can be as large as $f(n+1)/2$ if both prescribed endvertices have only few neighbors in F.

Theorem 1.4. *For every set F of at most $(n^2+n-4)/4$ vertices in Q_n and $n \geq 5$, the graph $Q_n - F$ contains a path of length at least $2^n - 2|F| - 2$ between every two vertices such that each of them has at most 3 neighbors in F.*

The general difficulty with quadratic bounds on $|F|$ in Theorems 1.1 and 1.4 is that the hypercube cannot be always split into subcubes so that the bounds hold in each subcube.

Thus, the standard induction technique fails. We introduce up to our knowledge a new technique of so called *potentials* which allows us to effectively deal with such situations.

Furthermore, in the proof of Theorem 1.4 we need to consider the following extension of the studied problem for two paths. Assume that we have two different (but not necessarily disjoint) sets $A = \{u, v\}$ and $B = \{x, y\}$ of vertices of $Q_n - F$. A path P between a vertex of A and a vertex of B is called an *AB-path*. Its length $|P|$ is the number of edges in P. A pair P_1, P_2 of vertex-disjoint AB-paths in $Q_n - F$ is called an *F-free AB-routing in Q_n*. Moreover, it is said to be *long* if $|P_1| + |P_2| \geq 2^n - 2|F| - 3$. Note that if A and B are not disjoint, say $A \cap B = \{u = x\}$, then any long F-free AB-routing consists of the uu-path of length 0 and an vy-path of length at least $2^n - 2|F| - 3$.

We studied those problems separately in [5] where we obtained the following results.[1]

Theorem 1.5 ([5])**.** *For every set F of at most $n - 3$ vertices in Q_n and $n \geq 4$, there exists a long F-free AB-routing in Q_n between every two different sets $A, B \subseteq V(Q_n) \setminus F$ such that $|A| = |B| = 2$ and $A \cup B$ is not monopartite.*

As a consequence, if $F \cup \{u, v\}$ is not monopartite, we obtain an uv-path in $Q_n - F$ of length at least $2^n - 2|F| - 1$, which is more than is guaranteed by long paths.

Corollary 1.6 ([5])**.** *For every set F of at most $n - 2$ vertices of Q_n and $n \geq 4$, the graph $Q_n - F$ has an uv-path of length at least $2^n - 2|F| - 1$ for every two vertices $u, v \in V(Q_n) \setminus F$ such that $F \cup \{u, v\}$ is not monopartite.*

From Theorem 1.1 it follows that the decision problem whether the hypercube Q_n for the given set F of faulty vertices contains an F-free cycle has a trivial answer if $|F| \leq \binom{n}{2} - 2$. On the other hand, Dvořák and Koubek [4] showed that this problem is NP-hard if $|F|$ is unbounded. Moreover, they [4] presented a function $\phi(n) = \Theta(n^6)$ such that the problem remains NP-hard even if $|F| \leq \phi(n)$. Furthermore, Dvořák and Koubek [3] described a polynomial algorithm for the similar decision problem of long F-free paths between given vertices in Q_n if $|F| \leq n^2/10 + n/2 + 1$.

For the completeness, let us also mention that there are many related results on similar problems of bipanconnectivity, bipancyclicity, long cycles, and long paths in various modifications of faulty hypercubes, see a survey of Xu and Ma [14] for further references.

2 Preliminaries

The n-dimensional hypercube Q_n is the (bipartite) graph with all binary vectors of length n as vertices and edges joining every two vertices that differ in exactly one coordinate. Let $\mathbf{0}$ denote the vertex of Q_n consisting of all 0's. For every $i \in [n] = \{1, 2, \ldots, n\}$ let e_i denote the vertex with 1 exactly in the i-th coordinate. Furthermore, for every distinct $i, j \in [n]$ let $e_{i,j}$ denote the vertex with 1 exactly in the i-th and j-th coordinate.

Let $d(u, v)$ be the (Hamming) distance of vertices u and v in Q_n, i.e. the number of coordinates where u and v differ. Recall that the *weight* $|u|$ of a vertex u is the number of 1's in u, i.e. $|u| = d(u, \mathbf{0})$. The vertices of even and odd weight, respectively, form bipartite classes of Q_n. The *parity* of a vertex u is the parity of its weight $|u|$. Hence, two vertices have the same parity if and only if they are in the same bipartite class. The *k-th level* of Q_n is the set of vertices of weight k for $0 \leq k \leq n$.

[1] This paper has not been published yet, so we include the proofs for the purpose of referee in the appendix.

Clearly, Q_n has a regular degree n. Let $N(u)$ be the set of neighbors of a vertex u in Q_n, and let $N^+(u)$ and $N^-(u)$ be the sets neighbors of u with weight $|u|+1$ and $|u|-1$, respectively. It is well-known that every two vertices of Q_n have 0 or 2 common neighbors.

In order to apply induction, we need to split the hypercube Q_n into two $(n-1)$-dimensional subcubes $Q_{i:L}$ and $Q_{i:R}$. This is obtained by fixing some coordinate $i \in [n]$. Formally, we define the subcube $Q_{i:L}$ as the subgraph of Q_n induced by vertices that have 0 on the i-th coordinate. Similarly, the subcube $Q_{i:R}$ is the subgraph of Q_n induced by vertices that have 1 on the i-th coordinate. For a vertex x of $Q_{i:L}$, let x_R be the (only) neighbor of x in $Q_{i:R}$. Similarly for a vertex x of $Q_{i:R}$, let x_L be the (only) neighbor of x in $Q_{i:L}$.

Assume that F is a given set of faulty vertices of Q_n. The vertices of Q_n which are not in F are called *F-free*. For every $i \in [n]$ we define $F_{i:L}$ and $F_{i:R}$ to be the sets of faulty vertices in $Q_{i:L}$ and $Q_{i:R}$, respectively. Let F^k be the set of vertices of F from level k (i.e. of weight k) for $0 \leq k \leq n$. Similarly, let $F^{\geq k}$ be the set of vertices of F from level at least k. Furthermore, we define $F^k_{i:L} = F^k \cap F_{i:L}$ and $F^k_{i:R} = F^k \cap F_{i:R}$. For a vertex u of Q_n let $F(u)$ be the set of faulty neighbors of u, i.e. $F(u) = F \cap N(u)$.

Let A_F be the $|F| \times n$ matrix whose rows are the binary vectors representing the vertices of F. Let $|A_F|$ be the number of ones in A_F. Clearly, $|A_F|$ is the sum of $|x|$ over all $x \in F$. Note that $|F_{i:L}|$ and $|F_{i:R}|$ are the numbers of zeros and ones, respectively, in the i-th column of A_F. By symmetry of Q_n, we assume that

$$|F_{i:L}| \geq |F_{i:R}| \text{ for every dimension } i \in [n]. \tag{1}$$

Indeed, by exchanging zeros and ones in those columns $i \in [n]$ where $|F_{i:L}| < |F_{i:R}|$ we obtain an automorphism of Q_n that maps the set F to a new set satisfying the condition (1).

To apply Theorem 1.2 we need to bound the number $\alpha(F)$ of vertices of Q_n that have at least 4 neighbors in F.

Proposition 2.1. *For every set $F \subseteq V(Q_n)$ it holds that*

$$\alpha(F) \leq \min\left\{\frac{n|F|}{4}, \frac{\binom{|F|}{2}}{3}\right\}.$$

Proof. Every vertex from F has n neighbors in Q_n, but every vertex x with $|F(x)| \geq 4$ has at least 4 neighbors in F. Hence, $\alpha(F) \leq n|F|/4$.

In order to prove the second inequality of this proposition we compute the number p of pairs of incident edges ux and vx of Q_n where $u, v \in F$ are distinct neighbors of x. Since every two vertices u and v of Q_n have at most 2 neighbors in common, we have $p \leq 2\binom{|F|}{2}$. On the other hand, every vertex x with $|F(x)| \geq 4$ has at least $\binom{4}{2} = 6$ pairs of vertices from F in its neighborhood, so $6\alpha(F) \leq p$. Hence, $\alpha(F) \leq \binom{|F|}{2}/3$. $\square$

Proposition 2.2. *For every set $F \subseteq V(Q_n)$ with $|F| \leq 6$ it holds that $\alpha(F) \leq 2$.*

Proof. Suppose for a contradiction that there exist three vertices a, b and c in Q_n such that $|F(a)|, |F(b)|, |F(c)| \geq 4$. Without lost of generality we assume that $a = \mathbf{0}$. Hence, there are at least 4 faulty vertices in the first level. Since there remain at most two vertices in $F \setminus F(a)$, the vertices b and c both share exactly 2 faulty neighbors with the vertex a, so they are in the second level. Furthermore, it follows that the vertices b and c share two neighbors $x, y \in F^3$, so (b, x, c, y) forms a cycle of length 4. But this contradicts the structure of Q_n since every cycle of length 4 in Q_n is contained in exactly 3 consecutive levels. $\square$

3 Overview of the proofs

In this section we give an overview of main proofs and explain the general ideas.

The proofs of Theorems 1.1 and 1.4 have very similar structure. In both theorems we are given a set of faulty vertices F in Q_n, but the maximal cardinality of F differs. For general purposes, let us denote the maximal cardinality of F by $z(n)$. In Theorem 1.1 we have $z(n) = \binom{n}{2} - 2$, and in Theorem 1.4 we have $z(n) = \left\lfloor \frac{n^2+n-4}{4} \right\rfloor$.

Both proofs proceed by induction on the dimension n. Fortunately, the base of induction for $n = 5$ is already known in both cases. For Theorem 1.1 it directly follows from the following result.

Theorem 3.1 (Castañeda and Gotchev [1]). *For every set F of at most $3n-7$ vertices in Q_n and $n \geq 5$, the graph $Q_n - F$ contains a cycle of length at least $2^n - 2|F|$.*

For Theorem 1.4, the base of induction follows from Theorem 1.2 since $2n-4 = \left\lfloor \frac{n^2+n-4}{4} \right\rfloor$ for $n = 5$, and the condition that $|F(u)|, |F(v)| \leq 3$ implies that $N(u) \not\subseteq F \cup \{v\}$ and $N(v) \not\subseteq F \cup \{u\}$ for $n = 5$.

Hence, our task remains to prove the induction step for both Theorems 1.1 and 1.4. Although they are applied in the proofs of each other, note that it is done in a correct way, since the induction steps proceed together. That is, the statements of Theorem 1.1 and Theorem 1.4 for n requires only that

$$\text{the statements of Theorem 1.1 and Theorem 1.4 hold for } n-1. \tag{2}$$

In the first part of the induction steps we assume that

$$\text{there exists a dimension } i \in [n] \text{ such that } |F_{i:L}|, |F_{i:R}| \leq z(n-1). \tag{3}$$

In this case in Theorem 1.4 we proceed directly by applying induction (2) on both $Q_{i:L}$ and $Q_{i:R}$. In Theorem 1.1 we obtain from (1) that[2]

$$|F_{i:R}| \leq \left\lfloor \frac{|F|}{2} \right\rfloor \leq \left\lfloor \frac{\binom{n}{2} - 2}{2} \right\rfloor = \left\lfloor \frac{(n-1)^2 + (n-1) - 4}{4} \right\rfloor .$$

Therefore, we may directly apply induction (2): Theorem 1.1 in $Q_{i:L}$ and Theorem 1.4 in $Q_{i:R}$.

3.1 Potentials

In the second part of both proofs we assume that (3) does not hold. The assumption (1) implies that

$$|F_{i:L}| > z(n-1) \text{ for every dimension } i \in [n]. \tag{4}$$

Now we introduce up to our knowledge a new method of so called *potentials* which is used in the both proofs of Theorems 1.1 and 1.4.

Let $k(n) = z(n) - z(n-1) - 1$. Note that if (4) holds, then $|F_{i:R}| = |F| - |F_{i:L}| \leq k(n)$ for every dimension $i \in [n]$. We define the *potentials* of the set F as follows:

[2] This explains why we consider at most $\left\lfloor \frac{n^2+n-4}{4} \right\rfloor$ faulty vertices in Theorem 1.4.

- $\phi_0(F) = 2(1 - |F^0|) = \begin{cases} 0 & \text{if } \mathbf{0} \in F \\ 2 & \text{if } \mathbf{0} \notin F, \end{cases}$

- $\phi_1(F)$ is the number of F-free vertices in the first level, i.e. $\phi_1(F) = n - |F^1|$,

- $\phi_{\geq 3}(F)$ is the sum of $|x| - 2$ over all faulty vertices x in level at least 3,

- $\phi_{dim}(F)$ is the sum of $|F_{i:L}| - z(n-1) - 1$ over all dimensions $i \in [n]$,

- $\phi(F) = \phi_0(F) + \phi_1(F) + \phi_{\geq 3}(F) + \phi_{dim}(F)$.

Clearly, $\phi_0(F)$, $\phi_1(F)$, $\phi_{\geq 3}(F)$ are non-negative. Furthermore, it follows from (4) that $\phi_{dim}(F)$ is non-negative. Consequently, $\phi(F)$ is non-negative.

Intuitively, the potential $\phi_0(F) + \phi_1(F) + \phi_{\geq 3}(F)$ determines how much the set F differs from a set F' with a minimal number of ones in the matrix $A_{F'}$. If $\mathbf{0} \notin F$, we pay by $\phi_0(F) = 2$; otherwise, $\phi_0(F) = 0$. For every vertex of weight 1 which is not in F, we pay by 1 in $\phi_1(F)$. For every vertex of F which has weight at least 3, we pay its distance to the second level in $\phi_{\geq 3}(F)$. Finally, for every dimension $i \in [n]$ we know that $|F_{i:L}| > z(n-1)$ since we assume (4), therefore we pay in $\phi_{dim}(F)$ the number of vertices which could be moved from $F_{i:L}$ to $F_{i:R}$ so that (4) remains satisfied.

Observe that the definition of $\phi_{dim}(F)$ and (4) implies that if $\phi_{dim}(F) < n$, then there exists a dimension $i \in [n]$ such that $|F_{i:L}| = z(n-1) + 1$. Now, we compute the potential $\phi(F)$ of the set F. Note that the potential $\phi(F)$ depends only on $|F|$, $z(n)$ and $z(n-1)$.

Proposition 3.2. *If $|F| \leq z(n)$ and $|F_{i:L}| > z(n-1)$ for every dimension $i \in [n]$, then*

$$\phi(F) = nk(n) - 2z(n) + n + 2 - (n-2)(z(n) - |F|).$$

Proof. We prove the requested equality by double-counting the number of 1's in the matrix A_F. First, we sum up 1's by columns. Since

$$|F_{i:R}| = |F| - |F_{i:L}| = k(n) - (z(n) - |F|) - (|F_{i:L}| - z(n-1) - 1),$$

we have

$$|A_F| = \sum_{i \in [n]} |F_{i:R}| = nk(n) - n(z(n) - |F|) - \phi_{dim}(F).$$

Now, we sum up 1's by rows.

$$\begin{aligned} |A_F| &= \sum_{x \in F} |x| = 0|F^0| + 1|F^1| + 2|F^2| + 2|F^{\geq 3}| + \phi_{\geq 3}(F) \\ &= \phi_0(F) + \phi_1(F) + \phi_{\geq 3}(F) + 2|F| - n - 2. \end{aligned}$$

The requested equality follows. □

Let us explain informally how potentials are useful for us. Below in Proposition 4.2 we compute the particular value of $\phi(F)$ for paths when $z(n) = \left\lfloor \frac{n^2+n-4}{4} \right\rfloor$; and in Proposition 5.2 we compute it for cycles when $z(n) = \binom{n}{2} - 2$. We will see that $\phi(F)$ is small in both cases. This allows us to split Q_n into $Q_{i:L}$ and $Q_{i:R}$ so that $|F_{i:L}| = z(n-1) + 1$, i.e. there is one faulty vertex more in $F_{i:L}$ than is allowed for applying induction. In such situations we ignore one properly chosen vertex $x \in F_{i:L}$ and try to proceed directly. If the vertex x belongs to the obtained path (or cycle), we attempt to detour it.

However, those detours may also fail because of another vertex $y \in F_{i:R}$. Nevertheless, if this happens, the vertex y must contribute into $\phi_{\geq 3}(F)$. By combination of those methods we either find a long F-free path in Q_n or obtain a contradiction with a small potential $\phi(F)$.

4 Long paths

In this section we prove Theorem 1.4. In what follows assume that F is a set of at most $z(n) = \left\lfloor \frac{n^2+n-4}{4} \right\rfloor$ vertices of Q_n, $n \geq 5$, and u, v are distinct vertices of $Q_n - F$ with $|F(u)|, |F(v)| \leq 3$. Recall that Theorem 1.4 says that $Q_n - F$ contains a path between u and v of length at least $2^n - 2|F| - 2$. Such path is called a long F-free uv-path.

The proof proceeds by induction on the dimension n. For $n = 5$ the statement follows from Theorem 1.2 since $|F| \leq z(5) = 6$. Now, we prove the induction step for $n \geq 6$. We divide the proof into two main parts.

4.1 Induction-friendly split

In the first part, we consider the case when Q_n can be split into $Q_{i:L}$ and $Q_{i:R}$ by a dimension $i \in [n]$ such that $|F_{i:L}|, |F_{i:R}| \leq z(n-1)$; see (3). In this case, we apply induction directly.

Lemma 4.1. *Let Q_n be split into subcubes $Q_{i:L}$ and $Q_{i:R}$ so that $|F_{i:L}|, |F_{i:R}| \leq z(n-1)$. Then there exists a long F-free uv-path P in Q_n. Moreover, if $|F^1_{i:L}| \geq n-2$, then $\mathbf{0} \notin P$.*

Proof. Since the dimension i is fixed, in this proof we omit the index i to simplify the notation. We distinguish two cases regarding the position of vertices u and v in Q_L and Q_R.

Case 1: If u, v are in different subcubes, say $u \in V(Q_L)$ and $v \in V(Q_R)$, then our aim is to find a vertex x in Q_L of opposite parity to the parity of u such that $x \notin F_L$, $x_R \notin F_R \cup \{v\}$ and $|F_L(x)|, |F_R(x_R)| \leq 3$. If there is a such vertex x, then by induction (2), Q_L has a long F_L-free ux-path P_L of length at least $2^{n-1} - 2|F_L| - 1$, Q_R has a long F_R-free $x_R v$-path P_R of length at least $2^{n-1} - 2|F_R| - 2$. Hence, their concatenation by the edge xx_R is the requested long F-free uv-path P in Q_n since

$$|P| = |P_L| + |P_R| + 1 \geq 2^{n-1} - 2|F_L| - 1 + 2^{n-1} - 2|F_R| - 2 + 1 = 2^n - 2|F| - 2.$$

Let A be the set of 2^{n-2} vertices x in Q_L with the opposite parity to the parity of u. We count for how many vertices x from A at least one of the following conditions fails: $x \notin F_L$, $x_R \notin F_R \cup \{v\}$, and $|F_L(x)|, |F_R(x_R)| \leq 3$. First, we find an upper bound on the number of vertices from A such that $x \in F_L$ or $|F_L(x)| \geq 4$.

Every vertex of $F_L \setminus A$ has $n-1$ neighbors in A, so there are at most $\frac{n-1}{4}|F_L \setminus A|$ vertices x of A such that $|F_L(x)| \geq 4$. Furthermore, we have $|F_L \cap A|$ vertices in A such that $x \in F_L$. Thus, the number of vertices x of A such that $x \in F_L$ or $|F_L(x)| \geq 4$ is at most

$$\frac{n-1}{4}|F_L \setminus A| + |F_L \cap A| \leq \frac{n-1}{4}|F_L|$$

since $n \geq 6$.

Similarly, the number of vertices x of A such that $x_R \in F_R$ or $|F_R(x_R)| \geq 4$ is at most $\frac{n-1}{4}|F_R|$. Finally, at most one vertex x of A has $x_R = v$. Altogether, we have at most $\frac{n-1}{4}|F|+1 \le \frac{n-1}{4}z(n)+1$ vertices x in A such that $x \notin F_L$, $x_R \notin F_R \cup \{v\}$, and $|F_L(x)|, |F_R(x_R)| \le 3$, which is less than $|A| = 2^{n-2}$ for $n \ge 6$. Therefore, the desired vertex x exists.

Case 2: If u, v are in the same subcube, say $u, v \in V(Q_L)$, then by induction (2), there exists a long F_L-free uv-path P_L in Q_L. Our aim is to find an edge xy of P_L such that $x_R, y_R \notin F_R$ and $|F_R(x_R)|, |F_R(y_R)| \le 3$. If there is such edge xy, then by induction, Q_R contains a long F_R-free $x_R y_R$-path P_R. By replacing the edge xy in P_L with the path (x, P_R, y), we obtain the requested long F-free uv-path P in Q_n since

$$|P| = |P_L| + |P_R| + 1 \ge 2^{n-1} - 2|F_L| - 2 + 2^{n-1} - 2|F_R| - 1 + 1 = 2^n - 2|F| - 2.$$

The path P_L has at least $2^{n-1} - 2|F_L| - 2$ edges. Every vertex z in Q_R such that $z \in F_R$ or $|F_R(z)| \ge 4$ can block at most two edges xy of P_L. We find an upper bound on the number of such vertices z.

By Proposition 2.1, there are $\alpha(F_R) \le \min\left\{\frac{n-1}{4}|F_R|, \binom{|F_R|}{2}/3\right\}$ vertices z in Q_R such that $|F_R(z)| \ge 4$. Hence, the number of edges xy of P_L such that $x_R, y_R \notin F_R$ and $|F_R(x_R)|, |F_R(y_R)| \le 3$ is at least

$$|P_L| - 2(|F_R| + \alpha(F_R)) \ge 2^{n-1} - 2|F| - 2\alpha(F_R) - 2 \ge \begin{cases} 2^{n-1} - 2z(n) - 2\binom{z(n-1)}{2}/3 - 2, & \text{and} \\ 2^{n-1} - 2z(n) - \frac{n-1}{2}z(n-1) - 2, \end{cases}$$

which is positive for $n = 6$ in the first case, and for $n \ge 7$ in the latter one. Therefore, the desired edge xy exists.

It remains to prove the second part of the statement. Assume that $|F_L^1| \ge n-2$. Since the vertex $\mathbf{0}$ has $n-1$ neighbors in Q_L, at most one of them is F_L-free. Recall that each endvertex of the path P_L has at most 3 neighbors in F_L and $n \ge 6$. Hence, the path P_L does not contain the vertex $\mathbf{0}$, and therefore also $\mathbf{0} \notin P$. □

4.2 Potentials

In the second part of the proof of Theorem 1.4 we assume that (3) fails, i.e. (4) holds.

By substituting $z(n) = \left\lfloor \frac{n^2+n-4}{4} \right\rfloor$ and $k(n) = z(n) - z(n-1) - 1$ into Proposition 3.2 we immediatelly obtain the following table of values of the potential $\phi(F)$ for $n = 4m + (n \bmod 4)$ where $m = \lfloor m/4 \rfloor$. Note that $k(n) \le \left\lfloor \frac{n-1}{2} \right\rfloor$ in the all four cases.

n	$z(n)$	$k(n)$	$\phi(F)$
$4m$	$4m^2+m-1$	$2m-1$	$4-2m-(n-2)(z(n)-\lvert F\rvert)$
$4m+1$	$4m^2+3m-1$	$2m-1$	$4-4m-(n-2)(z(n)-\lvert F\rvert)$
$4m+2$	$4m^2+5m$	$2m$	$4-2m-(n-2)(z(n)-\lvert F\rvert)$
$4m+3$	$4m^2+7m+2$	$2m+1$	$4-(n-2)(z(n)-\lvert F\rvert)$

Lemma 4.2. *If $|F_{i:L}| > z(n-1)$ for every dimension $i \in [n]$, then $|F| = z(n)$. Moreover, $\phi(F) = 2$ for $n = 6$ and $\phi(F) \le 4$ for $n \ge 7$.*

Proof. Since $\phi(F) \ge 0$ and $n \ge 6$, we have $(n-2)(z(n) - |F|) = 0$ in the above table, so $|F| = z(n)$. The above table also implies the second part of this statement. □

In the rest of the proof we proceed by contradiction, so let us suppose that F is a set of at most $z(n)$ vertices of Q_n and u, v are distinct vertices with $|F(u)|, |F(v)| \leq 3$ such that

$$Q_n \text{ does not contain a long } F\text{-free } uv\text{-path.} \tag{5}$$

Recall that Lemma 4.1 implies that the assumption (3) fails. In the next lemma we consider the configurations when faulty vertex $\mathbf{0}$ has at most two F-free neighbors in Q_n.

Lemma 4.3. *$\mathbf{0} \notin F$ or $|F^1| \leq n-2$.*

Proof. For a contradiction, suppose $\mathbf{0} \in F$ and $|F^1| \geq n-1$. Since $n \geq 6$ and $\phi_{dim}(F) \leq 4$ by Lemma 4.2, there exists $i \in [n]$ such that $|F_{i:L}| = z(n-1)+1$. It follows that (3) holds for the set $F' = F \setminus \{\mathbf{0}\}$ as $\mathbf{0} \in F_{i:L}$. Thus, there exists a long F'-free uv-path P in Q_n by Lemma 4.1. Since $|F^1_{i:L}| \geq n-2$, Lemma 4.1 implies that the path P does not contain the vertex $\mathbf{0}$. Therefore, P is also a long F-free uv-path contrary to (5). □

Corollary 4.4. *$\phi_{\geq 3}(F) \leq 2$ for $n \geq 7$, and $\phi_{\geq 3}(F) = 0$ for $n = 6$.*

Proof. Lemma 4.3 implies that $\phi_0(F) + \phi_1(F) \geq 2$. The rest follows from Lemma 4.2. □

The following corollary shows that we can use Theorem 1.1 to find a long $F_{i:L}$-free cycle in $Q_{i:L}$ for every dimension $i \in [n]$.

Corollary 4.5. *$|F_{i:L}| \leq \binom{n-1}{2} - 2$ for every $i \in [n]$.*

Proof. For a contradiction, suppose $|F_{i:L}| > \binom{n-1}{2} - 2$ for some $i \in [n]$. Since $|F_{i:L}| \leq |F| = z(n) = \left\lfloor \frac{n^2+n-4}{4} \right\rfloor$ and $n \geq 6$, the only possible values are $n = 6$ and $|F_{i:L}| = z(6) = 9$. Thus $|F_{i:R}| = 0$, and consequently, $\phi_{dim}(F) \geq 2$. But this contradicts $\phi(F) = 2$ from Lemma 4.2 and $\phi_0(F) + \phi_1(F) \geq 2$ from Lemma 4.3. □

Lemma 4.6. *If $\phi_{\geq 3}(F) \geq 2$ or $n = 6$, then $|F^1| = n$.*

Proof. If $\phi_{\geq 3}(F) \geq 2$ or $n = 6$, then by Lemma 4.2,

$$\phi_0(F) + \phi_1(F) + \phi_{dim}(F) \leq 2. \tag{6}$$

Thus, if $\mathbf{0} \notin F$, then $|F^1| = n$ by the definition of potentials $\phi_0(F)$ and $\phi_1(F)$.

Now suppose that $\mathbf{0} \in F$. Consequently, $|F^1| = n-2$ by Lemma 4.3 and (6). Let $i \in [n]$ be such that $e_i \notin F^1$. Since $\phi_{dim}(F) = 0$ by (6), we have $|F_{i:L}| = z(n-1)+1$. It follows that (3) holds for the set $F' = F \setminus \{\mathbf{0}\}$. Hence, there exists a long F'-free uv-path P in Q_n by Lemma 4.1. Moreover, since $\left|F^1_{i:L}\right| = n-2$, the path P does not contain the vertex $\mathbf{0}$ by the second part of Lemma 4.1. Therefore, P is also a long F-free uv-path, which is contrary to (5). □

In the next lemma we consider the configurations when u or v is $\mathbf{0}$ or there exists a dimension $i \in [n]$ such that $u, v \in V(Q_{i:R})$.

Lemma 4.7. *$u, v \neq \mathbf{0}$ and for every $i \in [n]$ it holds that $u_i = 0$ or $v_i = 0$.*

Proof. Without lost of generality, suppose for a contradiction that $u = \mathbf{0}$. Then $\phi_0(F) + \phi_1(F) \geq n-1$ by the definition of potentials $\phi_0(F)$ and $\phi_1(F)$ since $|F^1| = |F(u)| \leq 3$, which contradicts Lemma 4.2. Thus, the first part holds.

For the second part, suppose that $u_i = v_i = 1$ for some $i \in [n]$, so $u, v \in V(Q_{i:R})$. Since $|F_{i:L}| \leq \binom{n-1}{2} - 2$ by Corollary 4.5, there is a long $F_{i:L}$-free cycle C_L in $Q_{i:L}$ by induction (2). Let ab be an edge of C_L such that $a_R, b_R \notin F_{i:R}$ and $\{a_R, b_R\} \neq \{u, v\}$, and put $A = \{a_R, b_R\}$, $B = \{u, v\}$. Note that such edge ab exists since $|C_L| \geq 2^{n-1} - 2|F_{i:L}|$, every vertex of $F_{i:R} \cup \{u, v\}$ blocks at most 2 edges of C_L, and $2^{n-1} - 2|F| - 4 \geq 1$ for $n \geq 6$. Since $|F_{i:R}| \leq k(n) \leq \lfloor \frac{n-1}{2} \rfloor \leq n-3$, by Theorem 1.5 there is a long $F_{i:R}$-free AB-routing P_1, P_2 in $Q_{i:R}$. After interconnecting the path $C_L - \{ab\}$ and P_1, P_2 with the edges aa_R, bb_R we obtain an uv-path in $Q_n - F$ of length

$$|C_L| + |P_1| + |P_2| + 1 \geq 2^{n-1} - 2|F_{i:L}| + 2^{n-1} - 2|F_{i:R}| - 3 + 1 = 2^n - 2|F| - 2,$$

which contradicts with (5). $\square$

Next, we describe a construction based on long $F_{i:L}$-free cycles in $Q_{i:L}$. Without loss of generality, we assume that

$$\text{if } |u| = 1 \text{ or } |v| = 1, \text{ then } |u| = 1; \tag{7.1}$$

$$\text{if } |u|, |v| \geq 2 \text{ and, } |u| \geq 3 \text{ or } |v| \geq 3, \text{ then } |u| \geq 3; \tag{7.2}$$

$$\text{if } |u| = |v| = 2, \text{ then } |F^1 \cap N(u)| \geq |F^1 \cap N(v)|; \tag{7.3}$$

otherwise, we switch the roles of u and v. The last condition says that the vertex u has at least the same number of faulty neighbors in the first level as the vertex v.

By Lemma 4.7, there exists a dimension $i \in [n]$ such that $u_i = 0$ and $v_i = 1$, so $u \in V(Q_{i:L})$ and $v \in V(Q_{i:R})$. Since $|F_{i:L}| \leq \binom{n-1}{2} - 2$ by Corollary 4.5, there is an $F_{i:L}$-free cycle C_L in $Q_{i:L}$ by induction (2). For the rest of this section, this splitting of Q_n into $Q_{i:L}$ and $Q_{i:R}$, and the cycle C_L are fixed. For ease of notation, we omit the index i in the rest of this section.

For a vertex $z \in C_L$ let $c(z), a(z), z, b(z), d(z)$ be a subpath of C_L, and let $M(z) = \{a(z), b(z), c(z), d(z)\}$. For example, see the set $M(u)$ on Figure 1(a). We say that a vertex x of Q_L is *blocked* if $x_R \in F_R \cup \{v\}$. Furthermore, we say that $M(z)$ is blocked if every vertex of $M(z)$ is blocked. The following proposition gives a sufficient condition which guarantees that the vertex x cannot be blocked by the vertex v.

Proposition 4.8. *For every vertex x of Q_L, if $|x| \geq d(x, u)$, then $x_R \neq v$.*

Proof. Recall that $i \in [n]$ is the fixed splitting dimension of Q_n into Q_L and Q_R, so $u_i = x_i = 0$ and $v_i = 1$. If $|x| \geq d(x, u)$, then there exists $j \in [n] \setminus \{i\}$ such that $u_j = x_j = 1$ since $u \neq \mathbf{0}$ by Lemma 4.7. Furthermore, $v_j = 0$ by Lemma 4.7. Hence $d(x, v) \geq 2$. $\square$

The next construction gives us many blocked vertices. For a vertex $x \in V(Q_L) \setminus F_L$ and the cycle C_L let $\mathcal{S}(x)$ denote the following statement:

$$\mathcal{S}(x) := \begin{cases} M(x) \text{ is blocked} & \text{if } x \in C_L, \\ x \text{ is blocked} & \text{if } x \notin C_L. \end{cases}$$

Lemma 4.9. *Let C_L be a long F_L-free cycle in Q_L, $u \in V(Q_L)$, and $v \in V(Q_R)$. Then $\mathcal{S}(u)$ holds. Moreover, if $u \notin C_L$, then $\mathcal{S}(z)$ holds also for every neighbor z of u in $Q_L - F_L$.*

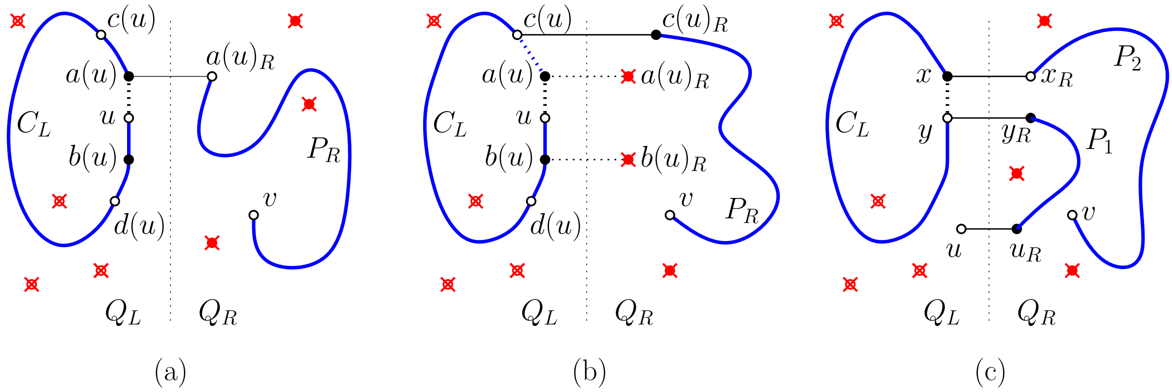

Figure 1: The construction in Lemma 4.9.

Proof. Case 1: $u \in C_L$. First, suppose that $a(u)$ or $b(u)$ is not blocked, say $a(u)_R \notin F_R \cup \{v\}$. See Figure 1(a) for an illustration. Then Q_R contains a long F_R-free $a(u)_R v$-path P_R by Corollary 1.3 since $|F_R| \le k(n) \le \left\lfloor \frac{n-1}{2} \right\rfloor$. By connecting P_R and the path $C_L - \{ua(u)\}$ with the edge $a(u)a(u)_R$ we obtain an uv-path in $Q_n - F$ of length

$$|C_L| + |P_R| \ge 2^{n-1} - 2|F_L| + 2^{n-1} - 2|F_R| - 2 = 2^n - 2|F| - 2,$$

which is a contradiction with (5).

Second, suppose that $c(u)$ or $d(u)$ is not blocked, say $c(u)_R \notin F_R \cup \{v\}$. See Figure 1(b) for an illustration. Since $a(u), b(u)$ are blocked, it follows that $F_R \cup \{c(u)_R, v\}$ is not monopartite. Thus, by Corollary 1.6 there is an $c(u)_R v$-path P_R in $Q_R - F_R$ of length at least $2^{n-1} - 2|F_R| - 1$. By connecting P_R and the path $C_L \setminus \{ua(u), a(u)c(u)\}$ with the edge $c(u)c(u)_R$ we obtain an uv-path in $Q_n - F$ of length

$$|C_L| + |P_R| - 1 \ge 2^{n-1} - 2|F_L| + 2^{n-1} - 2|F_R| - 2 = 2^n - 2|F| - 2,$$

which is a contradiction with (5).

Case 2: $u \notin C_L$. Next, suppose that the vertex u is not blocked. Then, we choose an edge xy on C_L such that $x_R, y_R \notin F_R$. Note that such edge xy exists since $|C_L| \ge 2^{n-1} - 2|F_L|$, every vertex of F_R blocks at most 2 edges of C_L, and $2^{n-1} - 2|F| \ge 1$ for $n \ge 6$. See Figure 1(c) for an illustration. For sets $A = \{x_R, y_R\}$, $B = \{u_R, v\}$ we have that $A \ne B$ and $A \cup B$ is not monopartite. Hence, by Theorem 1.5 there is a long F_R-free AB-routing P_1, P_2 in Q_R. By connecting u, the path $C_L - \{xy\}$, and P_1, P_2 with the edges xx_R, yy_R, uu_R, we obtain an uv-path in $Q_n - F$ of length

$$|C_L| + |P_1| + |P_2| + 2 \ge 2^{n-1} - 2|F_L| + 2^{n-1} - 2|F_R| - 1 = 2^n - 2|F| - 1,$$

which is contradiction with (5). Therefore, the statement $\mathcal{S}(u)$ is established.

Finally, suppose that $\mathcal{S}(z)$ does not hold for some neighbor $z \in V(Q_L) \setminus F_L$ of u. Then, by the same constructions as above, there is a long F-free zv-path P in Q_n. Note that $u \notin P$. By prolonging P with the edge uz we obtain a long F-free uv-path in Q_n, contrary to (5). □

In the next two lemmas we consider the configurations when the weight of the vertex u or v is not 2.

Lemma 4.10. *$|u|, |v| \geq 2$.*

Proof. Recall that $|u|, |v| \geq 1$ by Lemma 4.7. Suppose that $|u| = 1$ or $|v| = 1$, so $|u| = 1$ by the assumption (7.1). It follows that $|F^1| \leq n-1$, so $n \geq 7$ by Lemma 4.6. First, we assume that $u \in C_L$. Then $M(u)$ is blocked by Lemma 4.9. Clearly, at least one of $a(u)$ and $b(u)$ has weight 2, say $a(u)$, and $b(u)$ has weight 0 or 2. If $|b(u)| = 2$, then $a(u)_R, b(u)_R \in F_R$ by Proposition 4.8 and consequently, $\phi_{\geq 3}(F) \geq 2$ contrary to Lemma 4.6. Otherwise $|b(u)| = 0$ and consequently, $\mathbf{0} \notin F$, $|F^1| \leq n-2$, and $\phi_{\geq 3}(F) \geq 1$ since $a(u)_R \in F_R$ by Proposition 4.8. Hence $\phi_0(F) + \phi_1(F) + \phi_{\geq 3}(F) \geq 5$, which contradicts Lemma 4.2.

Now, we have $u \notin C_L$. If u has a neighbor z on C_L with $|z| = 2$, then $M(z)$ is blocked by Lemma 4.9. Note that $a(z)$ or $b(z)$ belong to the third level, say $|a(z)| = 3$, since z has exactly two neighbors in the first level and one of them is $u \notin C_L$. Hence, we have $a(z)_R \in F_R$ by Proposition 4.8 and consequently, $\phi_{\geq 3}(F) \geq 2$, which contradicts Lemma 4.6.

Otherwise, no neighbor z of u in $Q_L - F_L$ with $|z| = 2$ belongs to C_L. Since $|F(u)| \leq 3$, the vertex u has at least $n-5$ neighbors z in $Q_L - F_L$ with $|z| = 2$. By Lemma 4.9, they are all blocked, but by Proposition 4.8, they are not blocked by the vertex v. Hence, $\phi_{\geq 3}(F) \geq n-5 \geq 2$ which contradicts Lemma 4.6. $\square$

Lemma 4.11. *$|u|, |v| \leq 2$.*

Proof. Suppose that $|u| \geq 3$ or $|v| \geq 3$, so $|u| \geq 3$ by the assumption (7.2). First, we consider the case when $u \in C_L$. Then $M(u)$ is blocked by Lemma 4.9. Since $a(u)$ and $b(u)$ belong to level at least 2, we have $a(u)_R, b(u)_R \in F_R$ by Proposition 4.8, so we obtain that $\phi_{\geq 3}(F) \geq 2$. Thus, $|F^1| = n$ by Lemma 4.6. Hence, the vertices $c(u)$ and $d(u)$ have weight at least 2, and they are not blocked by the vertex v by Proposition 4.8. Consequently $\phi_{\geq 3}(F) \geq 4$, which contradicts Corollary 4.4.

Now, we have $u \notin C_L$, so the vertex u is blocked by Lemma 4.9. Since $u_R \in F_R$ by Proposition 4.8, we have $\phi_{\geq 3}(F) \geq 2$ and consequently, $|F^1| = n$ by Lemma 4.6. Furthermore, for an arbitrary neighbor $z \in V(Q_L) \setminus F_L$ of u we obtain from Lemma 4.9 that z is blocked if $z \notin C_L$, or $a(z)$ is blocked if $z \in C_L$. In both cases have another blocked vertex at distance at most 2 from u and in level at least 2, so $\phi_{\geq 3}(F) \geq 3$ by Proposition 4.8, which contradicts Corollary 4.4. $\square$

By the previous two lemmas we have $|u| = |v| = 2$. Let u_1, u_2 and v_1, v_2 be the neighbors of u and v of weight 1, respectively. Note that from Lemma 4.7 it follows that these four vertices are distinct.

Lemma 4.12. *$u_1 \in F$ or $u_2 \in F$.*

Proof. Suppose that $u_1, u_2 \notin F$. From the assumption (7.3) it follows that also $v_1, v_2 \notin F$. Thus, $\phi_1(F) \geq 4$. If $u \in C_L$, then $M(u)$ is blocked by Lemma 4.9, and $c(u)$ or $d(u)$ is in level at least 2, say $|c(u)| \geq 2$, since they have the same parity as u. By Proposition 4.8 we have $c(u)_R \in F_R$ and consequently, $\phi_{\geq 3}(F) \geq 1$. Hence, we obtain that $\phi_1(F) + \phi_{\geq 3}(F) \geq 5$, a contradiction with Lemma 4.2.

If $u \notin C_L$, the vertex u is blocked by Lemma 4.9. By Proposition 4.8 we have $u_R \in F_R$ and consequently, $\phi_{\geq 3}(F) \geq 1$. Similarly as above, we obtain that $\phi_1(F) + \phi_{\geq 3}(F) \geq 5$, a contradiction with Lemma 4.2. $\square$

The end of the proof of Theorem 1.4. If $u \in C_L$, then $M(u)$ is blocked by Lemma 4.9. From Lemma 4.12 it follows that $a(u)$ or $b(u)$ is in the third level, say $|a(u)| = 3$. Furthermore, $|c(u)| \geq 2$. Since $a(u)_R, c(u)_R \in F_R$ by Proposition 4.8, we have $\phi_{\geq 3}(F) \geq 3$, which contradicts Corollary 4.4.

Finally, if $u \notin C_L$, then u is blocked by Lemma 4.9. Let $z \in V(Q_L) \setminus F_L$ be an arbitrary neighbor of u with $|z| = 3$. Then by Lemma 4.9, z is blocked, or the vertices $a(z)$ and $b(z)$ of weight at least 2 are blocked. By Proposition 4.8, $u_R, z_R \in F_R$ in the first case, and $u_R, a(z)_R, b(z)_R \in F_R$ in the latter case. Altogether, we obtain that $\phi_{\geq 3}(F) \geq 3$, which is a final contradiction with Corollary 4.4.

Therefore, we conclude that the contradicted assumption (5) is false, i.e. the statement of Theorem 1.4 holds. □

5 Long cycles

In this section we prove the main Theorem 1.1 which says that for every set of faulty vertices F of Q_n of size at most $\binom{n}{2} - 2$ there exists a cycle in $Q_n - F$ of length at least $2^n - 2|F|$, where $n \geq 4$. Such cycle is called a long F-free cycle.

Fu [7] proved that there exists a long F-free cycle if $|F| \leq 2n - 4$, where $n \geq 3$, which implies that Theorem 1.1 holds for $n = 4$. Theorem 3.1 implies the base of induction of Theorem 1.1 for $n = 5$.

In the induction step of the proof of Theorem 1.1 for n, we assume that both Theorems 1.1 and 1.4 hold for $n - 1$; see (2). Let us consider a fixed set F of at most $\binom{n}{2} - 2$ faulty vertices in Q_n, where $n \geq 6$. Furthermore, we assume that $|F_{i:L}| \geq |F_{i:R}|$ for every dimension $i \in [n]$; see (1).

5.1 Induction-friendly split

In the first part of the proof of Theorem 1.1 we assume that there exists a dimension $i \in [n]$ such that $|F_{i:L}|, |F_{i:R}| \leq \binom{n-1}{2} - 2$; see (3). In this case we apply induction (2) in both $Q_{i:L}$ and $Q_{i:R}$ to construct a long F-free cycle Q_n. Moreover, the following lemma also considers other conditions in which we can simply find a long F-free cycle in the same way. Those conditions are useful later.

Lemma 5.1. *If there exists a dimension $i \in [n]$ such that at least one of the following conditions holds, then there exists a long F-free cycle in Q_n.*

(i) There exists a long $F_{i:L}$-free cycle C_L in $Q_{i:L}$;

(ii) $|F_{i:L}| \leq \binom{n-1}{2} - 2$;

(iii) $|F_{i:L}| = \binom{n-1}{2} - 1$ and there exists $x \in F_{i:L}$ having at most one $F_{i:L}$-free neighbor in $Q_{i:L}$.

Proof. Our first aim is to find a long $F_{i:L}$-free cycle C_L in $Q_{i:L}$. If the condition (i) is satisfied, then the cycle is given. If the condition (ii) is satisfied, then the cycle exists by induction (2).

Let us assume that the condition (iii) is satisfied. Let $F' = F_{i:L} \setminus \{x\}$. By induction (2), there exists a long F'-free cycle C_L in $Q_{i:L}$. Since no cycle of $Q_{i:L} - F'$ contains x, the cycle C_L is also $F_{i:L}$-free.

Our next aim is to find an edge xy of C_L such that

$$x_R, y_R \notin F_{i:R} \text{ and } |F_{i:R}(x_R)|, |F_{i:R}(y_R)| \leq 3. \tag{8}$$

If there exists an edge xy satisfying (8), then by induction (2), there is a long $F_{i:R}$-free $x_R y_R$-path P_R in $Q_{i:R}$ since $|F_{i:R}| \leq \left\lfloor \frac{|F|}{2} \right\rfloor \leq \left\lfloor \frac{(n-1)^2+(n-1)-4}{4} \right\rfloor$. We replace the edge xy in C_L by a path (x, x_R, P_R, y_R, y) and we obtain an F-free cycle in Q_n of length at least

$$(2^{n-1} - 2|F_{i:L}| - 1) + 2 + (2^{n-1} - 2|F_{i:R}| - 1) = 2^n - 2|F|.$$

It remains to show that there exists an edge xy satisfying (8). Recall that $\alpha(F_{i:R})$ is the number of vertices z in $Q_{i:R}$ with $|F_{i:R}(z)| \geq 4$. There are at most $|F_{i:R}| + \alpha(F_{i:R})$ vertices that cannot be used as end-vertices of a long $F_{i:R}$-free path in $Q_{i:R}$. Since the length of C_L is at least $2^{n-1} - 2|F_{i:L}|$, the number of edges xy satisfying (8) is at least

$$2^{n-1} - 2|F_{i:L}| - 2\left(|F_{i:R}| + \alpha(F_{i:R})\right) \geq 2^{n-1} - 2|F| - 2\alpha(F_{i:R}) \geq 1.$$

The last inequality follows from $|F_{i:R}| \leq |F|/2$ and from

- Proposition 2.2 for $n = 6$;
- the inequality $\alpha(F_{i:R}) \leq \binom{|F_{i:R}|}{2}/3$ by Proposition 2.1 for $n = 7$;
- the inequality $\alpha(F_{i:R}) \leq \frac{(n-1)|F_{i:R}|}{4}$ by Proposition 2.1 for $n \geq 8$.

□

5.2 Potentials

In the second part of the proof of Theorem 1.1 we assume that (3) fails, i.e. (4) holds.

Let us recall that we use the following potentials, where now we have $z(n) = \binom{n}{2} - 2$.

$$\phi(F) = \phi_0(F) + \phi_1(F) + \phi_{\geq 3}(F) + \phi_{dim}(F),$$
$$\phi_0(F) = 2 - 2|F^0|, \qquad \phi_1(F) = n - |F^1|,$$
$$\phi_{\geq 3}(F) = \sum_{x \in F^{\geq 3}} (|x| - 2), \qquad \phi_{dim}(F) = \sum_{i \in [n]} (|F_{i:L}| - z(n-1) - 1).$$

By substituting $z(n) = \binom{n}{2} - 2$ and $k(n) = z(n) - z(n-1) - 1 = n - 2$ into Proposition 3.2, the next lemma follows immediately.

Lemma 5.2. *Let F be a set of faulty vertices of Q_n of size at most $\binom{n}{2} - 2$. If $|F_{i:L}| \geq \binom{n-1}{2} - 1$ for every dimension $i \in [n]$, then $|F| \geq \binom{n}{2} - 3$ and*

$$\phi(F) = \begin{cases} 6 & \text{if } |F| = \binom{n}{2} - 2 \\ 8 - n & \text{if } |F| = \binom{n}{2} - 3. \end{cases}$$

In the rest of this section we proceed by contradiction. Therefore, we consider a set of vertices F of Q_n of size at most $\binom{n}{2} - 2$ such that

$$\text{there is no long } F\text{-free cycle in } Q_n. \tag{9}$$

From the assumption (ii) of Lemma 5.1 it follows that $|F_{i:L}| \geq \binom{n-1}{2} - 1$ and $|F_{i:R}| \leq n-2$ for every dimension $i \in [n]$; see (4).

It follows from Lemma 5.2 that there cannot be too many vertices in $F^{\geq 3}$ and they cannot be too far from $\mathbf{0}$. Now, we present a construction which gets a faulty vertex a and gives us another faulty vertex b_R in the level $|a|$ or $|a|+2$.

Lemma 5.3. *Let $i \in [n]$ be a dimenstion and let a be a given vertex of $F^k_{i:L}$. Let one of the two following conditions hold.*

(i) $|F_{i:L}| = \binom{n-1}{2} - 1$,

(ii) $|F_{i:L}| = \binom{n-1}{2}$, $\left|F^1_{i:L} \setminus \{a\}\right| \geq n-2$, $\mathbf{0} \in F$ *and* $a \neq \mathbf{0}$.

Then, there exists $b \in V(Q_{i:L}) \cap N(a)$ such that $b_R \in F_{i:R}$. Hence, $|b_R| \in \{k, k+2\}$.

Moreover, if at least one of the three following conditions holds, then $|b_R| = k+2$.

(iii) Every vertex $x \in N^-(a)$ is faulty,

(iv) for every $x \in N^-(a)$ the vertex x_R is $F_{i:R}$-free,

(v) $\left|F^1_{i:L}\right| = n-1$ *and* $k = 1$.

Proof. Let

$$F' = \begin{cases} F_{i:L} \setminus \{a\} & \text{if (i) holds,} \\ F_{i:L} \setminus \{a, \mathbf{0}\} & \text{if (ii) holds.} \end{cases}$$

By induction (2), there exists a long F'-free cycle C_L in $Q_{i:L}$. If (ii) holds, then $\mathbf{0} \notin C_L$ because $\mathbf{0}$ has at most one F'-free neighbor in $Q_{i:L}$. Since there is no long $F_{i:L}$-free cycle in $Q_{i:L}$ by the assumption (i) of Lemma 5.1 and by the contradicted assumption (9), the vertex a is contained in C_L.

Let b and c be two neighbors of a on C_L. If $b_R, c_R \notin F_{i:R}$, then by Theorem 1.2 there exists a long $F_{i:R}$-free $b_R c_R$-path P_R in $Q_{i:R}$ since $|F_{i:R}| \leq n-2$ and b_R, c_R are not adjacent. Hence, the length of an F-free cycle obtained from C_L by removing edges ba, ac and inserting a path (b, b_R, P_R, c_R, c) is at least

$$(2^{n-1} - 2\left|F'\right|) - 2 + 2 + (2^{n-1} - 2|F_{i:R}| - 2) \geq 2^n - 2|F|.$$

Therefore, at least one of b_R and c_R belongs into $F_{i:R}$, say $b_R \in F_{i:R}$, which implies the first part of the statement.

Now, we prove the second part. Note that

$$|b_R| = \begin{cases} k & \text{if } b \in N^-(a), \\ k+2 & \text{if } b \in N^+(a). \end{cases}$$

If $b \in N^-(a)$, then neither the condition (iii) nor (iv) is satisfied since $b \notin F_{i:L}$ and $b_R \in F_{i:R}$. If (v) holds, then $b \in N^+(a)$; otherwise, the vertex a is the only F'-free neighbor of $b = \mathbf{0}$ in $Q_{i:L}$, and there is no cycle in $Q_{i:L} - F'$ containing $\mathbf{0}$, but $b \in C_L$. □

This lemma is useful to find a faulty vertex in $F^{\geq 3}$ which increases the potential $\phi_{\geq 3}(F)$. We often combine this lemma with other observations to show that the potential $\phi(F)$ is greater than the value given by Lemma 5.2 which provides us with a contradiction. One such example follows, compare it with Lemma 4.3 in the previous section.

For practical purposes, we say that we use Lemma 5.3 with the assumption (i) on a vertex $x \in F_{i:L}$ to obtain a vertex $y \in V(Q_{i:L})$. This only means that Q_n is split by the dimension i, and we apply Lemma 5.3 for the given vertex $a = x$ such that the assumption (i) is satisfied. Then, y is the vertex b obtained by Lemma 5.3. Similarly, we say that we use Lemma 5.3 with the assumption (ii) and (iii) on a vertex $x \in F_{i:L}$ to obtain a vertex $z \in F_{i:R}$. This only means that the dimension i and the vertex $a = x$ satisfy both conditions (ii) and (iii) and z is the vertex $b_R \in F_{i:R}$ in level $|a| + 2$ obtained by Lemma 5.3. Note that $d(x, z) = 2$.

Lemma 5.4. *$\mathbf{0} \notin F$ or $|F^1| \leq n - 2$.*

Proof. For a contradiction, let us suppose that $\mathbf{0} \in F$ and $|F^1| \geq n - 1$.

If there exists a dimension i such that $|F_{i:L}| = \binom{n-1}{2} - 1$, then by Lemma 5.1 with the assumption (iii) for $x = \mathbf{0} \in F$, which has at most one $F_{i:L}$-free neighbor in $Q_{i:L}$, we obtain a long F-free cycle in Q_n which is a contradiction with (9).

Now, we assume that there is no dimension $i \in [n]$ such that $|F_{i:L}| = \binom{n-1}{2} - 1$, so $\phi_{dim}(F) \geq n$. This is possible, by Lemma 5.2, only if $\phi_{dim}(F) = n = 6 = \phi(F)$ and hence by the definition of $\phi(F)$ we have that $|F^1| = 6$, $\mathbf{0} \in F$ and $F^{\geq 3} = \emptyset$. Note that in this case $|F_{i:L}| = \binom{n-1}{2}$ for every dimension $i \in [n]$, so we use Lemma 5.3 with the assumptions (ii) and (iii) on some vertex $a \in F^1_{i:L}$ to obtain a vertex in $F^3_{i:R}$, which is a contradiction with $F^{\geq 3} = \emptyset$. □

Lemma 5.4 implies that $\phi_0(F) + \phi_1(F) \geq 2$. Hence, $\phi_{\geq 3}(F) + \phi_{dim}(F) \leq 4$ by Lemma 5.2 which implies that

$$\text{there exists a dimension } i \in [n] \text{ such that } |F_{i:L}| = \binom{n-1}{2} - 1, \tag{10}$$

since $n \geq 6$. Moreover, the definition of $\phi_{dim}(F)$ implies for a given vertex x of Q_n that

$$\text{if } \phi_{dim}(F) + |x| < n, \text{ then } \exists i \in [n] \text{ such that } |F_{i:L}| = \binom{n-1}{2} - 1 \text{ and } x \in V(Q_{i:L}), \tag{11}$$

because at least $n - \phi_{dim}(F)$ dimensions $i \in [n]$ satisfy $|F_{i:L}| = \binom{n-1}{2} - 1$, and at most $|x|$ of those dimensions volatile $x \in V(Q_{i:L})$.

Our proof still proceeds by contradiction (9). In the following two lemmas we prove that $\phi_0(F) + \phi_1(F) \geq 3$. In the first one we consider the case when $\mathbf{0} \notin F$ and $|F^1| = n$; and in the second one, the case when $\mathbf{0} \in F$ and $|F^1| = n - 2$.

Lemma 5.5. *$|F^1| \leq n - 1$.*

Proof. For a contradiction we suppose that $|F^1| = n$. Hence, $\mathbf{0} \notin F$ by Lemma 5.4. We proceeds in three steps. First, we prove that $|F^3| \geq 1$. Next, we prove that $|F^4| \geq 1$, which we finally improve to $|F^4| \geq 2$. This is a contradiction to Lemma 5.2.

By (10) we split Q_n by such dimension $i \in [n]$ that $|F_{i:L}| = \binom{n-1}{2} - 1$. We use Lemma 5.3 with the assumptions (i) and (v) on some vertex of $F^1_{i:L}$ to obtain $|F^3_{i:R}| \geq 1$. By Lemma 5.2 we know that $|F| = \binom{n}{2} - 2$ since $\phi_0(F) + \phi_{\geq 3}(F) \geq 3$ and $n \geq 6$.

We observe that $F^2_{i:L} \neq \emptyset$; otherwise $\left|F^{\geq 3}_{i:L}\right| = |F_{i:L}| - \left|F^1_{i:L}\right| = \binom{n-1}{2} - 1 - (n-1) \geq 4$ which implies $\phi_{\geq 3}(F) \geq \left|F^{\geq 3}_{i:L}\right| + \left|F^3_{i:R}\right| \geq 5$, contrary to Lemma 5.2. Hence, we use Lemma 5.3 with the assumptions (i) and (iii) on some vertex of $F^2_{i:L}$ to obtain a vertex $x \in F^4_{i:R}$.

Now, we know that F^3, $F^4 \neq \emptyset$ and $0 \notin F$ which implies $\phi_{dim}(F) \leq 1$ by Lemma 5.2. Therefore, there exists a dimension j such that $|F_{j:L}| = \binom{n-1}{2} - 1$ and $x \in F^4_{j:L}$ by (11). We use Lemma 5.3 with the assumption (i) on the vertex x to obtain a vertex in $F^{\geq 4}_{j:R}$. Hence, $\left|F^{\geq 4}\right| \geq 2$ and $\left|F^3\right| \geq 1$, so $\phi_{\geq 3}(F) \geq 5$. It implies $\phi(F) \geq \phi_0(F) + \phi_{\geq 3}(F) \geq 7$, which is a contradiction with Lemma 5.2. □

Lemma 5.6. *If* $\mathbf{0} \in F$, *then* $\left|F^1\right| \leq n - 3$.

Proof. For a contradiction we suppose that $\mathbf{0} \in F$ and $\left|F^1\right| = n - 2$. First, we prove that $\phi_{dim}(F) \geq 2$. Next, we prove that there exist two vertices x and y in $F^3_{d:R}$ for some $d \in [n]$. Finally, we show that there exist 4 distinct dimensions $d_1, d_2, d_3, d_4 \in [n]$, satisfying $x \in F^3_{d_l:R}$ for $l \in [4]$ which implies that $|x| \geq 4$, contrary to $x \in F^3$.

Let e_i and e_j be the (only) two F-free vertices in the first level. We observe that $|F_{i:L}|, |F_{j:L}| \geq \binom{n-1}{2}$; otherwise we use Lemma 5.1 with the assumption (iii) on the vertex $\mathbf{0}$ to obtain a contradiction with (9). Therefore, $\phi_1(F) + \phi_{dim}(F) \geq 4$; and consequently, $\phi_{\geq 3}(F) \leq 2$ by Lemma 5.2.

We split Q_n by a dimension $d \in [n]$ so that $|F_{d:L}| = \binom{n-1}{2} - 1$ by (10). Let a^1, a^2, a^3 be arbitrary distinct vertices of $F^1_{d:L}$. Note that such vertices exist since $\left|F^1_{d:L}\right| = n - 3$ and $n \geq 6$. We use Lemma 5.3 with the assumptions (i) and (iii) for every vertex a^m to obtain $b^m \in V(Q_{d:L})$ such that $b^m_R \in F^3_{d:R}$, where $m \in [3]$. Note that $\left|\left\{b^1_R, b^2_R, b^3_R\right\}\right| \leq 2$ since $\phi_{\geq 3}(F) \leq 2$. On the other hand, if $b^1_R = b^2_R = b^3_R$, then $b^1 = b^2 = b^3$; so $b^1 \in N^+(a^1) \cap N^+(a^2) \cap N^+(a^3)$, but $\left|N^-(b^1)\right| = 2$. Hence, b^1_R, b^2_R and b^3_R are two different vertices; say x_d and y_d. Furthermore, $\left|F^3\right| = \phi_{\geq 3}(F) = \phi_{dim}(F) = 2$.

Since $\phi_{dim}(F) = 2$ and $n \geq 6$, there are at least 4 distinct dimensions $d_l \in [n]$, $l \in [4]$, such that $|F_{d_l:L}| = \binom{n-1}{2} - 1$. For every dimension d_l we obtain vertices $x_{d_l}, y_{d_l} \in F^3_{d_l:R}$ in the same way as described in the previous paragraph. Since $\left|F^3\right| = 2$, the pairs of vertices x_{d_l} and y_{d_l} are the same for all $l \in [4]$; say $x_{d_1} = x_{d_2} = x_{d_3} = x_{d_4}$. But $x_{d_1} \in F^3_{d_l:R}$ for all $l \in [4]$ implies $|x_{d_1}| \geq 4$ which contradicts $x_{d_1} \in F^3$. □

Note that from Lemmas 5.5 and 5.6 it follows that $\phi_1(F) + \phi_0(F) \geq 3$. Therefore, Lemma 5.2 implies the following statement since $n \geq 6$.

Corollary 5.7. $\phi_{\geq 3}(F) + \phi_{dim}(F) \leq 3$ *and* $|F| = \binom{n}{2} - 2$.

Consequently, from (11) we obtain that for every vertex $a \in F$

$$\text{there exists a dimension } i \in [n] \text{ such that } a \in F_{i:L} \text{ and } |F_{i:L}| = \binom{n-1}{2} - 1. \tag{12}$$

Let $u \vee v$ denote the vertex $w = (w_1, w_2, \ldots, w_n)$ with $w_i = u_i \vee v_i$ for all $i \in [n]$, where $\vee$ is the logical disjunction. Note that $w \in Q_{i:L}$ if and only if $u, v \in Q_{i:L}$ for every dimension $i \in [n]$.

Lemma 5.8. $F^{\geq 3} = \emptyset$.

Proof. For a contradiction, let us suppose that there exists a vertex $a \in F^{\geq 3}$. We proceed in 4 steps. First, we prove that $F^{\geq 4} = \emptyset$. Each of next three steps splits Q_n and uses Lemma 5.3 to obtain a new vertex in F^3, which implies that $\left|F^3\right| \geq 4$, contrary to Corollary 5.7. Note that those three splits use different dimensions.

If $|a| \geq 4$, then we split Q_n so that $a \in F_{i:L}$ and $|F_{i:L}| = \binom{n-1}{2} - 1$ by (12). Then we use Lemma 5.3 with the assumption (i) on the vertex a to obtain another faulty vertex in level at least 4, which is a contradiction with Corollary 5.7. Therefore, we assume that $F^{\geq 4} = \emptyset$ and $a \in F^3$.

We split Q_n so that $a \in V(Q_{i:L})$ and $|F_{i:L}| = \binom{n-1}{2} - 1$ by (12). By Lemma 5.3 with the assumption (i), there exists $b \in F_{i:R}$ such that $|b| = 3$ and $d(a,b) = 2$. Hence, $\phi_{\geq 3}(F) \geq 2$.

Let $x = a \vee b$. Since $d(a,b) = 2$ and $|a| = |b| = 3$, we have $|x| = 4$. Since $\phi_{dim}(F) \leq 1$ by Corollary 5.7, there exists a dimension j such that $x \in V(Q_{j:L})$ and $|F_{j:L}| = \binom{n-1}{2} - 1$ by (11). Hence $a, b \in F_{j:L}$. We use Lemma 5.3 with the assumption (i) twice on both a and b to obtain $c, d \in F^3_{j:R}$ such $d(a,c) = d(b,d) = 2$. Since $\phi_{\geq 3}(F) \leq 3$ by Corollary 5.7, we have $c = d$. Hence, $|a| = |b| = |c| = 3$ and $d(a,b) = d(a,c) = d(b,c) = 2$.

Let $y = a \vee b \vee c$. Similarly, we have $|y| \leq 5$ and $\phi_{dim}(F) = 0$, so there exists a dimension d such that $y \in V(Q_{d:L})$ and $|F_{d:L}| = \binom{n-1}{2} - 1$ by (11). Using Lemma 5.3 with the assumption (i) on the vertex a we obtain a faulty vertex in $F^3_{d:R}$; so $\left|F^3\right| \geq 4$, which is a contradiction with Corollary 5.7. $\square$

Lemma 5.9. $\mathbf{0} \notin F$.

Proof. For a contradiction we suppose that $\mathbf{0} \in F$. Hence, $\left|F^1\right| \leq n - 3$ by Lemma 5.6.

We observe that $F^1 = \emptyset$, otherwise we choose $x \in F^1$, we split Q_n so that $x \in F_{i:L}$ and $|F_{i:L}| = \binom{n-1}{2} - 1$ by (12), and by Lemma 5.3 with the assumptions (i) and (iii) we obtain $F^{\geq 3} \neq \emptyset$, contrary to Lemma 5.8. Hence, $\phi_1(F) = n$ which is possible only if $n = 6$, $\left|F^1\right| = 0$, $\left|F^2\right| = 12$ and $\phi_{dim}(F) = 0$.

Since $\phi_{dim}(F) = 0$, we have $|F_{i:R}| = k(n) = n - 2 = 4$ for every dimension $i \in [n]$. Since $\left|F^1\right|, \left|F^{\geq 3}\right| = \emptyset$, only one vertex of $N^+(e_i)$ is F-free for every vertex e_i of the first level in Q_n. Therefore, for every dimension j there exists exactly one other dimension k such that $e_{j,k} \notin F$, so all dimensions are split into three pairs $\{j_1, k_1\}$, $\{j_2, k_2\}$ and $\{j_3, k_3\}$ such that $e_{j_1,k_1}, e_{j_2,k_2}, e_{j_3,k_3} \notin F$. This is satisfied up to isomorphism only by one set of faulty vertices F: the set of all vertices of level 0 or 2 except the vertices $e_{1,2}$, $e_{3,4}$ and $e_{5,6}$. By Lemma 5.1 with the assumption (i), it suffices to find a long $F_{6:L}$-free cycle in $Q_{6:L}$ which is presented on Figure 2. Thus, we obtain a contradiction with (9). $\square$

Finally, we prove the last simple lemma which leads to a contradiction with (9).

Lemma 5.10. *For every dimension $i \in [n]$, if $e_i \notin F$, then $|F_{i:L}| \geq \binom{n-1}{2}$.*

Proof. Let us consider a vertex $e_i \notin F$ such that $|F_{i:L}| = \binom{n-1}{2} - 1$. There exists a vertex $x \in F^1_{i:L}$, because $\phi_1(F) \leq 4 \leq n - 2$ by Lemmas 5.2 and 5.9. We use Lemma 5.3 with the assumptions (i) and (iv) on the vertex x to obtain $\left|F^3\right| \geq 1$, which is a contradiction to Lemma 5.8. $\square$

The end of the proof of Theorem 1.1. Recall that $\phi_0(F) = 2$ by Lemma 5.9, which implies that $\phi_{dim}(F) + \phi_1(F) \leq 4$ by Lemma 5.2. Lemma 5.10 says that $\phi_{dim}(F) \geq \phi_1(F)$ which implies that $\left|F^1\right| \geq n - 2$. On the other hand, we know that $\left|F^1\right| \leq n - 1$ by Lemma 5.5. Moreover, $|F| = \binom{n}{2} - 2$ and every faulty vertex is in the level 1 or 2 by Lemmas 5.8 and 5.9.

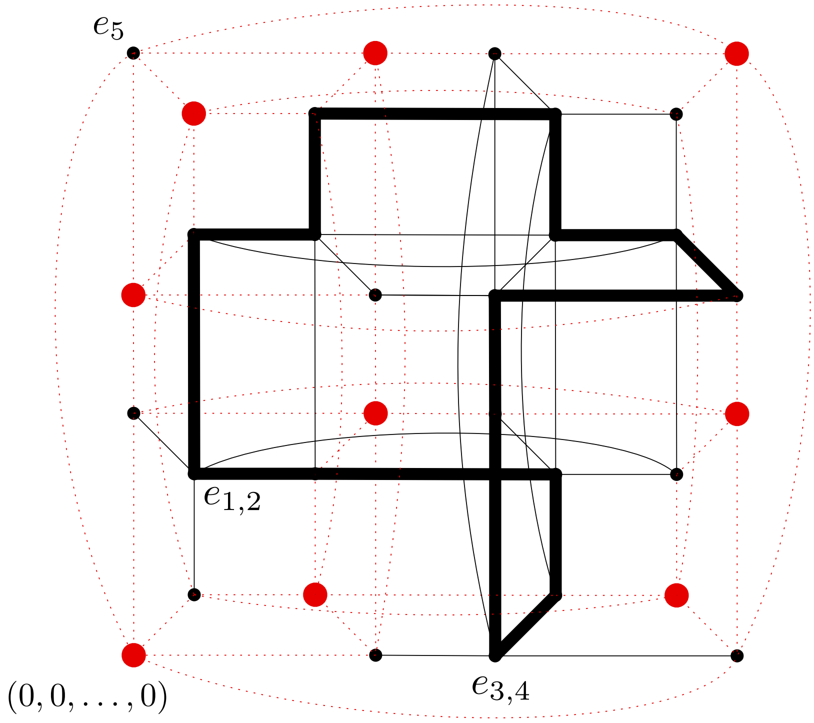

Figure 2: Bold points are faulty vertices and bold lines form a long $F_{6:L}$-free cycle in $Q_{6:L}$ for Lemma 5.9.

If there exists a vertex $a \in F^2$ such that both vertices in $N^-(a)$ are faulty, then we split Q_n so that $a \in F_{i:L}$ and $|F_{i:L}| = \binom{n-1}{2} - 1$ by (12). Then, we use Lemma 5.3 with the assumptions (i) and (iii) to obtain $\left|F^4\right| \geq 1$, which is a contradiction to Lemma 5.8. Hence, every vertex of F^2 is above some F-free vertex of level 1.

Lemma 5.10 also implies that there are at most $n-3$ faulty vertices above every F-free vertex in level 1. Since there are at most two F-free vertices in level 1, we have $\left|F^2\right| \leq 2(n-3)$. This leads to the final contradiction $3n - 7 \geq \left|F^1\right| + \left|F^2\right| = |F| = \binom{n}{2} - 2$ since $n \geq 6$, which finishes the proof of the main Theorem 1.1. □

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6 Appendix: Long routings with two paths

In this appendix we include the proofs of Theorem 1.5 and Corollary 1.6 from the paper [5]. The authors would like to study similar problems more intensively and publish them separately, but Theorem 1.5 and Corollary 1.6 are used in this paper, so their proofs are included for the purpose of referee.

Recall that for a set $F \subseteq V(Q_n)$ and two different sets $A, B \subseteq V(Q_n) \setminus F$ with $|A| = |B| = 2$, a pair P_1, P_2 of vertex-disjoint AB-paths in $Q_n - F$ is called a long F-free AB-routing if $|P_1| + |P_2| \geq 2^n - 2|F| - 3$. Note that if P_1 and P_2 have moreover both even, or both odd length, then actually $|P_1| + |P_2| \geq 2^n - 2|F| - 2$.

The proof is straightforward and apart from standard induction, it uses the following results.

Proposition 6.1 (Lewinter and Widulski [12]). *Let $n \geq 2$ and u, v, w be distinct vertices in Q_n such that u and v have the same parity opposite to the parity of w. Then, $Q_n - \{w\}$ has a Hamiltonian uv-path.*

Proposition 6.2 (Hung et al. [9]). *Let $n \geq 4$, $F \subseteq V(Q_n)$ such that $|F| \leq n-2$ and F is not monopartite, and let $u, v \in V(Q_n) \setminus F$ be distinct vertices. Then, $Q_n - F$ has an uv-path of length at least $2^n - 2|F|$.*

In the following two lemmas we start with dimensions $n = 3$ and $n = 4$. Note that Lemma 6.3 is needed for Lemma 6.4, whereas Lemma 6.4 serves us as a base of induction for Theorem 1.5.

Lemma 6.3. *For every set F of at most 1 vertex of Q_3, there exists a long F-free AB-routing in Q_3 between every two disjoint sets $A, B \subseteq V(Q_3) \setminus F$ such that $|A| = |B| = 2$ and $A \cup B$ is not monopartite.*

Proof. It is trivial to verify the statement by inspection of all cases. First, consider all possible sets A, B in case $F = \emptyset$ when we search for AB-routing P_1, P_2 in Q_3 such that $|P_1| + |P_2| \geq 5$. Then, consider the case $|F| = 1$ when we need $|P_1| + |P_2| \geq 3$. □

Note that the disjointness of the sets A and B is necessary in Lemma 6.3. Indeed, for $A = \{001, 110\}$, $B = \{111, 110\}$, and $F = \{000\}$, observe that there is no path between 001 and 111 in $Q_3 - \{000, 110\}$ of length at least 3, and consequently, no long F-free AB-routing in Q_3.

Lemma 6.4. *For every set F of at most 1 vertex of Q_4, there exists a long F-free AB-routing in Q_4 between every two different sets $A, B \subseteq V(Q_4) \setminus F$ such that $|A| = |B| = 2$ and $A \cup B$ is not monopartite.*

Proof. Case 1: First, we consider the case when $A = \{u, v\}$ and $B = \{x, v\}$ intersect at some vertex v. Then, we can treat v as a new faulty vertex in the set $F' = F \cup \{v\}$, so it suffices to find an ux-path in $Q_4 - F'$ of length at least $2^4 - 2|F'| - 1$. If u, x are of opposite parity, such path exists by Corollary 1.3. Now u and x are of the same parity.

If $F' = \{v\}$, then the requested ux-path exists by Proposition 6.1 since $A \cup B = \{u, x, v\}$ is not monopartite. Now we have $F' = \{f, v\}$. If f and v have opposite parity, then the requested path exists by Proposition 6.2.

Since $A \cup B$ is not monopartite, it remains to consider the case when f and v have the same parity opposite to the parity of u and x. We split Q_4 into Q_L and Q_R so that f and v are in separate subcubes, say $F'_L = \{f\}$ and $F'_R = \{v\}$, and we distinguish two subcases.

Subcase (i): If vertices u, x are in the same subcube, say $u, x \in V(Q_L)$, then from Proposition 6.1 we obtain ux-path P_L in $Q_L - F'_L$ of length 6. Let ab be an edge of P_L such that $a_R, b_R \neq v$. From Corollary 1.3 we obtain $a_R b_R$-path P_R in $Q_R - F'_R$ of length 5. After interconnecting P_R and $P_L - ab$ by edges aa_R, bb_R we get the desired ux-path in $Q_4 - F'$ of length $12 \geq 2^4 - 2|F'| - 1$.

Subcase (ii): Now vertices u, x are in different subcubes, say $x \in V(Q_L)$ and $u \in V(Q_R)$. We choose a vertex $a \in V(Q_L)$ with the opposite parity than u, $a \neq f$, and $a_R \neq u$. Note that $a \neq x$ and $a_R \neq v$. From Corollary 1.3 we obtain ax-path P_L in $Q_L - F'_L$ of length 5, and from Proposition 6.1 we obtain ua_R-path P_R in $Q_R - F'_R$ of length 6. By interconnecting these paths with the edge aa_R we obtain the desired ux-path in $Q_4 - F'$ of length $12 \geq 2^4 - 2|F'| - 1$.

Case 2: Second, we consider the case when $A = \{u, v\}$ and $B = \{x, y\}$ are disjoint. Then, we split Q_4 into Q_L and Q_R so that x, y are in different subcubes, say $x \in V(Q_L)$ and $y \in V(Q_R)$, and we distinguish two subcases depending on the vertices of A.

Subcase (i): If vertices u, v are in the same subcube, say $A \subseteq V(Q_L)$, we choose a vertex $a \in V(Q_L) \setminus F_L$ with the same parity as y, $a_R \notin F_R$, and $a \notin \{u, v, x\}$. Note that such vertex exists, since there are 4 candidate vertices in Q_L with the same parity as y, the set F blocks at most one of them, and the set $\{u, v, x\}$ blocks at most two of them, otherwise $A \cup B$ would be monopartite. For a set $B' = \{x, a\}$ it follows that A, B' are disjoint and $A \cup B'$ is not monopartite. Hence by Lemma 6.3, there is an AB'-routing P'_1, P'_2 in $Q_L - F_L$ such that $|P'_1| + |P'_2| \geq 2^3 - 2|F_L| - 3$. Assume that a is the endvertex of the path P'_1. By Corollary 1.3, there is an $a_R y$-path in $Q_R - F_R$ of length at least $2^3 - 2|F_R| - 1$ since a_R and y have opposite parity. By interconnecting P'_1 and P_R with the edge aa_R, we obtain AB-routing P_1, P'_2 in $Q_4 - F$ such that $|P_1| + |P'_2| = |P'_1| + |P_R| + 1 + |P'_2| \geq 2^4 - 2|F| - 3$.

Subcase (ii): Now vertices u, v are in different subcubes, say $u \in V(Q_L)$ and $v \in V(Q_R)$. If u and x, or v and y are of opposite parity, then from Corollary 1.3 we obtain a long F_L-free ux-path P_L in Q_L and a long F_R-free vy-path P_R in Q_R such that $|P_L| + |P_R| \geq 2^4 - 2|F| - 3$. Hence P_L, P_R is a long F-free AB-routing in Q_4.

Since $A \cup B$ is not monopartite, it remains to consider the case when u and x have the same parity opposite to the parity of v and y. We choose two vertices $a, b \in V(Q_L) \setminus F_L$ with the same parity opposite to the parity of u, and $a_R, b_R \notin F_R$. Note that such vertices exist since there are 4 candidate vertices in Q_L with the parity opposite to u and the set F blocks at most one of them. It follows that $A_L = \{u, x\}$, $B_L = \{a, b\}$ are disjoint and $A_L \cup B_L$ is not monopartite. Hence, by Lemma 6.3 there is a long F_L-free $A_L B_L$-routing P'_1, P'_2 in Q_L. Moreover, since both paths P'_1, P'_2 have odd length, we have $|P'_1| + |P'_2| \geq 2^3 - 2|F_L| - 2$. Assume that the $A_L B_L$-routing joins the vertex u with b, otherwise we switch the roles of a and b in what follows. By the definition of a, b, the sets $A_R = \{b_R, v\}$, $B_R = \{a_R, y\}$ are disjoint and $A_R \cup B_R$ is not monopartite. Hence, by Lemma 6.3 there is a long F_R-free $A_R B_R$-routing P'_3, P'_4 in Q_R. By interconnecting P'_1, P'_2 and P'_3, P'_4 with edges aa_R, bb_R we obtain AB-routing P_1, P_2 in $Q_4 - F$ such that $|P_1| + |P_2| = |P'_1| + |P'_2| + |P'_3| + |P'_4| + 2 \geq 2^4 - 2|F| - 2$. □

Now we are ready to prove Theorem 1.5, which says that for every set F of at most $n - 3$ vertices in Q_n and $n \geq 4$, there exists a long F-free AB-routing in Q_n between every two different sets $A, B \subseteq V(Q_n) \setminus F$ such that $|A| = |B| = 2$ and $A \cup B$ is not monopartite.

Proof of Theorem 1.5. We proceed by induction on the dimension n. For $n = 4$ we apply Lemma 6.4. Now assume $n \geq 5$.

First, we split Q_n into Q_L and Q_R such that we separate two arbitrarily chosen faulty vertices from F if $|F| \geq 2$, otherwise we split Q_n arbitrarily. It follows that $|F_L|, |F_R| \leq n-4$. Thus, we may apply induction both in Q_L and Q_R. We consider the following cases.

Case 1: If both A, B are in one subcube, say $A, B \subseteq V(Q_L)$, then by induction, there is a long F_L-free AB-routing P_1', P_2' in Q_L. Let ab be an edge of P_1' or P_2', such that $a_R, b_R \notin F_R$. Such edge exists, otherwise $2^{n-1} - 2|F_L| - 3 \leq |P_1'| + |P_2'| \leq 2|F_R|$, which yields a contradiction $2^{n-1} - 3 \leq 2|F| \leq 2n-6$ for $n \geq 5$. From Corollary 1.3 we obtain an $a_R b_R$-path P_R in $Q_R - F_R$ of length $2^{n-1} - 2|F_R| - 1$ since a_R and b_R have different parity. After interconnecting P_R and P_1' or P_2' with the edges aa_R, bb_R we get the AB-routing P_1, P_2 in $Q_n - F$ such that $|P_1| + |P_2| = |P_1'| + |P_2'| + |P_R| + 1 \geq 2^n - 2|F| - 3$.

Case 2: If A is in one subcube and B in the other subcube, say $A = \{u, v\} \subseteq V(Q_L)$ and $B = \{x, y\} \subseteq V(Q_R)$, we distinguish two subcases.

Subcase (i): If u and v have different parity, then from Corollary 1.3 we obtain an uv-path P_L in $Q_L - F_L$ of length at least $2^{n-1} - 2|F_L| - 1$. Let ab be an edge of P_L such that $A' = \{a_R, b_R\}$ is disjoint with F_R and $A' \neq B$. Such edge exists, otherwise $|P_R| \leq 2|F_R| + 1$, which yields a contradiction $2^{n-1} - 2 \leq 2|F| \leq 2n - 6$ for $n \geq 5$. Since $A' \cup B$ is not monopartite, there is a long F_R-free $A'B$-routing P_1', P_2' in Q_R. By interconnecting $P_L - ab$ and P_1', P_2' with the edges aa_R, bb_R, we get an AB-routing P_1, P_2 in $Q_n - F$ such that $|P_1| + |P_2| = |P_L| + |P_1'| + |P_2'| + 1 \geq 2^n - 2|F| - 3$.

Subcase (ii): Now u and v are of the same parity. We choose vertices $B' = \{a, b\} \subseteq V(Q_L) \setminus F_L$ of the same parity opposite to the parity of u such that $A' = \{a_R, b_R\}$ is disjoint with F_R. Such vertices exists, since there are 2^{n-2} candidates in Q_L with parity opposite to the parity of u, and at most $n-3$ of them are blocked by F. Clearly, $A \neq B'$ and $A \cup B'$ is not monopartite. Thus, there is a long F_L-free AB'-routing P_1', P_2' in Q_L. Moreover, since both P_1', P_2' have odd length, we have $|P_1'| + |P_2'| \geq 2^{n-1} - 2|F_L| - 2$. In the other subcube Q_R, at least one vertex of $B = \{x, y\}$ has the opposite parity to the parity of a_R, b_R, u, and v. It follows that $A' \neq B$ and $A' \cup B$ is not monopartite, and hence, there is a long F_R-free $A'B$-routing P_3', P_4' in Q_R. By interconnecting P_1', P_2' and P_3', P_4' with edges aa_R, bb_R we get an AB-routing P_1, P_2 such that $|P_1| + |P_2| = |P_1'| + |P_2'| + |P_3'| + |P_4'| + 2 \geq 2^n - 2|F| - 3$.

Case 3: If A is one subcube, and B in both subcubes, say $A = \{u, v\} \subseteq V(Q_L)$, $x \in V(Q_L)$, $y \in V(Q_R)$, then we proceed similarly as in Case 2, Subcase (i) of Lemma 6.4. We choose a vertex $a \in V(Q_L) \setminus F_L$ with the same parity as y, $a_R \notin F_R$, and $a \notin \{u, v, x\}$. Note that such vertex exists, since there are 2^{n-2} candidate vertices in Q_L with the same parity as y, the faulty vertices block at most $n-3$ of them, the set $\{u, v, x\}$ blocks at most 3 of them, and $2^{n-2} - (n-3) - 3 \geq 1$ for $n \geq 5$. For a set $B' = \{x, a\}$ it follows that A, B' are disjoint and $A \cup B'$ is not monopartite. Hence by induction, there is an AB'-routing P_1', P_2' in $Q_L - F_L$ such that $|P_1'| + |P_2'| \geq 2^{n-1} - 2|F_L| - 3$. Assume that a is the endvertex of the path P_1'. By Corollary 1.3, there is an $a_R y$-path in $Q_R - F_R$ of length at least $2^{n-1} - 2|F_R| - 1$ since a_R and y have opposite parity. By interconnecting P_1' and P_R with the edge aa_R, we obtain AB-routing P_1, P_2' in $Q_n - F$ such that $|P_1| + |P_2'| = |P_1'| + |P_R| + 1 + |P_2'| \geq 2^n - 2|F| - 3$.

Case 4: If A, B are both subcubes, say $u, x \in V(Q_L)$ and $v, y \in V(Q_R)$, then we proceed similarly as in Case 2, Subcase (ii) of Lemma 6.4. If u and x, or v and y are of opposite parity, then from Corollary 1.3 we obtain a long F_L-free ux-path P_L in Q_L and a long F_R-free

vy-path P_R in Q_R such that $|P_L| + |P_R| \geq 2^n - 2|F| - 3$. Hence P_L, P_R is a long F-free AB-routing in Q_n.

Since $A \cup B$ is not monopartite, it remains to consider the case when u and x have the same parity opposite to the parity of v and y. We choose two vertices $a, b \in V(Q_L) \setminus F_L$ with the same parity opposite to the parity of u, and $a_R, b_R \notin F_R$. Note that such vertices exist since there are 2^{n-2} candidate vertices in Q_L with the parity opposite to the parity of u, the faulty vertices block at most $n-3$ of them, and $2^{n-2} - (n-3) \geq 2$ for $n \geq 5$. It follows that $A_L = \{u, x\}$, $B_L = \{a, b\}$ are disjoint and $A_L \cup B_L$ is not monopartite. Hence, by induction there is a long F_L-free $A_L B_L$-routing P_1', P_2' in Q_L. Moreover, since both paths P_1', P_2' have odd length, we have $|P_1'| + |P_2'| \geq 2^{n-1} - 2|F_L| - 2$. Assume that the $A_L B_L$-routing joins the vertex u with b, otherwise we switch the roles of a and b in what follows. By the definition of a, b, the sets $A_R = \{b_R, v\}$, $B_R = \{a_R, y\}$ are disjoint and $A_R \cup B_R$ is not monopartite. Hence, by induction there is a long F_R-free $A_R B_R$-routing P_3', P_4' in Q_R. By interconnecting P_1', P_2' and P_3', P_4' with edges aa_R, bb_R we obtain AB-routing P_1, P_2 in $Q_n - F$ such that $|P_1| + |P_2| = |P_1'| + |P_2'| + |P_3'| + |P_4'| + 2 \geq 2^n - 2|F| - 2$. □

Finally, we prove Corollary 1.6 that says for every set F of at most $n-2$ vertices of Q_n and $n \geq 4$, the graph $Q_n - F$ has an uv-path of length at least $2^n - 2|F| - 1$ for every two vertices $u, v \in V(Q_n) \setminus F$ such that $F \cup \{u, v\}$ is not monopartite.

Proof of Corollary 1.6. If $F = \emptyset$, then u and v have opposite parity, and the statement follows from a well-known fact that Q_n contains a Hamiltonian path between every two vertices of opposite parity. Otherwise, there exists $f \in F$ such that $\{u, v, f\}$ is not monopartite. Applying Theorem 1.5 for $A = \{u, f\}$, $B = \{v, f\}$, $F' = F \setminus \{f\}$ we obtain vertex-disjoint paths P_1, P_2 such that P_1 joins u and v, P_2 contains only f, and $|P_1| + |P_2| \geq 2^n - 2|F'| - 3$. Hence $|P_1| \geq 2^n - 2|F| - 1$, and P_1 is the desired path. □