Long paths and cycles in faulty hypercubes: existence, optimality, complexity

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Abstract

A fault-free cycle in the n-dimensional hypercube Q_n with f faulty vertices is *long* if it has length at least $2^n - 2f$. If all faulty vertices are from the same bipartite class of Q_n, such length is the best possible. We prove a conjecture of Castañeda and Gotchev [2] asserting that $f_n = \binom{n}{2} - 2$ where f_n is the largest integer such that for every set of at most f_n faulty vertices, there exists a long fault-free cycle in Q_n. Furthermore, we present several results on similar problems of long paths and long routings in faulty hypercubes and their complexity.

Keywords: hypercube, faulty vertex, long path, long cycle, NP-hard

1 Introduction

The n-dimensional hypercube Q_n is a (bipartite) graph with all binary vectors of length n as vertices and edges joining every two vertices that differ in exactly one coordinate. The application of hypercubes as interconnection networks inspired research of their fault-tolerant properties. Here we consider a problem

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of long fault-free cycles and long fault-free paths between two given vertices of a hypercube in which some vertices are faulty.

Let F be a set of faulty vertices of Q_n. A cycle C of Q_n is a *long fault-free* cycle if it does not contain any faulty vertex and the length of C is at least $2^n - 2|F|$. A path P of Q_n between vertices u and v is a *long fault-free* path if it does not contain any faulty vertex and the length of P is at least $2^n - 2|F| - 2$. These concepts are motivated by the observation that if all faulty vertices belong to the same bipartite class of Q_n, then every long fault-free cycle and long fault-free path are the longest possible.

Fu [6] proved that there exists a long fault-free cycle if $|F| \leq 2n - 4$. Castañeda and Gotchev [2] improved the bound to $|F| \leq 3n - 7$ for $n \geq 5$. The similar problem for paths was first studied by Fu [7] who showed that there is a long fault-free path in Q_n between every two fault-free vertices if $|F| \leq n - 2$. Recently, the bound of Fu was improved by Kueng, Liang, Hsu, and Tan [9] to $|F| \leq 2n - 5$, but with an additional (strong) condition that every vertex has at least two fault-free neighbors.

2 Long paths

A vertex u is *surrounded* by F if F contains all neighbors of u. Note that if a vertex u is surrounded by $F \cup \{v\}$ or v is surrounded by $F \cup \{u\}$, then there is no long fault-free path of length at least 2 between u and v. We proved that this necessary condition is also sufficient if $|F| \leq 2n - 4$.

Theorem 2.1 ([4]) *Let F be a set of at most $2n - 4$ faulty vertices of Q_n where $n \geq 5$. Then for every two fault-free vertices u and v, there exists a long fault-free path between u and v in Q_n if and only if u is not surrounded by $F \cup \{v\}$ in Q_n and v is not surrounded by $F \cup \{u\}$ in Q_n.*

Moreover, the bound on $|F|$ is tight, as for every $n \geq 5$ there is a configuration of $2n - 3$ faulty vertices and two fault-free vertices u, v satisfying the above necessary condition, but no fault-free path between them is long. Therefore, a natural question arises whether there exists a simple condition on neighbors of end-vertices which allows to significantly increase the upper bound on the number of faulty vertices.

Theorem 2.2 ([3]) *Let F be a set of at most $\frac{n^2}{10} + \frac{n}{2} + 1$ faulty vertices of Q_n where $n \geq 15$. Then there is a long fault-free path between every pair of distinct vertices of the largest biconnected component of $Q_n - F$.*

The bound on $|F|$ is asymptotically optimal in the following sense. Let

$\psi(n)$ ($\phi(n)$) be the largest integer such that for every set F of at most $\psi(n)$ (resp. $\phi(n)$) faulty vertices of Q_n there exists a long fault-free path (resp. between every pair of distinct vertices) in the largest biconnected component of $Q_n - F$.

Theorem 2.3 **([3])** *$\phi(n) \leq \binom{n}{2} - 2$ for $n \geq 4$ and $\psi(n) \leq 2\binom{n}{2} - 1$ for $n \geq 6$.*

Putting together Theorems 2.2 and 2.3, we conclude that $\phi(n), \psi(n) \in \Theta(n^2)$. Imposing even more specific condition on end-vertices, we can still increase the upper bound on $|F|$ by a multiplicative constant.

Theorem 2.4 **([5])** *If $n \geq 5$, $|F| \leq \frac{n^2+n-4}{4}$, and u, v are two fault-free vertices such that both have at most 3 faulty neighbors, then there exists a long fault-free path between u and v.*

Furthermore, another such a condition is on the minimal distance in F. Clearly, if every two vertices from F are at (Hamming) distance at least 3, then the minimal degree $\delta(Q_n - F)$ in $Q_n - F$ is at least $n - 1$.

Theorem 2.5 **([8])** *Let $F \subseteq V(Q_n)$ where $n \geq 1$ such that $\delta(Q_n - F) \geq n - 1$. Then $Q_n - F$ contains a long uv-path for every pair of fault-free vertices u, v.*

3 Long cycles

Let f_n be the largest integer such that for every set of at most f_n faulty vertices of Q_n there exists a long fault-free cycle. Castañeda and Gotchev [2] noticed, independently on us, that for $n \geq 4$ there is a set F of $\binom{n}{2} - 1$ faulty vertices such that Q_n has no long fault-free cycle. Such a set F may be, for example, formed by all but one vertex at distance 2 from a fixed vertex.

Moreover, they conjectured [2] that $f_n = \binom{n}{2} - 2$. Recently, we proved that the conjecture holds.

Theorem 3.1 **([5])** *For every set of at most $\binom{n}{2} - 2$ faulty vertices, $n \geq 3$, there is a long fault-free cycle in Q_n.*

Furthermore, Theorem 2.5 implies that there is a long cycle in $Q_n - F$ if $\delta(Q_n - F) \geq n - 1$ and $n \geq 3$.

4 Routing

We also extend the problem for more paths that interconnect two given sets A, B of vertices with $A \neq B$ and $|A| = |B| = k$. An *AB-path* is a path between a vertex of A and a vertex of B. An *AB-routing* is a collection of k vertex-disjoint AB-paths. An AB-routing $P_1, P_2, \ldots, P_k$ in $Q_n - F$ is *long* if

$$|P_1| + |P_2| + \cdots + |P_k| \geq 2^n - 2|F| - k - 1.$$

Note that for $k = 1$, this definition corresponds to long paths. For general k, it can be shown that a long AB-routing exists in $Q_n - F$ for any F only if

$$b(A \cup B) + b(A \cap B) \leq 2 \tag{1}$$

where $b(C) = \big||C \cap X| - |C \cap Y|\big|$ is the *balance* of C, and X, Y are the bipartite classes of Q_n.

In particular, the (necessary) condition (1) for $k = 2$ is equivalent to $b(A \cup B) < |A \cup B|$, i.e. $A \cup B$ is not monopartite. A set C is called *monopartite* if all vertices in C have the same parity.

Theorem 4.1 ([5]) *Let $n \geq 5$, $F \subseteq V(Q_n)$, $A, B \subseteq V(Q_n - F)$ be such that $|F| \leq \lfloor n/2 \rfloor$, $|A| = |B| = 2$, $A \neq B$, and $A \cup B$ is not monopartite. Then $Q_n - F$ has a long AB-routing.*

As a consequence, if $F \cup \{u, v\}$ is not monopartite and $|F| \leq \lfloor n/2 \rfloor + 1$, we obtain uv-paths in $Q_n - F$ of length at least $2^n - 2|F| - 1$, which is more than is guaranteed by long paths.

5 Complexity

We conclude this survey with a study of computational complexity of our main problems, denoted by **LC**, **LP** and **LPP**, and formulated in the following way: Given n and a set F of faulty vertices of Q_n (and vertices u, v for the last problem), is there a long fault-free cycle, a long fault-free path, or a long fault-free path between u and v, respectively? Considering the applications in data compression [1], for each problem $P \in \{\mathbf{LC}, \mathbf{LP}, \mathbf{LPP}\}$ it is useful to deal also with the variant P' where the input is formed by the set of fault-free vertices.

If $|F| \leq \binom{n}{2} - 2$, the problems **LC**, **LC$'$**, **LP**, **LP$'$** are (trivially) decidable in polynomial time by Theorem 3.1. The next theorem shows that the same conclusion holds even for **LPP** and **LPP$'$** under a slightly weaker, but still quadratic bound on $|F|$. Moreover, it also shows that the construction of a long fault-free path or cycle takes only constant amortized time per one vertex of the output.

Theorem 5.1 ([3]) *Let Q_n contain at most $\frac{n^2}{10} + \frac{n}{2} + 1$ faulty vertices. There is an algorithm deciding in $O(n^8)$ time whether a long fault-free path P between a given pair of distinct vertices exists. If P exists, the algorithm constructs P in linear time with respect to its length.*

On the other hand, in case that the number of faults is not limited, all the problems are intractable.

Theorem 5.2 *For every $P \in \{\mathbf{LC}, \mathbf{LP}, \mathbf{LPP}\}$, P is NP-hard and P' is NP-complete.*

Moreover, there exists a polynomial $p(n)$ of degree 6 such that **LC**, **LP**, **LPP** remain NP-hard even if the number of faults is at most $p(n)$.

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