TOWARDS A PROBLEM OF RUSKEY AND SAVAGE ON MATCHING EXTENDABILITY

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Abstract. Does every matching in the n-dimensional hypercube Q_n extend to a Hamiltonian cycle? This question was raised by Ruskey and Savage in 1993 and even though a positive answer in known in some special cases, the problem still remains open in general. In this paper we present recent results on extendability of matchings in hypercubes to Hamiltonian cycles and paths as well as on the computational complexity of these problems, motivated by the Ruskey-Savage question. Moreover, we verify the conjecture of Vandenbussche and West saying that every matching in Q_n, $n \geq 2$, extends to a 2-factor.

1. Introduction

The n-dimensional hypercube Q_n is the graph on the set of all n-bit strings with edges joining two vertices whenever they differ in exactly one bit. There is a large literature on structural properties of this class of graphs which comes from research on the topological structure and analysis of hypercubic interconnection networks [24].

It is well known that Q_n is Hamiltonian for every $n \geq 2$. Among a number of appealing problems related to Hamiltonicity of hypercubes, the most prominent role was played by the notorious Middle Levels Conjecture, recently resolved by Mütze [17]. But there is another long-standing question, raised in 1993 by Ruskey and Savage [20], asking whether every matching in Q_n extends to a Hamiltonian cycle. A positive solution has been verified for $n \leq 5$ by a computer search [25], but for larger values of n, the answer is known only in several special cases. It may be of interest that matchings in hypercubes that can be avoided by a Hamiltonian cycle were characterized in [2].

The purpose of this paper is to present recent results on extendability of matchings in hypercubes to Hamitonian cycles, Hamiltonian paths and 2-factors as well as on the computational complexity of these problems, motivated by the Ruskey-Savage question.

2. Preliminaries

Throughout this paper, n always denotes an integer such that $n \geq 2$. Given an n-bit string u, we use u_i to denote the i-th element of the sequence $u_1 \ldots u_n = u$. Vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively.

Given the n-dimensional hypercube Q_n, the *parity* $\chi(v)$ of a vertex v of Q_n is defined by $\chi(v) = \prod_{i=1}^{n} (-1)^{v_i}$. A set $S \subseteq V(Q_n)$ is called *balanced* if $\sum_{v \in S} \chi(v) = 0$. We use $d(u, v)$ to denote the Hamming distance of $u, v \in V(Q_n)$, i.e. $d(u, v) = |\{i \mid u_i \neq v_i\}|$. The *dimension* of an edge uv of Q_n is defined as the integer i such that $u_i \neq v_i$. If $u_i = 0 \neq v_i$, the parity of an edge uv is defined as the parity of

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u. The set of all edges of Q_n of the same dimension and the same parity is called a *half-layer*.

Let $K(Q_n)$ denote the complete graph on the set of vertices of Q_n. We use $B(Q_n)$ to denote the spanning subgraph of $K(Q_n)$ containing only edges $uv \in E(K(Q_n))$ such that $d(u, v)$ is odd. While $K(Q_n)$ is a completion of Q_n, $B(Q_n)$ may be viewed as a bipartite completion of Q_n.

A *matching* is a set of edges without common vertices. A matching M in a graph G is called *perfect* if every vertex of G is incident with an edge of M. A matching M in $K(Q_n)$ is called *balanced* if the set of all vertices that are incident with an edge of M forms a balanced subset of $V(Q_n)$. We say that a set of edges P of $K(Q_n)$ is *extendable* if there exists a set of edges R of Q_n such that $P \cup R$ is a Hamiltonian cycle of $K(Q_n)$.

3. (Nearly) perfect matchings

An affirmative answer to the Ruskey-Savage problem in the case of perfect matchings was obtained a decade ago by Fink [6] who thus verified a conjecture published by Kreweras [16] and popularized by Knuth [14]. Fink's theorem actually provides a slightly stronger statement saying that every matching in $K(Q_n)$ is extendable [6, 7]. This result inspired several generalizations [1, 12], e.g. the authors of [1] showed that the Kreweras conjecture also holds for sparse spanning regular subgraphs of hypercubes. Note that Fink's theorem implies a positive solution for every matching that can be extended to a perfect matching, which includes e.g. every induced matching [21]. However, it does not settle the problem in general, as hypercubes may contain matchings that are maximal with respect to inclusion but still not perfect [10].

It is well known that Q_n is not only Hamiltonian, but moreover, there is a Hamiltonian path between vertices x and y if and only if $\chi(x) \neq \chi(y)$ [13]. In joint work with Škrekovski [11], we extend Fink's theorem in a similar fashion. In particular, the following result provides a necessary and sufficient condition for the existence of a Hamiltonian path in Q_n between given vertices x and y containing a given perfect matching. Similarly as Fink's theorem extends matchings in $K(Q_n)$, we extend matchings in the complete bipartite graph $B(Q_n)$.

Theorem 3.1 ([11]). *Let P be a perfect matching of $B(Q_n)$, $n \geq 5$, and $x, y \in V(Q_n)$. Then P may be extended by edges of Q_n into a Hamiltonian path in $B(Q_n)$ between x and y if and only if*

- $\chi(x) \neq \chi(y)$,
- $xy \notin P$,
- $(P \cup \{x'y'\}) \setminus \{xx', yy'\}$ *does not contain a half-layer,*

where x', y' are the vertices of Q_n such that $xx', yy' \in P$.

It should be noted that the proof of Theorem 3.1 is computer-assisted: the case $n = 5$, which serves as the basis for the inductive proof, was verified by a computer search [19]. The assumption $n \geq 5$ is essential, as for $n = 4$ the computer search identified 8 non-isomorphic counterexamples.

Theorem 3.1 can be also viewed as an extension of Fink's theorem to perfect matchings with one additional edge. In this equivalent formulation it says that $P \cup \{xy\}$ is extendable whenever the three conditions of Theorem 3.1 hold. We can provide another extension of Fink's theorem to perfect matchings with one edge missing; note that this is not an easy corollary of Theorem 3.1.

Theorem 3.2. *Every matching of $B(Q_n)$ which covers all but two vertices of Q_n is extendable.*

4. Small matchings

As far as arbitrary matchings are concerned, a positive solution to the Ruskey-Savage problem was obtained only for matchings of linear size, the most recent upper bound being $3n-10$ due to Wang and Zhang [23]. We are able to improve this result in two ways.

Firstly, matchings of quadratic size were studied by the second author of this paper in [3]. However, they were only extended to long cycles which need not necessarily cover all the vertices.

Theorem 4.1 ([3]). *For every matching M in $K(Q_n)$ of size at most $\frac{n^2}{12}+\frac{n}{4}$ there is a set $S \subseteq E(Q_n)$ such that $M \cup S$ forms a cycle in $K(Q_n)$ of length at least $\frac{3}{4}|V(Q_n)|$.*

Building on this result, we derive another quadratic upper bound on the size of a matching, but this time the matching extends to a Hamiltonian cycle.

Theorem 4.2 ([4]). *Every matching in Q_n of size at most $\frac{n^2}{16}+\frac{n}{4}$ is extendable.*

5. Other variants of the problem

Vandenbussche and West [22] conjectured that every matching in the hypercube extends into a 2-factor, which is a weaker variant of the Ruskey-Savage problem. We prove this conjecture.

Theorem 5.1 ([8]). *Every matching in the hypercube can be extended into a 2-factor.*

The basic idea of the proof of Theorem 5.1 is considering the hypercube Q_n as the Cartesian product $Q_2 \,\square\, Q_{n-2}$. The only property of Q_{n-2} used in the proof is the bipartiteness of Q_{n-2}. Therefore, it proves in fact a stronger statement: every matching of $Q_2 \,\square\, G$ can be extended into a 2-factor for every bipartite graph G. Furthermore, with a more involved argument we can show that a similar statement holds for non-bipartite G as well.

Theorem 5.2. *Every matching in $Q_2 \,\square\, G$ can be extended into a 2-factor for every graph G.*

One may ask whether the graph Q_2 in Theorem 5.2 can be replaced by some other graph. In general, the characterization of all graphs H such that every matching of $H \,\square\, G$ can be extended into a 2-factor for every graph G is still open.

On the other hand, inspired by the previous results, namely Theorems 3.1, 3.2 and 4.1, we may be tempted to suggest a stronger variant of the Ruskey-Savage problem as follows: Is it true that every matching in $B(Q_n)$ is extendable? We can show that the answer is positive only for small values of n.

Theorem 5.3 ([4]). *Every balanced matching M in $K(Q_n)$ such that $|M| \leq \max(8, n-1)$ is extendable.*

Hence every matching in $B(Q_n)$ is extendable for $n \leq 4$. This is, however, not true in general, as demonstrated by the following result.

Theorem 5.4 ([4]). *For every $n \geq 9$ there is a matching $M \subseteq E(B(Q_n)) \setminus E(Q_n)$ of size $2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}+1$ which is not extendable.*

Note that the matching of size $\Theta(2^n/\sqrt{n})$ from Theorem 5.4 lies in $B(Q_n)$ but not in Q_n. Hence it does not answer the Ruskey-Savage problem which still remains open.

6. Algorithms

When the existence of a combinatorial object is proven, it is natural to ask if there also exists an efficient algorithm for finding this object. For example, this paper is based on the existence of Hamiltonian cycles in hypercubes, while a survey of efficient algorithms that generate such cycles may be found in [14]. When Mütze [17] proved the Hamiltonicity of the middle levels graph, he and Nummenpalo [18] also presented an algorithm that generates such a Hamiltonian cycle in an efficient way.

Motivated by these natural algorithmic questions, Knuth [15] asked whether a Hamiltonian cycle extending a given perfect matching of the hypercube can be found in linear time. We present such an algorithm [9] using the Random Access Machine (RAM) with n-bit words as the model of computation.

Theorem 6.1 ([9])**.** *There exists an algorithm which, given a perfect matching P of $K(Q_n)$, finds a perfect matching R of Q_n such that $P \cup R$ is a Hamiltonian cycle of $K(Q_n)$. The time complexity on n-bit word RAM is linear in the number of vertices of the hypercube Q_n.*

However, this algorithm does not generate edges of a Hamiltonian cycle extending a given perfect matching in the order in which these edges lie on the cycle. Although it is possible to store all generated edges and then list them in the appropriate order, this is not an efficient way to find only a few first edges. On the other hand, efficient algorithms for finding edges in the order of a Hamiltonian cycle are known both for the hypercube [5, 14] and for the middle levels graph [18].

Motivated by this fact, we develop an algorithm which

- given a perfect matching P of Q_n
- iteratively finds a perfect matching R of Q_n
- so that edges of R are generated in the order of the Hamiltonian cycle $P \cup R$
- and the first k edges, $k = 1, 2, \dots, 2^n$, are found in time $\mathcal{O}(k^2 \operatorname{poly}(n))$.

Since the size of the given perfect matching P is exponential in n, it is assumed that P is given by an oracle which for a given vertex u of Q_n returns the edge uv of P.

Theorem 6.2 ([9])**.** *There exists an algorithm which for a perfect matching P of $K(Q_n)$ given by an oracle finds a perfect matching $R = \{u_1v_1, \dots, u_{2^{n-1}}v_{2^{n-1}}\}$ of Q_n such that $P \cup R$ forms a Hamiltonian cycle of $K(Q_n)$ and $v_i u_{i+1} \in P$ for every $i = 1, \dots, 2^{n-1} - 1$. Furthermore, the algorithm finds the first k edges $u_1v_1, \dots, u_kv_k$ in time $\mathcal{O}(k^2 \operatorname{poly}(n))$.*

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