Matching graphs of Hypercubes and Complete bipartite graphs

Jiří Fink*

Department of Applied Mathematics
Faculty of Mathematics and Physics
Charles University
Malostranské náměstí 25
118 00 Prague 1
E-mail: fink@kam.mff.cuni.cz

Abstract

Kreweras' conjecture [9] asserts that every perfect matching of the hypercube Q_d can be extended to a Hamiltonian cycle of Q_d. We [5] proved this conjecture but here we present a simplified proof.

The *matching graph* $\mathcal{M}(G)$ of a graph G has a vertex set of all perfect matchings of G, with two vertices being adjacent whenever the union of the corresponding perfect matchings forms a Hamiltonian cycle of G. We show that the matching graph $\mathcal{M}(K_{n,n})$ of a complete bipartite graph is bipartite if and only if n is even or $n = 1$. We prove that $\mathcal{M}(K_{n,n})$ is connected for n even and $\mathcal{M}(K_{n,n})$ has two components for n odd, $n \geq 3$. We also compute distances between perfect matchings in $\mathcal{M}(K_{n,n})$.

1 Introduction

A set of edges $P \subseteq E$ of a graph $G = (V, E)$ is a *matching* if every vertex of G is incident with at most one edge of P. If a vertex v of G is incident with an edge of P, then v is *covered* by P. A matching P is *perfect* if every vertex of G is covered by P.

The *d-dimensional hypercube* Q_d is a graph whose vertex set consists of all binary vectors of length d, with two vertices being adjacent whenever the corresponding vectors differ at exactly one coordinate.

It is well known that Q_d is Hamiltonian for every $d \geq 2$. This statement can be traced back to 1872 [7]. Since then the research on Hamiltonian cycles in hypercubes satisfying certain additional properties has received considerable attention. An interested reader can find more details on this topic in the survey of Savage [11]. Dvořák [2] showed that every set of at most $2d - 3$ edges of Q_d ($d \geq 2$) that induces vertex-disjoint paths is contained in a Hamiltonian cycle. Dimitrov et al.[1] proved that for every perfect matching P of Q_d ($d \geq 3$) there exists some Hamiltonian cycle that faults P, if and only if P is not a set all edges of Q_d of one dimension.

Kreweras [9] [8, page 33, question 55] conjectured the following:

Conjecture 1. *Every perfect matching in the d-dimensional hypercube with $d \geq 2$ extends to a Hamiltonian cycle.*

We [5] proved this conjecture but here we present a simplified proof.

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The *matching graph* $\mathcal{M}(G)$ of a graph G on even number of vertices has a vertex set of all perfect matchings of G, with two vertices being adjacent whenever the union of the corresponding perfect matchings forms a Hamiltonian cycle. Note that Kreweras' conjecture 1 can be restated in the following way: *There is no isolated vertex in $\mathcal{M}(Q_d)$ for $d \geq 2$.*

There is a natural one-to-one correspondence between Hamiltonian cycles of G and edges of $\mathcal{M}(G)$. The enumeration of all Hamiltonian cycles of a hypercube is a well-known open problem. Feder and Subi [3] presented the following bounds

$$\left(\left(\frac{d\log 2}{e\log\log d}\right)(1-o(1))\right)^{(2^d)} \leq H_d \leq \frac{1}{2}(d!)^{\frac{2^d}{2d}}((d-1)!)^{\frac{2^d}{2(d-1)}}$$

where H_d is the number of Hamiltonian cycles of Q_d.

A partitioning of the edges of a graph G into perfect matchings is a *1-factorization*. A 1-factorization is *perfect* if the union of every pair of its perfect matchings forms a Hamiltonian cycle of G. Observe that k-regular G on even number of vertices has a perfect 1-factorization if and only if $\mathcal{M}(G)$ contains a complete graph on k vertices as a subgraph. Wanless [12] proved that $K_{p,p}$ and $K_{2p-1,2p-1}$ have perfect 1-factorization if p is a prime and proved that $K_{n,n}$ has no perfect 1-factorization if n is even and $n > 2$. Wanless [12] conjectured that $K_{n,n}$ has a perfect 1-factorization if n is odd and $n \geq 3$.

Let G be a graph. We say that a bijection $f : V(G) \to V(G)$ is an *automorphism* if $\{u, v\}$ is an edge of G if and only if $\{f(u), f(v)\}$ is an edge of G for every $u, v \in V(G)$.

We say that two perfect matchings P and R are isomorphic if there exists an automorphism $f : V(Q_d) \to V(Q_d)$ such that $f(u)f(v) \in R$ for every edge $uv \in P$. This relation of isomorphism is an equivalence and it partitions the set of all perfect matchings. Kreweras [9] considered a graph M_d which is obtained from $\mathcal{M}(Q_d)$ by contracting all vertices of each class of this equivalence.

Kreweras [9] proved by inspection of all perfect matchings of Q_3 and Q_4 that the graphs M_3 and M_4 are connected and he conjectured that M_d is connected for every $d \geq 3$. It is more general to also ask whether the graph $\mathcal{M}(Q_d)$ is connected since the connectivity of $\mathcal{M}(Q_d)$ implies the connectivity of M_d. The answer is negative for $d = 3$. However, we [4] proved that $\mathcal{M}(Q_d)$ is connected for $d \geq 4$. Since $\mathcal{M}(Q_d)$ is an induced subgraph of $\mathcal{M}(K_{2^{d-1},2^{d-1}})$, it is natural to ask whether $\mathcal{M}(K_{n,n})$ is connected.

For the study of properties of $\mathcal{M}(Q_d)$, one might ask what additional requirements can we pose on the extending perfect matching R in Theorem 3. For example, can we find R that satisfies Theorem 3 and contains only edges from a given list of dimensions of hypercube? A natural necessary condition says that the set D of allowed edges for R together with the prescribed matching P form a connected subgraph. The following result due to Gregor [6] shows that this condition is also sufficient in the case when D is formed by disjoint subcubes of (possibly different) dimensions. Let $K(A)$ be the complete graph on a set of vertices A.

Theorem 2 (Gregor [6])**.** *Let $A_1, \ldots, A_m \subseteq V(Q_d)$, $d \geq 2$, be pairwise disjoint subcubes of nonzero dimension. Let $A = \bigcup_{i\in[m]} A_i$, $D = \bigcup_{i\in[m]} E(A_i)$ and let P be a perfect matching of $K(A)$. There exists $R \subseteq D$ such that $P \cup R$ forms a Hamiltonian cycle of $K(A)$ if and only if $P \cup D$ is connected.*

In this article we prove that $\mathcal{M}(K_{n,n})$ is bipartite if and only if n is even or $n = 1$. If n is even or $n = 1$ then $\mathcal{M}(K_{n,n})$ is connected, otherwise $\mathcal{M}(K_{n,n})$ has two components. We proved that distance between every pair of perfect matchings in $\mathcal{M}(K_{n,n})$ is at most 3. Moreover, Theorem 8 presents exact distance between every pair of perfect matching in $\mathcal{M}(K_{n,n})$.

2 Perfect matchings extend to Hamiltonian cycles

Let $K(G)$ be the complete graph on the vertices of a graph G. Observe that the following theorem simply implies Kreweras' conjecture 1.

Theorem 3 (Fink [5]). *For every perfect matching P of $K(Q_d)$ there exists a perfect matching R of Q_d such that $P \cup R$ forms a Hamiltonian cycle of $K(Q_d)$ where $d \geq 2$.*

Proof. The proof proceeds by induction on d. The statement holds for $d = 2$. Let us assume that the statement is true for every hypercube Q_k with $2 \leq k \leq d-1$ and let us prove it for d.

Let P be a perfect matching of $K(Q_d)$ and let u_1u_2 be an edge of P. We divide the d-dimensional hypercube Q_d into two $(d-1)$-dimensional subcubes Q^1 and Q^2 such that $u_i \in V(Q^i)$ for $i \in \{1,2\}$. Let $K^i = K(Q^i)$ and $P^i = P \cap E(K^i)$ for $i \in \{1,2\}$. Since u_1u_2 does not belong to $P_1 \cup P_2$, both matchings P_1 and P_2 are not perfect.

The number of vertices of K^1 that are uncovered by P^1 is even and we choose an arbitrary perfect matching S^1 on those vertices. Hence, $P^1 \cup S^1$ is a perfect matchings of K^1. By induction there exists a perfect matching R^1 of Q^1 such that $(P^1 \cup S^1) \cup R^1$ forms a Hamiltonian cycle of K^1.

Our aim is to proceed in a similar way in Q^2: We find a matching S^2 of K^2 covering vertices that are uncovered by P^2. By induction we obtain a perfect matching R^2 of Q^2 such that $(P^2 \cup S^2) \cup R^2$ forms a Hamiltonian cycle of K^2. Clearly, $R := R^1 \cup R^2$ is a perfect matching of Q_d. The only obstacle proves that $P \cup R$ forms a Hamiltonian cycle of $K(Q_d)$. For this purpose we define S^2 to be the set of following *short cuts*

$$S^2 := \left\{ xy \in E(K^2) \;\middle|\; \begin{array}{c} \exists x', y' \in V(Q^1) \text{ such that } xx', yy' \in P \text{ and} \\ \text{there exists a path between } x' \text{ and } y' \text{ of } P^1 \cup R^1 \end{array} \right\}.$$

Observe that $P^1 \cup R^1$ is a partition of Q^1 into vertex-disjoint paths between vertices uncovered by P^1. For every path between x' and y' of this partition there exist vertices x and y of Q^2 such that $xx', yy' \in P$. Thus, the set of edges S^2 is a matching of K^2. Moreover, the set of edges $P^2 \cup S^2$ is a perfect matching of K^2 because S^2 covers each vertex covered by P but uncovered by P^2. By induction there exists a perfect matching R^2 of Q^2 such that $(P^2 \cup S^2) \cup R^2$ forms a Hamiltonian cycle of K^2. Let $R := R^1 \cup R^2$.

It remains to prove that $P \cup R$ forms a Hamiltonian cycle of $K(Q_d)$. Clearly, $P \cup R$ is a set of vertex-disjoint cycles covering all vertices. Suppose on the contrary that $P \cup R$ contains a cycle C that is not Hamiltonian. Notice that C cannot belong to K^1 or to K^2, because $P_1 \cup R_1$ and $P_2 \cup R_2$ have no cycle. Therefore, C has edges in both K^1 and K^2. We shorten C into a cycle C' of K^2 in the following way: We replace every path $xx' \cdots y'y$ such that $x, y \in V(Q^2)$; $x', y' \in V(Q^1)$; $xx', yy' \in P$ and $x' \cdots y'$ is a path of $P^1 \cup R^1$, by an edge $xy \in S^2$. Since C' is a cycle which contains only edges of $P^2 \cup S^2 \cup R^2$, both cycles C' and C cover all vertices of Q^2. Hence, C does not cover a vertex v of Q^1. Let x' and y' be the end-vertices of the maximal path of $P^1 \cup R^1$ that contains v. Let xx' and yy' be edges of P. Observe that $x, y \in V(K^2)$ and $xy \in S^2$. Since C does not contain whole path $xx' \ldots y'y$, the cycle C' does not cover x and y which is a contradiction. □

Ruskey and Savage [10, page 19, question 3] asked the following question:

Does every (not necessarily perfect) matching of Q_d for $d \geq 2$ extends to a Hamiltonian cycle of Q_d?

The statement can be shown to be true for $d = 2, 3, 4$. However, our approach does not seem to lead to proving this stronger statement.

3 Bipartiteness

The number components of a graph on edge set E is denoted by $c(E)$. Let P_1 and P_2 be two perfect matchings of the same graph. Note that $P_1 \cup P_2$ is a set of $c(P_1 \cup P_2)$ vertex-disjoint cycles where common edges $P_1 \cap P_2$ are considered as a cycle of length two.

There is a natural one-to-one correspondence between perfect matchings of the complete bipartite graph $K_{n,n}$ and permutations on a set of size n. Let I_n be the perfect matching on $K_{n,n}$ that corresponds to the identical permutation on a set of size n.

A permutation π on a set M of size n is called *transposition* if π swaps exactly two elements of M. A permutation is *even* if it can be expressed as a composition of even number of transpositions; otherwise permutation is *odd*. A *cycle* of a permutation is a subset of permutation whose elements trade places with one another. It can be shown that every permutation is either odd or even and it cannot be both. From known properties of permutations it follows that a permutation π on n-element set is even if and only if the number of cycles of π has the same parity as n. Moreover, it is known that composition $\pi_1 \circ \pi_2$ is even if and only if π_1 and π_2 have the same parity. Therefore, the *inverse* permutation π^{-1} has the same parity as π.

We say that a perfect matching of $K_{n,n}$ is *even* if its corresponding permutation is even; otherwise its parity is *odd*.

Let P_1 and P_2 be two perfect matchings of $K_{n,n}$ and π_1 and π_2 be their corresponding permutations. Note that cycles of a permutation π_1 correspond to cycles of a graph on edges $P_1 \cup I_n$. Moreover, cycles of a composition permutation $\pi_1 \circ \pi_2^{-1}$ correspond to cycles of $P_1 \cup P_2$. Therefore, the following statements are equivalent.

- P_1 and P_2 have the same parity.
- π_1 and π_2 have the same parity.
- π_1 and π_2^{-1} have the same parity.
- $\pi_1 \circ \pi_2^{-1}$ is an even permutation.
- The number of cycles of $\pi_1 \circ \pi_2^{-1}$ has the same parity as n.
- $c(P_1 \cup P_2) \equiv n \pmod 2$.

The following lemma summarizes above discussion.

Lemma 4. *Two perfect matchings P_1 and P_2 of $K_{n,n}$ have the same parity if and only if $c(P_1 \cup P_2) \equiv n \pmod 2$. Moreover, if $P_1 \cup P_2$ forms a Hamiltonian cycles of $K_{n,n}$ then P_1 and P_2 have the same parity if and only if n is odd.*

From the last lemma it follows that a matching graph $\mathcal{M}(K_{n,n})$ is bipartite for n even where one partite contains even perfect matchings and the other parity contains odd perfect matchings. A matching graph $\mathcal{M}(Q_d)$ is bipartite because it is an induced subgraph of $\mathcal{M}(K_{2^{d-1},2^{d-1}})$.

Theorem 5. *The graph $\mathcal{M}(Q_d)$ is bipartite.*

It is a natural question whether $\mathcal{M}(K_{n,n})$ is bipartite also for n odd. The answer is negative for $n > 1$. Let $b_0, \ldots, b_{n-1}$ be vertices of one color class of $K_{n,n}$ and $w_0, \ldots, w_{n-1}$ be vertices of the other color class. Let

$$Z_i = \{b_k w_{k+i \bmod n} \mid 0 \le k < n\},\ 0 \le i < p,$$

be perfect matchings of $K_{n,n}$, where p is the smallest prime that divides n.

Let $0 \le i < j < p$. A graph on edges $Z_i \cup Z_j$ has only one cycle

$$b_0, w_{j \bmod n}, b_{j-i \bmod n}, w_{(j-i)+j \bmod n}, b_{2(j-i) \bmod n}, \ldots, b_{n(j-i) \bmod n} = b_0$$

because $j - i$ and n are relatively prime numbers. Hence, $Z_0, \ldots, Z_{p-1}$ form a complete subgraph of $\mathcal{M}(K_{n,n})$.

Theorem 6. *The graph $\mathcal{M}(K_{n,n})$ is bipartite, if and only if n is even or $n = 1$.*

4 Connectivity

From Lemma 4 it follows that a union of two permutations P_1 and P_2 of $K_{n,n}$ with different parities cannot form a Hamiltonian cycle if n is at least 3 and odd. Hence, $\mathcal{M}(K_{n,n})$ is not connected for such n.

Lemma 7. *For every perfect matching P of $K_{n,n}$ such that $c(P \cup I_n) \equiv n \pmod 2$, $n \geq 2$, there exists a perfect matching R of $K_{n,n}$ such that $P \cup R$ and $R \cup I_n$ form Hamiltonian cycles of $K_{n,n}$.*

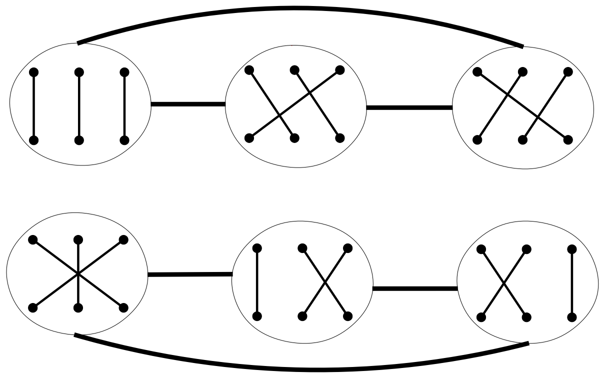

Figure 1: The circles and bold lines are vertices and edges of $\mathcal{M}(K_{3,3})$. The matching graph $\mathcal{M}(K_{3,3})$ has two components; the upper one contains even perfect matchings of $K_{3,3}$ and the lower one contains odd perfect matchings.

Proof. The proof proceeds by induction on n. The statement holds for $n \in \{2, 3\}$, see Figure 1 for $n = 3$. Let us suppose that the statement is true for every $K_{k,k}$ with $2 \leq k < n$ and let us prove it for $K_{n,n}$ where $n \geq 4$.

We divide the proof into three cases:

1. $P = I_n$
2. There exists a cycle of length at least six in $P \cup I_n$.
3. There exist at least two cycles of length four in $P \cup I_n$.

Every perfect matching P belongs to at least one case because there is no cycle of odd length in the bipartite graph $P \cup I_n$. If $P \cup I_n$ contains only one cycle of length four and no cycle of length at least 6, then $c(P \cup I_n) = n - 1$ which contradicts the assumption $c(P \cup I_n) \equiv n \pmod 2$.

The first case is simple because it is sufficient to choose an arbitrary perfect matching R of $K_{n,n}$ such that $R \cup I_n$ forms a Hamiltonian cycle.

Let us consider that the graph $P \cup I_n$ contains a cycle $b_a, w_b, b_b, w_c, b_c, w_d, \ldots, b_a$ of length at least six. Let $P' := (P \setminus \{b_a w_b, b_b w_c, b_c w_d\}) \cup \{b_a w_d\}$. The matching P' of $K_{n,n}$ does not cover vertices w_b, b_b, w_c and b_c so P' is a perfect matching of $K_{n-2,n-2}$ up to isomorphism. Moreover, $c(P \cup I_n) = c(P' \cup I_{n-2})$ because we only shorten the cycle. By induction there exists a perfect matching R' of $K_{n-2,n-2}$ such that $P' \cup R'$ and $R' \cup I_{n-2}$ form Hamiltonian cycles of $K_{n-2,n-2}$. Let b_e be the vertex of $K_{n-2,n-2}$ such that $b_e w_d \in R'$. Note that $R := (R' \setminus \{b_e w_d\}) \cup \{b_e w_c, b_c w_b, b_b w_d\}$ is a perfect matching of $K_{n,n}$. We observe that $R \cup I_n$ forms a Hamiltonian cycle of $K_{n,n}$ because we replace the edge $b_e w_d$ in $R' \cup I_{n-2}$ by the path $b_e, w_c, b_c, w_b, b_b, w_d$. Similarly, $P \cup R$ forms a Hamiltonian cycle of $K_{n,n}$ because we replace a path b_a, w_d, b_e in $P' \cup R'$ by the path $b_a, w_b, b_c, w_d, b_b, w_c, b_e$.

Now, we consider that P contains edges $b_a w_b, b_b w_a, b_c w_d$ and $b_d w_c$ which belong to two cycles of length four in $P \cup I_n$. Let us define a perfect matching $P' = (P \setminus \{b_a w_b, b_b w_a, b_c w_d, b_d w_c\}) \cup \{b_c w_c, b_d w_d\}$ of $K_{n-2,n-2}$. We again use the induction to find a perfect matching R' of $K_{n-2,n-2}$ such that $P' \cup R'$ and $R' \cup I_{n-2}$ form Hamiltonian cycles of $K_{n-2,n-2}$. Let w_e and b_f be vertices of

$K_{n-2,n-2}$ such that $b_d w_e, b_f w_d \in R'$. Note that $R := (R' \setminus \{b_d w_e, b_f w_d\}) \cup \{b_a w_e, b_b w_d, b_d w_a, b_f w_b\}$ is a perfect matching of $K_{n,n}$. The union $R \cup I_n$ forms a Hamiltonian cycle of $K_{n,n}$ because we replace a path b_f, w_d, b_d, w_e in $R' \cup I_{n-2}$ by a path $b_f, w_b, b_b, w_d, b_d, w_a, b_a, w_e$. Finally, $P \cup R$ is Hamiltonian cycle of $K_{n,n}$ because we replace an edge $b_c w_c$ by a path $b_c, w_d, b_b, w_a, b_d, w_c$ and a path b_f, w_d, b_d, w_e by b_f, w_b, b_a, w_e. $\square$

Let $f : V(K_{n,n}) \to V(K_{n,n})$ be an automorphism. We extend domain and codomain of f on edges of $K_{n,n}$ in this natural way: $f(\{u,v\}) = \{f(u), f(v)\}$, where $uv \in E(K_{n,n})$. Furthermore, we extend domain and codomain of f on perfect matchings of $K_{n,n}$ in this way: $f(P) = \{f(e) \mid e \in P\}$, where P is perfect matching of $K_{n,n}$. Note that for every two perfect matchings P and R of $K_{n,n}$ there exists an automorphism f such that $f(P) = R$.

Theorem 8. *The distance between perfect matchings P and S of $K_{n,n}$, $n \geq 2$, in the matching graph $\mathcal{M}(K_{n,n})$ is the following.*

Conditions		*n is even*	*n is odd*
	$c(P \cup S) = n$	0	0
	$c(P \cup S) = 1$	1	1
$1 < c(P \cup S) < n$	$c(P \cup S) \equiv n \pmod 2$	2	2
	$c(P \cup S) \equiv n+1 \pmod 2$	3	∞

Proof. If $c(P \cup S) = n$, then $P = S$. If $c(P \cup S) = 1$, then $P \cup S$ forms a Hamiltonian cycle and there is an edge PS in the graph $\mathcal{M}(K_{n,n})$. Let us suppose that $1 < c(P \cup S) < n$. Hence, the distance between P and S is at least 2.

Let us consider that $c(P \cup S) \equiv n \pmod 2$. Let f be an automorphism on $K_{n,n}$ such that $f(I_n) = S$. We observe that $c(f^{-1}(P) \cup I_n) = c(P \cup S) \equiv n \pmod 2$. By Lemma 7, there exists a perfect matching R of $K_{n,n}$ such that $f^{-1}(P) \cup R$ and $R \cup I_n$ form Hamiltonian cycles of $K_{n,n}$. Hence, $P \cup f(R)$ and $f(R) \cup S$ form Hamiltonian cycles of $K_{n,n}$.

Let us consider that $c(P \cup S) \equiv n+1 \pmod 2$ and n is even. By Theorem 6, the distance between P and S is odd. Let R be a perfect matching of $K_{n,n}$ such that $P \cup R$ forms a Hamiltonian cycle. The distance between R and S is 2 because $c(R \cup S) \equiv n \pmod 2$. Hence, the distance between P and S is 3.

If n is odd then there is no pair of perfect matchings with different parity whose union forms a Hamiltonian cycle of $K_{n,n}$ by Lemma 4. Therefore, if $c(P \cup S) \equiv n+1 \pmod 2$ and n is odd, then P and S belong to different components of $\mathcal{M}(K_{n,n})$. $\square$

Corollary 9. *The graph $\mathcal{M}(K_{n,n})$ has one component for n even and two components for n odd, where $n \geq 2$.*

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