Gray codes with bounded weights

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Abstract

Given a set H of binary vectors of length n, is there a cyclic listing of H so that every two successive vectors differ in a single coordinate? The problem of existence of such a listing, which is called a *cyclic Gray code* of H, is known to be NP-complete in general. The goal of this paper is therefore to specify boundaries between its intractability and polynomial decidability.

For that purpose, we consider a restriction when the vectors of H are of a bounded weight. A *weight* of a vector $u \in \{0,1\}^n$ is the number of 1's in u. We show that if every vertex of H has weight k or $k+1$, our problem is polynomial for $k \leq 1$ and NP-complete for $k \geq 2$. Furthermore, if $k = 2$ and for every $i \in [n]$ there are at most m vectors of H of weight two having one in the i-th coordinate, then the problem becomes polynomial for $m \leq 3$ and NP-complete for $m \geq 13$.

The following complementary problem is also known to be NP-hard: given an $F \subseteq \{0,1\}^n$, which now plays the role of a set of faults to be avoided, is there a cyclic Gray code of $\{0,1\}^n \setminus F$? We show that if every vertex of F has weight at most k, the problem is polynomial for $k \leq 2$ and NP-hard for $k \geq 5$. It follows that there is a function $f(n) = \Theta(n^4)$ such that the existence of a cyclic Gray code of $\{0,1\}^n \setminus F$ for a given set $F \subseteq \{0,1\}^n$ of size at most $f(n)$ is NP-hard.

In addition, we study the cases when the Gray code does not have to be cyclic, and moreover, when the first and the last vectors of the code are prescribed. For these two modifications, all NP-hardness and NP-completeness results hold as well.

Keywords: Gray code, faulty vertex, hypercube, Hamiltonian path, Hamiltonian cycle, NP-hard, polynomial algorithm

1. Introduction

Let H be a set of binary vectors of length n. Is there a (cyclic) listing of all vectors of H so that every two successive vectors differ in a single coordinate? Such a listing, which corresponds to a Hamiltonian path (cycle) of the subgraph of the n-dimensional hypercube induced by H, is called a *(cyclic) Gray code* of H [13]. This problem, which has applications in the field of data compression [8, 11], is already known to be NP-complete [4]. Our main goal is therefore to specify boundaries between its intractability and polynomial decidability.

For that purpose, we consider a restriction when the vectors are of a bounded weight. A *weight* $|u|$ of a vector u of $\{0,1\}^n$ is the number of 1's in u. Using the graph-theoretic terminology, our problem may be formulated as follows: The *n-dimensional hypercube* Q_n is the graph with all n-bit vectors as vertices, an edge joining two vertices whenever they differ in exactly one bit. Let $\mathcal{L}_{a,b}$ be the family of all subgraphs of Q_n induced by vertices of weight at least a and at most b, where $n \geq b > a$. For a class of graphs $\mathcal{C}$ let HC($\mathcal{C}$), HP($\mathcal{C}$), HPE($\mathcal{C}$) be the decision problems whether a given graph from the class $\mathcal{C}$ has a Hamiltonian cycle, a Hamiltonian path, a Hamiltonian path between prescribed end-vertices, respectively.

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Theorem 1.1. *The problems* $\mathrm{HC}(\mathcal{L}_{k,k+1})$*,* $\mathrm{HP}(\mathcal{L}_{k,k+1})$ *and* $\mathrm{HPE}(\mathcal{L}_{k,k+1})$ *are* NP*-complete for* $k \geq 2$ *while they are polynomial for* $k = 0$ *and* $k = 1$*.*

Note that here we obtain a tight dichotomy. Moreover, we can show that $\mathrm{HC}(\mathcal{L}_{0,2})$, $\mathrm{HP}(\mathcal{L}_{0,2})$ and $\mathrm{HPE}(\mathcal{L}_{0,2})$ are decidable in polynomial time. On the other hand, we can provide even a more detailed insight into the complexity of $\mathrm{HC}(\mathcal{L}_{2,3})$. Let $\mathcal{L}_{2,3}^{m}$ be the family of all $H \in \mathcal{L}_{2,3}$ such that for every $d \in \{1, 2, \ldots, n\}$, at most m vertices of H of weight two have its d-th coordinate equal to one.

Theorem 1.2. *The problem* $\mathrm{HC}(\mathcal{L}_{2,3}^{13})$ *is* NP*-complete, while* $\mathrm{HC}(\mathcal{L}_{2,3}^{3})$ *is polynomial.*

Motivated by the study of fault-tolerance of hypercubic interconnection networks [15], we also consider the following complementary problem: If we remove from Q_n a given set F of faulty vertices, does the resulting graph $Q_n - F$ still contain a Hamiltonian cycle?

If the number of removed vertices is small, the answer is known. Clearly, it is necessary that the set F is balanced in the sense that it contains the same number of vertices from each class of bipartition of Q_n. Locke [10] conjectured that $Q_n - F$ contains a Hamiltonian cycle for every balanced set F with $|F| \leq 2n - 4$ and proved it for $|F| = 2$. Dvořák and Gregor [4] verified it for $|F| \leq \frac{n-5}{3}$. Furthermore, Gregor and Škrekovski [6] showed that it is possible to get far beyond the Locke's bound, if F forms a linear code with odd minimum distance at least 3, or if F induces a matching in Q_n with minimum distance at least 3. They also conjecture that $Q_n - F$ contains a Hamiltonian cycle for every balanced set F of vertices with minimum distance at least 3.

Let $\mathcal{F}_k$ be the family of all graphs $Q_n - F$ where $n \geq k$ and F is a set of vertices of Q_n of weight at most k. Let $\mathrm{FHC}(\mathcal{F}_k)$, $\mathrm{FHP}(\mathcal{F}_k)$ and $\mathrm{FHPE}(\mathcal{F}_k)$ be the decision problems whether for a given (n, F) where $Q_n - F \in \mathcal{F}_k$, the graph $Q_n - F$ contains a Hamiltonian cycle, Hamiltonian path and Hamiltonian path between given vertices, respectively. Note that the input of these problems consists of a pair (n, F) and therefore the number of vertices of $Q_n - F$ is not bounded by a polynomial function with respect the input.

It was shown in [4] that the problems $\mathrm{FHC}(\mathcal{F})$, $\mathrm{FHP}(\mathcal{F})$ and $\mathrm{FHPE}(\mathcal{F})$ where $\mathcal{F} = \bigcup_{k \in \mathbb{N}} \mathcal{F}_k$ are NP-hard. In this paper, we provide a further refinement to the complexity of these problems.

Theorem 1.3. *The problems* $\mathrm{FHC}(\mathcal{F}_k)$*,* $\mathrm{FHP}(\mathcal{F}_k)$ *and* $\mathrm{FHPE}(\mathcal{F}_k)$ *are* NP*-hard for* $k \geq 5$ *while the problem* $\mathrm{FHC}(\mathcal{F}_k)$ *is decidable in polynomial time for* $k \leq 2$*.*

The existence of a polynomial algorithm for $\mathrm{FHC}(\mathcal{F}_2)$ follows from the following characterization.

Theorem 1.4. *Let* $n \geq 5$ *and* F *be a subset of* $V(Q_n)$ *containing only vertices of weight at most* 2*. Then* $Q_n - F$ *contains a Hamiltonian cycle if and only if* F *is balanced and every vertex of* $Q_n - F$ *has degree at least* 2*.*

In addition, we characterize all forbidden configurations of faulty vertices for which $Q_n - F$ is not Hamiltonian.

It remains open where the dichotomy in Theorem 1.3 is. Furthermore, we believe that it may be of interest whether $\mathrm{FHC}(\mathcal{F}) \in \mathsf{NP}$. Note that the straightforward approach does not provide a non-deterministic polynomial-time algorithm, because for a given (n, F), a Hamiltonian cycle of $Q_n - F$ may have exponential length with respect to $|F|$.

The rest of this paper is laid out as follows. In Section 3 we prove the NP-completeness parts of Theorems 1.1 and 1.2, while Section 4 provides the NP-hardness part of Theorem 1.3. In Section 5 we prove Theorem 1.4, which implies the polynomial part of Theorem 1.3. The paper is concluded with Sections 6 and 7, which provide the polynomial parts of Theorems 1.1 and 1.2, respectively.

The notations and results used in this paper are presented in Section 2. Further sections may be read independently except Section 4 where a polynomial transformation continues on Section 3.

2. Preliminaries

Throughout this text, n always denotes a positive integer while $[n]$ stands for the set $\{1, 2, \ldots, n\}$. For $u, v \subseteq [n]$, let $u \triangle v$ denote the set $(u \setminus v) \cup (v \setminus u)$.

Vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Given a set $V \subseteq V(G)$, let $G[V]$ denote the subgraph of G induced by V while $G - V$ stands for $G[V(G) \setminus V]$. The distance of vertices u, v in G is denoted by $d_G(u, v)$, the subscript being omitted if no ambiguity may arise. The distance $d(\{u, v\}, \{x, y\})$ of edges $\{u, v\}, \{x, y\} \in E(G)$ is defined as $\min\{d(w, z) \mid w \in \{u, v\}, z \in \{x, y\}\}$.

A path with endvertices a and b is denoted by P_{a-b}. In particular, P_{a-a} denotes the path consisting of a single vertex a. A path P is called a *subpath* of a path P' (of a cycle C) if P forms a subgraph of P' (of C).

The *n-dimensional hypercube* Q_n is usually defined as the graph with all binary vectors of length n as vertices and edges joining every two vertices that differ in exactly one coordinate. However, in this paper we employ an alternative definition which says that Q_n is the graph with all subsets of $[n]$ as vertices and edges joining every two vertices $a, b \subseteq [n]$ such that $|a \triangle b| = 1$. Note that $\emptyset$ and $[n]$ are (antipodal) vertices whose binary representations consist only of zeros and ones, respectively. To simplify the notation, sometimes we denote vertex $\{u_1, u_2, \ldots, u_k\}$ of Q_n simply by a string $u_1 u_2 \cdots u_k$.

The *direction* $\mathrm{dir}(u, v)$ of an edge $\{u, v\}$ of Q_n is defined by $\mathrm{dir}(uv) = u \triangle v$. Given a vertex $v \in V(Q_n)$

- the size $|v|$ of v is called the *weight* of v,
- the *parity* $p(v)$ of v is defined by $p(v) = |v| \mod 2$,
- v^d denotes the vertex $v \triangle \{d\}$.

Note that Q_n is a bipartite graph whose partite classes are formed by vertices of even and odd parities.

For every $d \in [n]$ let $Q^{L,d}$ and $Q^{R,d}$ denote the subgraphs of Q_n induced by the sets $\{v \in V(Q_n) \mid d \notin v\}$ and $\{v \in V(Q_n) \mid d \in v\}$, respectively. The symbols L and R stand for 'left' and 'right', which corresponds to the presentation of $Q^{L,d}$ and $Q^{R,d}$ in our figures. Note that both $Q^{L,d}$ and $Q^{R,d}$ are isomorphic to Q_{n-1}.

Let F be a set of *faulty vertices* of Q_n. Vertices of $V(Q_n) \setminus F$ are then called *fault-free* or *healthy*. For every $d \in [n]$ and $1 \le i \le j \le n$ we put

$$\begin{aligned}
&F^{L,d} = F \cap V(Q^{L,d}), \quad F^{R,d} = F \cap V(Q^{R,d}),\\
&F_{i,j} = \{x \in F \mid i \le |x| \le j\}, \quad F_i = F_{i,i},\\
&F^{L,d}_{i,j} = F^{L,d} \cap F_{i,j}, \quad F^{L,d}_i = F^{L,d}_{i,i}.
\end{aligned}$$

F is called *balanced* if

$$|\{x \in F \mid p(x) = 0\}| = |\{x \in F \mid p(x) = 1\}| \ .$$

Let D^k_n, L^k_n and U^k_n denote the set of vertices of Q_n of weight at most k, exactly k and at least k, respectively.

Akiyama, Nishizeki and Saito [14] proved that the problem of deciding whether a 2-connected cubic bipartite planar graph or a 3-connected cubic bipartite graph contains a Hamiltonian cycle is NP-complete. For our purposes the following statement is sufficient. Let $\mathcal{B}$ be the class of all cubic bipartite graphs.

Theorem 2.1 (Akiyama at. al. [14]). *The problem* $\mathrm{HC}(\mathcal{B})$ *is NP-complete.*

An obvious necessary condition for the existence of a Hamiltonian path or cycle in a faulty hypercube may be formulated in terms of balance [4].

Proposition 2.2. *If $Q_n - F$ contains a Hamiltonian cycle, then F is balanced. If $Q_n - F$ contains a Hamiltonian path P_{u-v}, then*

- *either $p(u) \neq p(v)$ and F is balanced,*
- *or $p(u) = p(v)$ and $F \cup \{u\}$ is balanced.*

The following well-known folklore result on Hamiltonicity of hypercubes may be found e. g. in [7].

Lemma 2.3 ([7])**.** *The hypercube Q_n contains a Hamiltonian path P_{u-v} for every pair $u, v \in V(Q_n)$ such that $p(u) \neq p(v)$.*

There is a number of generalizations of this property to faulty hypercubes [2, 4]. We shall do only with two special cases: the case of one faulty vertex [9] and that of two adjacent faulty vertices [3].

Lemma 2.4 ([9])**.** *Let u, v, w be pairwise distinct vertices of Q_n such that $p(u) = p(v) \neq p(w)$. Then there is a Hamiltonian path P_{u-v} of $Q_n - \{w\}$.*

Lemma 2.5 ([3])**.** *Let $n \geq 3$ and x, y, u, v be pairwise distinct vertices of Q_n such that $\{x, y\} \in E(Q_n)$ and $p(u) \neq p(v)$. Then,*

- *either there exists a Hamiltonian path P_{u-v} of $Q_n - \{x, y\}$,*
- *or $n = 3$, $\{u, v\} \in E(Q_n)$ and $d(\{u, v\}, \{x, y\}) = 2$.*

3. Prescribed vertices of weight 2 and 3

In this section we prove that problems HC($\mathcal{L}_{k,k+1}$), HP($\mathcal{L}_{k,k+1}$) and HPE($\mathcal{L}_{k,k+1}$) are NP-complete for $k \geq 2$. This section first present a polynomial transformation from HC($\mathcal{B}$) to HC($\mathcal{L}_{2,3}$) and then it is simply generalized to other problems including HC($\mathcal{L}_{2,3}^{13}$).

Let $G = (A, B, E)$ be a given cubic bipartite graph on vertices $A \cup B$ and edges E joining vertices A and B. Our aim is to find a set of vertices V of the hypercube Q_n of weight 2 and 3 such that the graph $Q_n[V]$ contains a Hamiltonian cycle if and only if G contains a Hamiltonian cycle, where $n = \Theta(|V(G)|)$.

Every vertex $u \in A$, whose neighbours are a, b and c, is replaced by the gadget $GA(u)$; see Figure 1. Similarly, every vertex $a \in B$, whose neighbours are u, v and w, is replaced by the gadget $GB(u)$; see Figure 2. Gadgets in $Q_n[V]$ are interconnected by ports in the same way as vertices in G; see Figure 3.

In both gadgets we use two types of letters for directions. The first type is labeled by the Latin alphabet and it corresponds to vertices of the graph G. The second type is labeled the Greek alphabet. Let us point out that Greek letters mean different directions in different gadgets. Formally, we should use $\alpha^u_{a,b,c}$ instead of α in Figure 1 to emphasise that the direction α in $GA(u)$ is different from directions α in $GB(a)$ and $GA(v)$ etc. but the notation $\alpha^u_{a,b,c}$ would be very inconvenient.

Lemma 3.1. *Every gadget has Hamiltonian paths between every pair of ports.*

Proof. The path between ua and uc in the gadget $GA(u)$ is ua, $ua\gamma$, $u\gamma$, $uc\gamma$, $c\gamma$, $c\gamma\delta$, $c\delta$, $uc\delta$, $u\delta$, $ua\delta$, $a\delta$, $a\alpha\delta$, $a\alpha$, $ua\alpha$, $u\alpha$, $ub\alpha$, ub, $ub\delta$, $b\delta$, $b\beta\delta$, $b\beta$, $ub\beta$, $u\beta$, $uc\beta$, uc. The other two paths between ports of the gadget $GA(u)$ follows from symmetry.

Paths in the gadget $GB(a)$ are:

- $au\alpha$, $u\alpha$, $u\alpha\beta$, $\alpha\beta$, $w\alpha\beta$, $w\alpha$, $w\alpha\gamma$, $w\gamma$, $aw\gamma$, $a\gamma$, $a\beta\gamma$, $\beta\gamma$, $w\beta\gamma$, $w\beta$, $w\beta\epsilon$, $\beta\epsilon$, $a\beta\epsilon$, $a\beta$, $a\alpha\beta$, $a\alpha$, $a\alpha\delta$, $\alpha\delta$, $v\alpha\delta$, $v\delta$, $v\beta\delta$, $v\beta$, $av\beta$;

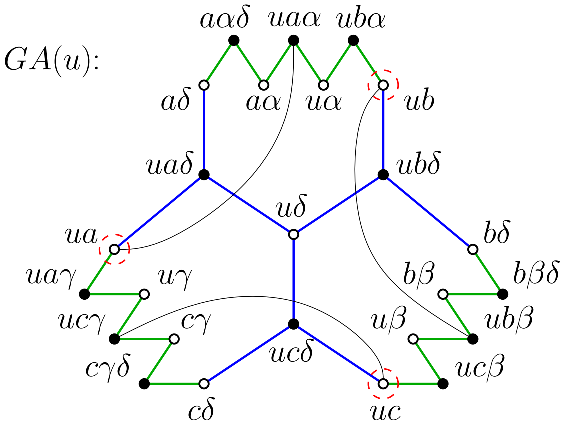


Figure 1: The gadget $GA(u)$ for a vertex $u \in A$ whose neighbours are a, b and c. The marked vertices ua, ub and uc are the ports.

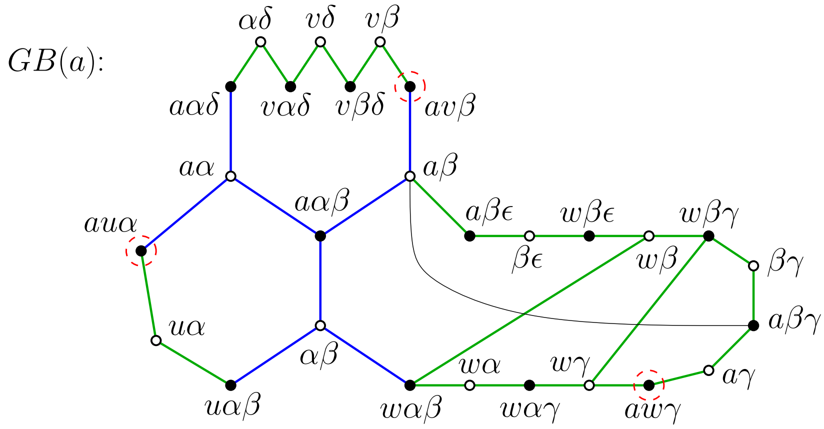


Figure 2: The gadget $GB(a)$ for a vertex $a \in B$ whose neighbours are u, v and w. The marked vertices $au\alpha$, $av\beta$ and $aw\gamma$ are the ports.

- *$au\alpha$, $u\alpha$, $u\alpha\beta$, $\alpha\beta$, $a\alpha\beta$, $a\alpha$, $a\alpha\delta$, $\alpha\delta$, $v\alpha\delta$, $v\delta$, $v\beta\delta$, $v\beta$, $av\beta$, $a\beta$, $a\beta\epsilon$, $\beta\epsilon$, $w\beta\epsilon$, $w\beta$, $w\alpha\beta$, $w\alpha$, $w\alpha\gamma$, $w\gamma$, $w\beta\gamma$, $\beta\gamma$, $a\beta\gamma$, $a\gamma$, $aw\gamma$;*
- *$av\beta$, $v\beta$, $v\beta\delta$, $v\delta$, $v\alpha\delta$, $\alpha\delta$, $a\alpha\delta$, $a\alpha$, $au\alpha$, $u\alpha$, $u\alpha\beta$, $\alpha\beta$, $a\alpha\beta$, $a\beta$, $a\beta\epsilon$, $\beta\epsilon$, $w\beta\epsilon$, $w\beta$, $w\alpha\beta$, $w\alpha$, $w\alpha\gamma$, $w\gamma$, $w\beta\gamma$, $\beta\gamma$, $a\beta\gamma$, $a\gamma$, $aw\gamma$.*

□

Gadgets in the graph $Q_n[V]$ are interconnected in a straightforward way through ports; see Figure 3. Recall that every vertex of $u \in A$ is replaced by the gadget $GA(u)$ of $Q_n[V]$ and every vertex $a \in B$ is replaced by the gadget $GB(a)$ of $Q_n[V]$. Every edge $\{a, u\}$ of G is replaced by the edge of $Q_n[V]$ that connects ports ua of $GA(u)$ and $ua\alpha$ of $GB(a)$. Note that there is an one-to-one correspondence between edges of G and edges connecting ports of $Q_n[V]$.

Lemma 3.2. *If G contains a Hamiltonian cycle C, then $Q_n[V]$ contains a Hamiltonian cycle C'.*

Proof. The cycle C' goes through all gadgets of $Q_n[V]$ in the same order as the cycle C goes through all corresponding vertices of G. Lemma 3.1 gives us paths in gadgets between all pairs of ports. □

Lemma 3.3. *There is no pair of gadgets sharing the same vertex. The only edges joining vertices of different gadgets correspond to edges of G.*

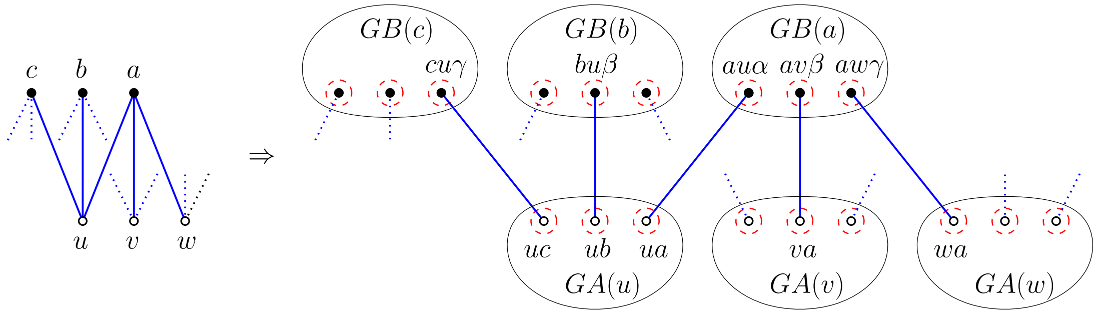


Figure 3: The interconnection of gadgets. Left figure presents a part of graph G and right figure presents corresponding part of graph $Q_n[V]$.

Proof. Observe on Figures 1 and 2 that every vertex of $Q_n[V]$ except ports has at least one Greek dimension which occurs only in its gadget. Every port vertex is determined by the corresponding edge in G. So, no vertex is shared by more gadgets.

Since $Q_n[V]$ is bipartite, every edge joins a vertex x of weight 2 with a vertex y of weight 3. If x is not port, then it has at least one Greek dimension, which implies that y shares at least one Greek dimension with x and both vertices belong into the same gadget. If x is a port of gadget $GA(u)$, then x is adjacent to three vertices of its gadget and one port of adjacent gadget. Therefore, adjacent vertices x and y belong to different gadgets only if edge $\{x, y\}$ of $Q_n[V]$ has a corresponding edge of G. □

Lemma 3.4. *If $Q_n[V]$ contains a Hamiltonian cycle C', then G contains a Hamiltonian cycle C.*

Proof. From Lemma 3.3 it follows that C' cannot visit any gadget twice, but it has to go through all vertices of the gadget and then continue through all vertices of an adjacent gadget and so on. Therefore, we obtain a Hamiltonian cycle C in G from C' by contracting all vertices of the same gadget. □

Let us consider one fix gadget $GB(a)$ of $Q_n[V]$ and its vertices $v\alpha\delta$, $v\delta$ and $v\beta\delta$. Let $V' = V \setminus \{v\delta\}$.

Lemma 3.5. *The following statements are equivalent.*

- *The graph G has a Hamiltonian cycle.*
- *The graph $Q_n[V]$ has a Hamiltonian cycle.*
- *The graph $Q_n[V']$ has a Hamiltonian path.*
- *The graph $Q_n[V']$ has a Hamiltonian path between vertices $v\alpha\delta$ and $v\beta\delta$.*

Proof. Lemmas 3.2 and 3.4 states that G is Hamiltonian if and only if $Q_n[V]$ is Hamiltonian. Since vertices $v\alpha\delta$, $v\delta$ and $v\beta\delta$ have degree two in $Q_n[V]$, all statements are equivalent. □

Theorem 3.6. *The problems* HC($\mathcal{L}_{2,3}$), HP($\mathcal{L}_{2,3}$) *and* HPE($\mathcal{L}_{2,3}$) *are NP-complete.*

Proof. Given a sequence of vertices of a graph of $\mathcal{L}_{2,3}$, we can easily verify in polynomial time whether the sequence forms a Hamiltonian cycle or Hamiltonian path (between prescribed end-vertices). Hence, the problems HC($\mathcal{L}_{2,3}$), HP($\mathcal{L}_{2,3}$) and HPE($\mathcal{L}_{2,3}$) belong into NP.

Since $Q_n[V]$ and $Q_n[V']$ are of polynomial size with respect to G, we have a polynomial reduction from the problem HC($\mathcal{B}$) to the problems HC($\mathcal{L}_{2,3}$), HP($\mathcal{L}_{2,3}$) and HPE($\mathcal{L}_{2,3}$) by Lemma 3.5. From Theorem 2.1 it follows that those problems is NP-complete. □

Since vertices of Q_n are subsets of $[n]$, the union $\cup$ of two vertices (or sets of directions) is also a vertex of Q_n.

Corollary 3.7 (The first part of Theorem 1.1). *The decision problems* $\mathrm{HC}(\mathcal{L}_{k,k+1})$, $\mathrm{HP}(\mathcal{L}_{k,k+1})$ *and* $\mathrm{HPE}(\mathcal{L}_{k,k+1})$ *are NP-complete for every* $k \geq 2$.

Proof. Let $Q_n[V]$ be a graph of $\mathcal{L}_{2,3}$. Let $D = [n+k-2] \setminus [n]$ and $V^\star = \{i \cup D;\ i \in V(Q_n[V])\}$. Note that vertices i and j are adjacent in $Q_n[V]$ if and only if $i \cup D$ and $j \cup D$ are adjacent in $Q_{n+k-2}[V^\star]$. Hence, the graphs $Q_n[V]$ and $Q_{n+k-2}[V^\star]$ are isomorphic. The rest of the statement follows. □

Note that there exists a constant m independent on the graph G such that for every $d \in [n]$ there are at most m vertices of $v \in V$ satisfying $d \in v$. Moreover, it is possible to set up ports of our gadgets in such a way that for every $d \in [n]$ there are at most 13 vertices $v \in V$ of weight 2 satisfying $d \in v$. This proves that $\mathrm{HC}(\mathcal{L}_{2,3}^{13})$ is NP-complete which is stated by the first part of Theorem 1.2.

4. Faulty vertices of weight at most 5

In this section we show that the decision problems $\mathrm{FHC}(\mathcal{F}_k)$, $\mathrm{FHP}(\mathcal{F}_k)$ and $\mathrm{FHPE}(\mathcal{F}_k)$ are NP-hard for $k \geq 5$. We use the construction from the previous section which transforms a cubic bipartite graph $G = (A, B, E)$ with $|A| = |B| = k$ into graph $Q_n[V']$. Lemma 3.5 states that G contains a Hamiltonian cycle if and only if $Q_n[V']$ contains a Hamiltonian path between vertices $v\alpha\delta$ and $v\beta\delta$. We find a set $F \subseteq D_n^5$ such that $Q_n{-}F$ contains a Hamiltonian cycle if and only if G contains a Hamiltonian cycle.

We remove all vertices of L_n^4 from Q_n except two vertices p and q which are neighbors of $v\alpha\delta$ and $v\beta\delta$, respectively. Then, we find a set $Z \subseteq L_n^5$ such that $Q_n[U_n^5 \setminus Z]$ has a Hamiltonian path P between two vertices r and s of L_n^5 which are neighbors of p and q, respectively. Finally, $F = (Z \cup D_n^4) \setminus (V' \cup \{p, q\})$ is the set of faulty vertices. Note, that G contains a Hamiltonian cycle if and only if $Q_n{-}F$ contains a Hamiltonian cycle.

Hence, our next aim is to study the following problem. Assume that r and s are two vertices of L_n^m where $n \geq 3$ and $1 \leq m \leq n-1$. Is there a set $Z \subseteq L_n^m$ such that the induced subgraph $Q_n[U_n^m \setminus Z]$ has a Hamiltonian path between r and s? Dvořák and Koubek [5] give a positive answer to this question (even in a stronger form). Since we are interested in algorithmic aspects, we present here also a (simplified) proof to show that the set Z can be found in $\mathcal{O}\left(n\binom{n}{m}\right)$ time.

For the purpose of induction, one needs a stronger statement with an additionally prescribed second vertex t for the Hamiltonian path. The following auxiliary proposition solves the base configurations.

Proposition 4.1. *Let* $n \geq 3$ *and* $m \in \{1, n-1\}$. *Let* r, s *be distinct vertices of* L_n^m *and let* $t \in L_n^{m+1}$ *be a neighbor of the vertex* r. *Then there exists a set* $Z \subseteq L_n^m$ *such that* $Q_n[U_n^m \setminus Z]$ *contains a Hamiltonian path* $P = (r, t, \ldots, s)$.

Proof. If $m = 1$, then we put $Z = \emptyset$. By the result of Dvořák [3] on Hamiltonian cycles in Q_n with prescribed edges, there exists a Hamiltonian cycle C of Q_n containing $(t, r, \emptyset, s)$ as a subpath. By removing the vertex $\emptyset$ from C we obtain the desired path P. If $m = n-1$, then $t = [n]$, and we put $Z = L_n^{n-1} \setminus \{r, s\}$ and $P = (r, t, s)$. □

Note that the set Z is constructed in $\mathcal{O}(1)$ time if $m = 1$, and in $\mathcal{O}(n)$ time if $m = n-1$. Next, we consider the general case.

Lemma 4.2. *Let* $n \geq 3$ *and* $1 \leq m < n$. *Let* r, s *be distinct vertices of* L_n^m *and let* $t \in L_n^{m+1}$ *be a neighbor of the vertex* r *such that* $t \neq r \cup s$. *There is an algorithm running in* $\mathcal{O}\left(n\binom{n}{m}\right)$ *time which finds a set* $Z \subseteq L_n^m$ *such that* $Q_n[U_n^m \setminus Z]$ *contains a Hamiltonian path* $P = (r, t, \ldots, s)$.

Proof. We proceed by induction on n and m. If $m = 1$ or $m = n-1$, then the statement holds by Proposition 4.1. Now we assume that $2 \leq m \leq n-2$, so $n \geq 4$. See Figure 4(a) for an illustration. Since $|t| = m+1 < n$ and $t \neq r \cup s$, there exists $d \in [n]$ such that $r, t \in V(Q^{L,d})$ and $s \in V(Q^{R,d})$.

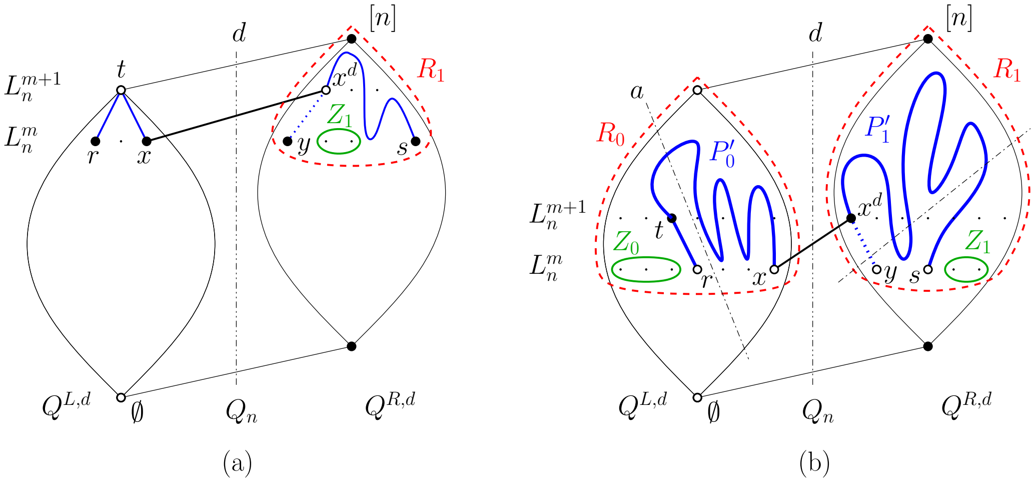


Figure 4: An illustration for Lemma 4.2: (a) the case $m = n - 2$, (b) the case $2 \leq m \leq n - 3$.

If $m = n - 2$ then there exists $x \in V(Q^{L,d})$ such that $x \neq r$, $|x| = m$ and $s \nsubseteq x^d$. Note that x and t are adjacent. Choose $y \in V(Q^{R,d})$ with $|y| = m$ and $y \subseteq x^d$. Let $R_1 = U_n^m \cap V(Q^{R,d})$, and let $\phi : V(Q_n) \to V(Q_{n-1})$ be the mapping defined by $\phi(a) = a \setminus \{d\}$. Note that ϕ is an isomorphism of $Q^{R,d}$ onto Q_{n-1}, and $\phi(R_1) = U_{n-1}^{m-1}$.

Hence by induction, there is a set $Z_1 = L_n^m \cap V(Q^{R,d})$ such that $Q^{R,d}[R_1 \setminus Z_1]$ has a Hamiltonian path $P_1 = (y, x^d, P_1', s)$. Observe that for the set $Z = (L_n^m \cap V(Q^{L,d}) \setminus \{x, r\}) \cup Z_1 \cup \{y\}$, the path $P = (r, t, x, x^d, P_1', s)$ is the desired Hamiltonian path of $Q_n[U_n^m \setminus Z]$. Thus we can assume that $2 \leq m \leq n - 3$.

Then there exist $a \in [n]$ such that $a \notin t$ and $a \neq d$, and $b \in [n]$ such that $b \in s$ and $b \neq a, d$. Select $x \in V(Q^{L,d})$ such that $|x| = m$, $a \in x$ and $b \notin x$. Then $t \neq r \cup x$ and $s \nsubseteq x^d$. Let $y \neq s$ be an arbitrary neighbor of x^d in $Q^{R,d}$ with $|y| = |s| = m$ then $x^d \neq y \cup s$.

Let $R_0 = U_n^m \cap V(Q^{L,d})$, $R_1 = U_n^m \cap V(Q^{R,d})$, and let $\phi : V(Q_n) \to V(Q_{n-1})$ be the mapping defined by $\phi(a) = a \setminus \{d\}$. Note that ϕ is an isomorphism of $Q^{L,d}$, $Q^{R,d}$ onto Q_{n-1}, and $\phi(R_0) = U_{n-1}^m$, $\phi(R_1) = U_{n-1}^{m-1}$.

Hence by induction, there are sets $Z_0 = L_n^m \cap V(Q^{L,d})$ and $Z_1 = L_n^m \cap V(Q^{R,d})$ such that $Q^{L,d}[R_0 \setminus Z_0]$ has a Hamiltonian path $P_0 = (r, t, P_0', x)$, and $Q^{R,d}[R_1 \setminus Z_1]$ has a Hamiltonian path $P_1 = (y, x^d, P_1', s)$. Observe that for the set $Z = Z_0 \cup Z_1 \cup \{y\}$, the path $P = (r, t, P_0', x, x^d, P_1', s)$ is the desired Hamiltonian path of $Q_n[U_n^m \setminus Z]$.

This provides a recursive algorithm to construct the set Z (without constructing the path P). Since the size of Z is bounded by $\mathcal{O}\left(\binom{n}{m}\right)$, it runs in time

$$T(n, m) = T(n-1, m) + T(n-1, m-1) + \mathcal{O}\left(\binom{n}{m}\right),$$
$$T(n, 1) = \mathcal{O}(1), \quad T(n, n-1) = \mathcal{O}(n).$$

Therefore, it follows directly that $T(n, m) = \mathcal{O}\left(n\binom{n}{m}\right)$. □

Note that the algorithm from Lemma 4.2 runs in polynomial time if m is constant. In order to prove that FHP($\mathcal{F}^5$) and FHPE($\mathcal{F}^5$) are NP-hard it suffices to process as in previous section: Let $F' = F \cup \{\beta\epsilon\}$, where $\beta\epsilon$ is a vertex of the fixed gadget $GB(a)$ of $Q_n[V]$. Note on Figure 2 that vertices $\alpha\beta\epsilon$, $\beta\epsilon$ and $w\beta\epsilon$ have degree 2. Hence, the following equivalent statement provides us the polynomial transformation.

- $Q_n - F$ contains a Hamiltonian cycle.
- $Q_n - F'$ contains a Hamiltonian path.

- $Q_n - F'$ contains a Hamiltonian path between vertices $\alpha\beta\epsilon$ and $w\beta\epsilon$.

Since $\mathcal{F}^5 \subseteq \mathcal{F}^k$ for $k \geq 5$, problems FHC($\mathcal{F}^k$), FHP($\mathcal{F}^k$) and FHPE($\mathcal{F}^k$) are NP-hard for $k \geq 5$. This concludes the first part of Theorem 1.3. Since a balanced $F \subseteq L_n^5$ has at most $\mathcal{O}(n^4)$ vertices, we have the following corollary.

Corollary 4.3. *There is a function $f(n) = \Theta(n^4)$ such that the following decision problem is* NP*-hard: Is the graph $Q_n - F$ Hamiltonian for given integer n and a set $F \subseteq V(Q_n)$ of size at most $f(n)$?*

5. Faulty vertices of weight at most 2

In this section we prove that FHC($\mathcal{F}_k$) is decidable in polynomial time for $k \leq 2$ (see Theorem 1.3). This statement follows from Theorem 1.4 which is a consequence of Theorem 5.4, the main result of this section.

We start with a characterization of all forbidden configurations of balanced $F \subseteq D_n^2$ which prevents the existence of a Hamiltonian cycle in $Q_n - F$.

5.1. Forbidden configurations

Now, we describe forbidden configurations of faulty vertices which, although balanced, do not allow the existence of a fault-free Hamiltonian cycle.

Note that $F = F_{1,2}$ means that F contains only vertices of weight 1 and 2, and so $\emptyset$ is healthy. For every $n \geq 2$ let

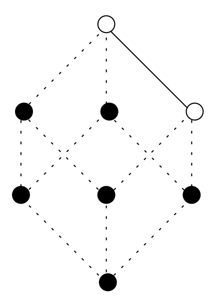
$$
\begin{aligned}
\mathrm{FC}_n^{i} &= \{F \subseteq V(Q_n) \mid F = F_{1,2}, |F_1| = n = |F_2|\},\\
\mathrm{FC}_n^{ii} &= \{F \subseteq V(Q_n) \mid F = F_{1,2}, |F_1| = n-1 = |F_2|\}\\
\mathrm{FC}_n^{iii} &= \{F \subseteq V(Q_n) \mid F = F_{0,2}, \emptyset \in F, |F_1| = n-1,\\
&\qquad F_2 \text{ consists of } n-2 \text{ neighbors of vertex } u \text{ such that } |u| = 1 \text{ and } u \notin F\}.
\end{aligned}
$$

The vertex $\emptyset$ is healthy in configurations FC_n^{i} and FC_n^{ii} but it has zero and one healthy neighbor in $Q_n - F$, respectively. In the configuration FC_n^{iii}, there is only one healthy vertex u of weight 1 but it has only one healthy neighbor which has weight 2.

Moreover, there is one special configuration for $n = 3$:

$$\mathrm{FC}_3^{e} = \{F \subseteq V(Q_3) \mid F = F_{0,2}, |F_{0,1}| = 4, |F_2| = 2\}.$$

Note that $Q_3 - F$ for $F \in \mathrm{FC}_3^{e}$ is a graph consisting of a single edge (see Figure 5); the superscript e means edge.



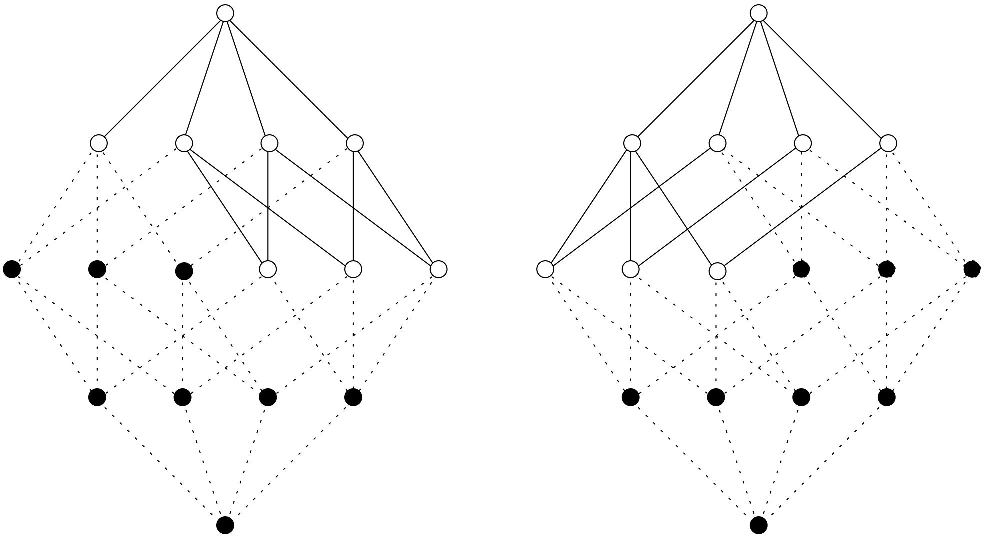
FC_3^{e}

Figure 5: A special forbidden configuration for $n = 3$ and $F \in \mathrm{FC}_3^{e}$. Faulty and healthy vertices are depicted as black and white, respectively. A solid line depicts the edge of the graph $Q_3 - F$, while the remaining edges of Q_3 are dotted.

Finally, there are two special configurations for $n = 4$:

$$\begin{aligned}
\mathrm{FC}_4^o = \{&F \subseteq V(Q_4) \mid F = F_{0,2}, |F_{0,1}| = 5,\\
&F_2 \text{ contains all vertices of weight 2 except the neighbors of some vertex } u,\ |u| = 1\},\\
\mathrm{FC}_4^t = \{&F \subseteq V(Q_4) \mid F = F_{0,2}, |F_{0,1}| = 5,\\
&F_2 \text{ consists of all neighbors of weight 2 of some vertex } u,\ |u| = 1\}.
\end{aligned}$$

Note that $Q_4 - F$ for $F \in \mathrm{FC}_4^o$ and $F \in \mathrm{FC}_4^t$ is a graph of minimum degree one and two, respectively (see Figure 6).



FC_4^o FC_4^t

Figure 6: Special forbidden configurations for $n = 4$. Black/white vertices and solid/dotted edges have the same meaning as in Figure 5.

To put all that together, for every $n \geq 2$ let

$$\mathrm{FC}_n = \begin{cases} \mathrm{FC}_n^i \cup \mathrm{FC}_n^{ii} \cup \mathrm{FC}_n^{iii} \cup \mathrm{FC}_3^e & \text{if } n = 3,\\ \mathrm{FC}_n^i \cup \mathrm{FC}_n^{ii} \cup \mathrm{FC}_n^{iii} \cup \mathrm{FC}_4^o \cup \mathrm{FC}_4^t & \text{if } n = 4,\\ \mathrm{FC}_n^i \cup \mathrm{FC}_n^{ii} \cup \mathrm{FC}_n^{iii} & \text{otherwise.} \end{cases}$$

Proposition 5.1. *If $F \in \mathrm{FC}_n$, then $Q_n - F$ is not Hamiltonian.*

Proof. $Q_n - F$ either contains a vertex of degree at most one (if $F \in \mathrm{FC}_n \setminus \mathrm{FC}_4^t$), or consists of three paths of length three and an edge, glued together at endvertices (if $F \in \mathrm{FC}_4^t$), and therefore it cannot be Hamiltonian. □

Recall the notation from Section 2, that a path with endvertices a and b is denoted by P_{a-b}.

Lemma 5.2. *Let $F \in \mathrm{FC}_3^e \cup \mathrm{FC}_4^o \cup \mathrm{FC}_4^t$. Then for every vertex $u \in Q_n - F$ of weight two there is vertex v such that $p(u) \neq p(v)$ and $Q_n - F$ contains a Hamiltonian path P_{u-v}.*

Proof. There is nothing to prove in case $n = 3$, as $Q_3 - F$ for $F \in \mathrm{FC}_3^e$ consists of a single edge. A solution to the case $n = 4$ is provided on Figure 7. □

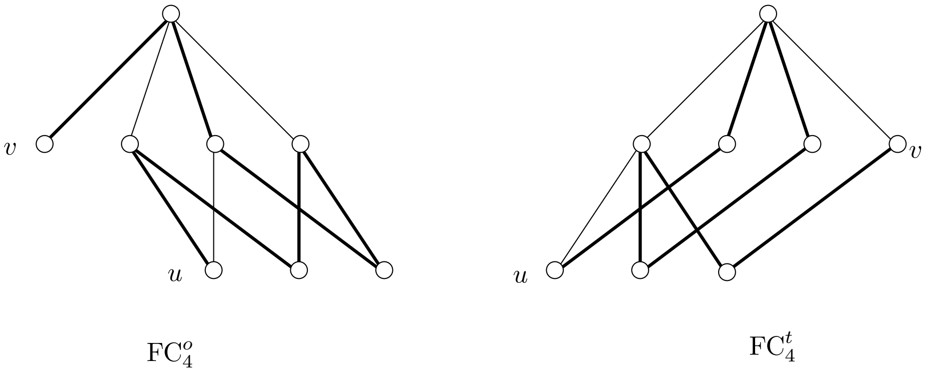


Figure 7: Proof of Lemma 5.2 in case $n = 4$. Heavy lines depict Hamiltonian paths of $Q_4 - F, F \in \mathrm{FC}_4^o \cup \mathrm{FC}_4^t$ between u and v for all (up to isomorphism) choices of vertex u of weight 2.

5.2. Main result

Lemma 5.3. *Let $n \geq 2$ and F be a balanced subset of $V(Q_n)$ such that $F = F_{0,2}$ and $F \notin \mathrm{FC}_n$. Then there exists $d \in [n]$ such that at least one of the following three conditions hold.*

(1) $F^{R,d} = \{x, y\}, |x| = 1, |y| = 2,$ *and* $F^L \notin \mathrm{FC}^i_{n-1} \cup \mathrm{FC}^{ii}_{n-1}$.

(2) $F^{R,d} = \{x\}, |x| = 1$.

(3) $F^{R,d} = \emptyset = F^{L,d}_0$, $Q_{n-1}\left[F^{L,d}_{1,2}\right]$ *is a 2-regular graph, and* $F^{L,d} \notin \mathrm{FC}^i_{n-1} \cup \mathrm{FC}^{iii}_{n-1}$.

Proof. If there is a vertex $v \in F_1$ which has at most one neighbor in F_2, put $d = \mathrm{dir}(\emptyset, v)$. Then $|F^{R,d}| \leq 2$. Moreover, in this case we have $F^{L,d} \notin \mathrm{FC}^i_{n-1} \cup \mathrm{FC}^{ii}_{n-1}$, for otherwise $F \in \mathrm{FC}^i_n \cup \mathrm{FC}^{ii}_n$, contrary to our assumption. This settles parts (1) and (2).

Otherwise each vertex of F_1 has at least two neighbors in F_2. Consequently,

$$2|F_1| \leq |\{\{u, v\} \mid u \in F_1, v \in F_2\}| \leq 2|F_2| \leq 2|F_1| \ .$$

The second inequality holds because each vertex of weight two has exactly two neighbors of weight one in Q_n. The last one follows from $|F_2| \leq |F_1|$, which is implied by the balance of F.

It follows that all the inequalities are actually equalities. In particular, $|F_1| = |F_2|$, and the balance of F implies that $\emptyset \notin F$. Moreover, each vertex of F_1 has exactly two neighbors in F_2, and each vertex of F_2 has exactly two neighbors in F_1. Consequently, $Q_n[F]$ is a two-regular graph. Since $F \notin \mathrm{FC}_n$, there must be a healthy vertex v of weight one. The two-regularity of $Q_n[F]$ implies that all neighbors of v are healthy, too. Hence putting $d = \mathrm{dir}(\emptyset, v)$ splits F so that $F^{R,d} = \emptyset$ while $Q_{n-1}[F^{L,d}] = Q_{n-1}[F^{L,d}_{1,2}]$ is a 2-regular graph. To complete this part, note that $F^{L,d} \notin \mathrm{FC}^i_{n-1} \cup \mathrm{FC}^{iii}_{n-1}$: Indeed, $F^{L,d} \notin \mathrm{FC}^{iii}_{n-1}$ since $\emptyset \notin F^{L,d}$. $F^{L,d} \in \mathrm{FC}^i_{n-1}$ is also impossible, as it would imply that $F \in \mathrm{FC}^{ii}_{n-1}$, contrary to our assumption. This settles part (3).

□

Theorem 5.4. *Let $n \geq 2$ and F be a subset of $V(Q_n)$ containing only vertices of weight at most 2. Then $Q_n - F$ contains a Hamiltonian cycle if and only if F is balanced and $F \notin \mathrm{FC}_n$.*

Proof. The necessity of the condition follows from Propositions 2.2 and 5.1. To verify the sufficiency, we argue by induction on n. Cases $n = 2, 3$ may be verified by inspection.

Let $n \geq 4$ and assume that F is balanced and $F \notin \mathrm{FC}_n$. First note that we may assume that $F_2 \neq \emptyset$. Indeed, if $F_2 = \emptyset$, then the fact that F is balanced implies that F is either empty, or consists of two adjacent vertices. In any case, the desired Hamiltonian cycle exists by Lemma 2.3 or 2.5.

Let $d \in [n]$ be the integer satisfying the conclusion of Lemma 5.3. To simplify the notation, we omit the superscript d in $Q^{L,d}, Q^{R,d}, F^{L,d}$ and $F^{R,d}$. We claim that $Q^L - F^L$ contains a Hamiltonian path P_{u-v} such that

(i) $u^d, v^d \notin F^R$, and

(ii) if $n = 4$, $\{u, v\} \in E(Q_n)$ and $F^R = \{x, y\}$, then $d\left(\left\{u^d, v^d\right\}, \{x, y\}\right) \neq 2$.

If the claim is true, we are done. Indeed, if $F^R = \{x\}$, the balance of F together with Proposition 2.2 imply that $p(u) = p(v) = p(x)$. Then $p(u^d) = p(v^d) \neq p(x)$ and therefore Lemma 2.4 guarantees the existence of a Hamiltonian path $P_{u^d - v^d}$ of $Q^R - F^R$. Otherwise Lemma 5.3 guarantees that F^R is either empty, or it consists of two adjacent vertices. Then Proposition 2.2 implies that $p(u) \neq p(v)$ and therefore there exists a Hamiltonian path $P_{u^d - v^d}$ of $Q^R - F^R$ by Lemma 2.3 or 2.5. In any case, the desired Hamiltonian cycle of $Q_n - F$ is formed by concatenation of P_{u-v} with $P_{u^d - v^d}$.

The rest of this proof is devoted to the verification of this claim. By Lemma 5.3, it suffices to consider the following three cases.

(Case 1) $F^R = \{x, y\}, |x| = 1, |y| = 2$: This means that $\{x, y\} \in E(Q_n)$. Note that then both F^L and F^R are balanced. By part (1) of Lemma 5.3, it suffices to consider the following four subcases.

(Case 1.1) $F^L \notin \mathrm{FC}_{n-1}$: Then, by the induction hypothesis, there is a Hamiltonian cycle C of $Q^L - F^L$. Since $v = [n] \setminus \{d\}$ is a healthy vertex of Q^L, Hamiltonian cycle C must pass through three consecutive vertices u, v, w. Then both pairs of vertices u, v and v, w must satisfy condition (i) and at least one of pairs u, v and v, w satisfies condition (ii). Therefore, at least one of the subpaths P_{u-v}, P_{v-w} of C is the Hamiltonian path that satisfies the claim.

(Case 1.2) $F^L \in \mathrm{FC}^{iii}_{n-1}$: Then there are adjacent vertices $u, v \in Q^L - F^L$, $|u| = 1, |v| = 2$, such that all neighbors of u in Q^L except v are faulty. Note that u^d must be healthy, for otherwise $F \in \mathrm{FC}^{iii}_n$, contrary to our assumption. Put $F' = F^L \cup \{u, v\}$.

(Case 1.2.1) $F' \notin \mathrm{FC}_{n-1}$ or $F' \in \mathrm{FC}^t_4 \cup \mathrm{FC}^e_3$: Select a neighbor w of v in $Q^L - F'$.

If $F' \notin \mathrm{FC}_{n-1}$, by the induction hypothesis, there is a Hamiltonian cycle C of $Q^L - F'$. Note that C must contain w followed by some vertex z. Let P_{w-z} be the subpath of C.

If $F' \in \mathrm{FC}^t_4 \cup \mathrm{FC}^e_3$, let P_{w-z} be a Hamiltonian path of $Q^L - F'$ which exists for some z by Lemma 5.2.

In both cases, $P_{u-z} = (u, v, P_{w-z})$ is a Hamiltonian path of $Q^L - F^L$. Moreover, u^d is healthy as noted above, while z^d is healthy because $|z^d| \geq 3$. It follows that condition (i) holds for u, z. Since $d(u, z) \geq 3$, condition (ii) holds as well. Hence P_{u-z} is the path that satisfies the claim.

(Case 1.2.2) $F' \in \mathrm{FC}_{n-1} \setminus (\mathrm{FC}^t_4 \cup \mathrm{FC}^e_3)$: This subcase cannot occur. Indeed, $F \notin \mathrm{FC}^o_4$, since F' contains vertex u of weight 1 together with all its neighbors in Q^L, while in FC^o_4, each vertex of weight one has a healthy neighbor. Next, since $\emptyset \in F'$, the set F' cannot belong to FC^i_{n-1} or FC^{ii}_{n-1}. And finally, F' contains all vertices of Q^L of weight one, and therefore $F' \notin \mathrm{FC}^{iii}_{n-1}$.

(Case 1.3) $F^L \in \mathrm{FC}^o_4 \cup \mathrm{FC}^t_4$: Here the claim holds by Lemma 5.2.

(Case 1.4) $F^L \in \mathrm{FC}^e_3$: Here $Q^L - F^L$ consists of an edge $\{u, v\}$. If $d(\left\{u^d, v^d\right\}, \{x, y\}) = 2$, then $F \in \mathrm{FC}^t_4$, contrary to our assumption. Therefore it must be the case that $d(\left\{u^d, v^d\right\}, \{x, y\}) \neq 2$ and the claim holds.

(Case 2) $F^R = \{x\}, |x| = 1$: Select a vertex $w \in F^L_2$ which exists since we assume that $F \neq \emptyset$ and put $F' = F^L \setminus \{w\}$. Note that F' is balanced.

(Case 2.1) $F' \notin \mathrm{FC}_{n-1}$: Hamiltonian cycle C of $Q^L - F'$, which exists by the induction hypothesis, passes through three consecutive vertices u, w, v. The subpath P_{u-v} of C is the desired Hamiltonian path of $Q^L - F^L$ which satisfies the claim.

(Case 2.2) $F' \in \mathrm{FC}^i_{n-1} \cup \mathrm{FC}^{ii}_{n-1}$: This subcase cannot occur, since then $F \in \mathrm{FC}^i_n \cup \mathrm{FC}^{ii}_n$, contrary to our assumption.

(Case 2.3) $F' \in \mathrm{FC}^{iii}_{n-1}$: Then there must be a vertex $u \in Q^L - F^L$ of weight 1, which has $n-3$ neighbors in F^L_2. Is suffices to select w as one of these neighbors. Then u has two neighbors of weight 2 not in F' and therefore F' cannot fall into FC^{iii}_{n-1}.

(Case 2.4) $F' \in \mathrm{FC}^t_4 \cup \mathrm{FC}^o_4$: In this case there is a vertex $u \in F$ which has only one (say w') neighbor in F_2. Setting d to $\mathrm{dir}(\emptyset, u)$ splits F so that $F^R = \{u, w'\}$ and Case 1 applies.

(Case 2.5) $F' \in \mathrm{FC}_3^e$: This subcase cannot occur, since then $F \in \mathrm{FC}_4^o$, contrary to our assumption.

(Case 3) $Q_{n-1}[F^L] = Q_{n-1}[F^L_{1,2}]$ is a 2-regular graph: By part (3) of Lemma 5.3, it suffices to consider the following three subcases.

(Case 3.1) $F^L \notin \mathrm{FC}_{n-1}$: Select an arbitrary pair u, v of consecutive vertices on a Hamiltonian cycle C of $Q^L - F'$, which exists by the induction hypothesis. The subpath P_{u-v} of C is the desired Hamiltonian path of $Q^L - F^L$ which satisfies the claim.

(Case 3.2) $F^L \in \mathrm{FC}^{ii}_{n-1}$: Let $u, v \in Q^L - F^L$, $|u| = 0, |v| = 1$. Note that as $Q_{n-1}[F^L] = Q_{n-1}[F^L_{1,2}]$ is a 2-regular graph, each faulty vertex of weight 2 has only faulty neighbors of weight 1. Consequently, vertex v has no neighbors in F^L_2. Put $F' = F^L \cup \{u, v\}$.

(Case 3.2.1) $F' \notin \mathrm{FC}_{n-1}$ or $F' \in \mathrm{FC}_4^o$: Select a neighbor w of v in $Q^L - F'$.

If $F' \notin \mathrm{FC}_{n-1}$, by the induction hypothesis there is a Hamiltonian cycle C of $Q^L - F'$. Note that C must contain w followed by some vertex z. Let P_{w-z} be the subpath of C.

If $F' \in \mathrm{FC}_4^o$, let P_{w-z} be a Hamiltonian path of $Q^L - F'$ which exists for some z by Lemma 5.2.

In both cases, $P_{u-z} = (u, v, P_{w-z})$ is the desired Hamiltonian path of $Q^L - F^L$ that satisfies the claim.

(Case 3.2.2) $F' \in \mathrm{FC}_{n-1} \setminus \mathrm{FC}_4^o$: This subcase cannot occur. Indeed, $F' \notin \mathrm{FC}^i_{n-1} \cup \mathrm{FC}^{ii}_{n-1}$, since $\emptyset \in F'$. Next, $F' \notin \mathrm{FC}^{iii}_{n-1}$, as $Q_n[F'_{1,2}]$ consists of cycles and one isolated vertex, which is not true for FC^3_{n-1}. And finally, note that $Q_n[F - \{u, v\}]$ where $F \in \mathrm{FC}_3^e \cup \mathrm{FC}_4^t$, $|u| = 0$ and $|v| = 1$, is never a 2-regular graph. Therefore $F' \notin \mathrm{FC}_3^e \cup \mathrm{FC}_4^t$ as well.

(Case 3.3) $F^L \in \mathrm{FC}_3^e \cup \mathrm{FC}_4^o \cup \mathrm{FC}_4^t$: The desired Hamiltonian path that satisfies the claim exists by Lemma 5.2. $\square$

Note that the assumption $n \geq 5$ in Theorem 1.4 is necessary, as $Q_4 - F$ for $F \in \mathrm{FC}_4^t$ is an example of a two-regular graph which is not Hamiltonian by Proposition 5.1.

6. Prescribed vertices of weight at most 2

In this section, we prove that the problems $\mathrm{HC}(\mathcal{L}_{0,2})$, $HP(\mathcal{L}_{0,2})$ and $\mathrm{HPE}(\mathcal{L}_{0,2})$ can be decided in polynomial time. This implies the second part of Theorem 1.1 since $\mathcal{L}_{0,1}, \mathcal{L}_{1,2} \subseteq \mathcal{L}_{0,2}$.

Clearly, a bipartite graph G of $\mathrm{HC}(\mathcal{L}_{k,k+1})$ is balanced if it has a Hamiltonian cycle. Since we can verify whether a graph G is balanced in a linear time, we assume that G is balanced. A balanced graph of $\mathcal{L}_{0,1}$ has at most two vertices, so there is no Hamiltonian graph in $\mathcal{L}_{0,1}$. The following proposition proves that the problem $\mathrm{HC}(\mathcal{L}_{1,2})$ is polynomial.

Proposition 6.1. *A graph G of $\mathcal{L}_{1,2}$ is Hamiltonian if and only if it is a cycle.*

Proof. Clearly, a cycle is a Hamiltonian graph. So, let us prove the other implication where G has a Hamiltonian cycle C. Let t be the number of vertices of weight 1 which is also the number of vertices of weight 2. Every vertex of G of weight 2 has degree at most 2, so G has at most $2t$ edges. But C has exactly $2t$ edges. So, G has only $2t$ edges and they belong to C. $\square$

In the same way we can prove the following proposition.

Proposition 6.2. *A graph G of $\mathcal{L}_{0,2}$ that contains the vertex $\emptyset$ is Hamiltonian if and only if $G - \emptyset$ is a path between vertices of weight* 1.

From Propositions 6.1 and 6.2 it follows the complexity of $\mathrm{HC}(\mathcal{L}_{0,2})$.

Corollary 6.3. *The problem $\mathrm{HC}(\mathcal{L}_{0,2})$ is decidable in polynomial time.*

Lemma 6.4. *Let G be a graph of $\mathcal{L}_{0,2}$ and x, y be vertices of G of weight* 1. *If G contains vertex xy, then G has a Hamiltonian path between vertices x and y if and only if G has only vertices x, y, and xy. Otherwise, G has a Hamiltonian path between vertices x and y if and only if $Q_n[V(G) \cup \{xy\}]$ is Hamiltonian.*

Proof. Both conditions for existence a Hamiltonian path between vertices x and y are clearly sufficient. If G contains a vertex xy, then xy has degree two, so desired path has to use edges incident with xy; and therefore, it can visit only vertices x, y and xy. If G has a Hamiltonian path between x and y, then we can prolong the path into a Hamiltonian cycle of $Q_n[V(G) \cup \{xy\}]$ using vertex xy. □

Proposition 6.5. *The problem* HPE($\mathcal{L}_{0,2}$) *is decidable in polynomial time.*

Proof. Let x, y be desired end-vertices of a graph G of $\mathcal{L}_{0,2}$. If both vertices x and y are of weight 1, then Lemma 6.4 provides a polynomial time decision algorithm.

Let x be a vertex of weight 1 and y be a vertex of weight 0 or 2. Clearly, G has a Hamiltonian path between x and y if and only if y has a neighbour y' such that $G - y$ has a Hamiltonian path between x and y'. Since y has at most n neighbours in G and all of them are of weight 1, we can verify in polynomial time whether there exists a neighbour y' of y such that $G - y$ has a Hamiltonian path between x and y'.

We can process in a similar way if neither x nor y is of weight 1. □

We can consider all pairs of vertices as end-vertices of a Hamiltonian path in order to decide whether a graph has a Hamiltonian path without prescribed end-vertices.

Corollary 6.6. *The problem* HP($\mathcal{L}_{0,2}$) *is decidable in polynomial time.*

7. HC($\mathcal{L}^3_{2,3}$) is polynomial

In this section we provide even a more detailed insight into the complexity of HC($\mathcal{L}_{2,3}$). First we show that HC($\mathcal{L}_{2,3}$) is equivalent to a restricted version of another well-known problem.

To that end, given a subgraph $P \in \mathcal{L}_{2,3}$ of Q_n, we define a graph G_P and a 3-uniform hypergraph H_P. The set of vertices of G_P and H_P is $[n]$. Vertices u, v are joined by an edge in G_P if uv is a vertex of P of weight 2. Vertices u, v, w are joined by a hyperedge in H_P if uvw is a vertex of P of weight 3. Since edges of G_P and H_P correspond to vertices of Q_n, we use the notation of uv and uvw rather than $\{u, v\}$ and $\{u, v, w\}$ also for edges of G_P and hyperedges H_P.

We say that an edge uv of G_P is contained in a hyperedge xyz of H_P if $\{u, v\} \subseteq \{x, y, z\}$.

When speaking about a *cyclic ordering* $e_1, e_2, \ldots, e_m$, we use e_j for $j > m$ to refer to $e_{(j \bmod m)+1}$. A cyclic ordering $e_1, e_2, \ldots, e_m$ of all m edges of a graph is called *sequential* if for every $i \in [m]$, e_i is incident with e_{i+1}. The next lemma shows that HC($\mathcal{L}_{i,j}$) is actually equivalent to the existence of a sequential ordering of $E(G_P)$, satisfying two additional conditions involving the hypergraph H_P.

Lemma 7.1. *Let $P \in \mathcal{L}_{2,3}$. Then P is Hamiltonian if and only if there is a sequential ordering $e_1, e_2, \ldots, e_m$ of $E(G_P)$ such that*

(1) *for every $i \in [m]$ there is a unique $t \in E(H_p)$ containing both e_i and e_{i+1},*

(2) *for every $t \in E(H_P)$ there is a unique $i \in [m]$ such that t contains both e_i and e_{i+1}.*

Proof. It is straightforward to verify that the existence of a sequential ordering satisfying (1) and (2) is necessary for the existence of a Hamiltonian cycle of P. To verify that these conditions are also sufficient, consider a sequential ordering $e_1, e_2, \ldots, e_m$ of $E(G_P)$. By (1), for every $i \in [m]$ there is $t_i \in E(H_P)$ containing e_i and e_{i+1}. Note that then $\{e_i, t_i\}$ and $\{t_i, e_{i+1}\}$ are edges of P. We claim that in

$$C = e_1, t_1, e_2, t_2, \ldots, e_m, t_m,$$

every vertex of $v \in P$ occurs exactly once. Indeed, if $|v| = 2$, then $v \in E(G_P)$ and the claim follows from $e_1, \ldots, e_m$ being an ordering of $E(G_P)$. Otherwise, $v \in E(H_P)$ and the claim follows from (1) and (2). Hence, C is a Hamiltonian cycle of P. □

It is easy to see that the existence of a sequential ordering of $E(G)$ is equivalent to the problem of Hamiltonicity of the line graph of G denoted by $\mathrm{HLG}(G)$. Lemma 7.1 therefore shows that $\mathrm{HC}(\mathcal{L}_{i,j})$ is actually a restricted version of $\mathrm{HLG}(G_P)$.

Moreover, the problem $\mathrm{HLG}(G)$ is known to be NP-complete [1] even in the case that the maximal degree $\Delta(G)$ is 3 [12]. Observe that $P \in \mathcal{L}_{2,3}^k$ if and only if $\Delta(G_P) \leq k$. Although HLG is intractable even for subcubic graphs, the main result of this section shows that in our variant of this problem, bounds between intractability and polynomial solvability are slightly different.

Theorem 7.2. $\mathrm{HC}(\mathcal{L}_{2,3}^3)$ *is polynomial.*

The rest of this section is devoted to the description of a polynomial-time algorithm for $\mathrm{HC}(\mathcal{L}_{2,3}^3)$. Regarding Lemma 7.1, it suffices to decide whether there exists a sequential ordering of edges of a given subcubic graph G, satisfying conditions (1) and (2) for a given 3-uniform hypergraph H. Our goal is to generate a sequential ordering of $E(G)$ so that at each step there is no more than one way to proceed to satisfy the two additional conditions. The next lemma resolves the case when $E(G)$ contains a triangle.

Lemma 7.3. *Let G be a graph with $\Delta(G) \leq 3$ and $|E(G)| \geq 8$, H be a 3-uniform hypergraph with $V(H) = V(G)$, and $e_1, e_2, \ldots, e_m$ be a sequential ordering of $E(G)$ satisfying conditions* (1) *and* (2) *of Lemma 7.1. Let e, e', e'' be edges of G forming a triangle. Then there exist $i \in [m]$ and distinct edges $xy, yz, xz, x\bar{x}, y\bar{y}, z\bar{z} \in E(G)$ such that $\{e, e', e''\} = \{xy, yz, xz\}$, $e_i = \bar{x}w$ for some $w \notin \{x, y, z\}$, $e_{i+1} = \bar{x}x$ and exactly one of the following cases occurs (see Figure 8):*

(a) *$e_{i+2}, \ldots, e_{i+6} = xy, xz, z\bar{z}, yz, y\bar{y}$ and $\bar{x} \neq \bar{y}, \bar{y} \neq \bar{z}$,*

(b) *$e_{i+2}, \ldots, e_{i+6} = xy, y\bar{y}, yz, xz, z\bar{z}$ and $\bar{x} \neq \bar{y}, \bar{x} \neq \bar{z}$,*

(c) *$e_{i+2}, \ldots, e_{i+6} = xz, xy, y\bar{y}, yz, z\bar{z}$ and $\bar{x} \neq \bar{z}, \bar{y} \neq \bar{z}$,*

(d) *$e_{i+2}, \ldots, e_{i+6} = xz, z\bar{z}, yz, xy, y\bar{y}$ and $\bar{x} \neq \bar{y}, \bar{x} \neq \bar{z}$.*

Proof. The following simple observation shall be useful later in the proof:

$$\text{For any } j \in [m], \text{ edges } e_j, e_{j+1} \text{ and } e_{j+2} \text{ never form a triangle in } G. \qquad (*)$$

Indeed, if e_j, e_{j+1}, e_{j+2} form a triangle in G, then $e_j \cup e_{j+1} = e_{j+1} \cup e_{j+2}$, which contradicts the requirement of uniqueness, imposed by conditions (1)-(2).

Now we can proceed to the proof of the lemma. First note that as $m \geq 8$, there must be an $i \in [m]$ such that e_i is not incident with our triangle while e_{i+1} is. Therefore $e_i = w\bar{x}$, $e_{i+1} = \bar{x}x$ and $\bar{x} \notin \{x, y, z\}$.

If e_{i+2} is incident with $\bar{x}$, let $j > i + 2$ be minimal such that e_j equals $\bar{y}y$ or $\bar{z}z$. Assuming without loss of generality the former, note that then $e_{j+1}, e_{j+2} = yz, xy$ or $e_{j+1}, e_{j+2} = xy, yz$. While in the former case there is no choice for e_{j+3}, in the latter we have $e_{j+3} \neq xz$ by $(*)$, which leads to a contradiction, as then there is no way to visit xz.

We can therefore conclude that e_{i+2} equals xy or xz. Since both cases are entiraly symmetrical, we can assume that the former occurs. If $e_{i+3} = yz$, then $e_{i+4} = xz$ is excluded by $(*)$ while $e_{i+4} \neq xz$ is impossible as well, as then there is no way to visit xz. Hence it must be the case that $e_{i+3} \in \{xz, y\bar{y}\}$.

If $e_{i+3} = xz$, then $(*)$ implies that $e_{i+4} = z\bar{z}$. To visit edge yz, there are two options left. Either $e_j = \bar{y}y, e_{j+1} = yz$ for some $j > i+4$, but then there is no choice for e_{j+2}, or $e_{i+5} = yz, e_{j+6} = y\bar{y}$, which leads to case (a).

If $e_{i+3} = y\bar{y}$, then there are again two options to visit the remaining edges of our triangle. Either $e_j = \bar{z}z$ while $\{e_{j+1}, e_{j+2}\} = \{yz, xz\}$ for some $j > i + 3$, but then there is no choice for e_{j+3}, or $e_{i+4}, e_{i+5}, e_{i+6} = yz, xz, z\bar{z}$, which leads to case (b). The remaining two cases (c)-(d) are just symmetrical versions of cases (a)-(b), obtained by setting $e_{i+2} = xz$.

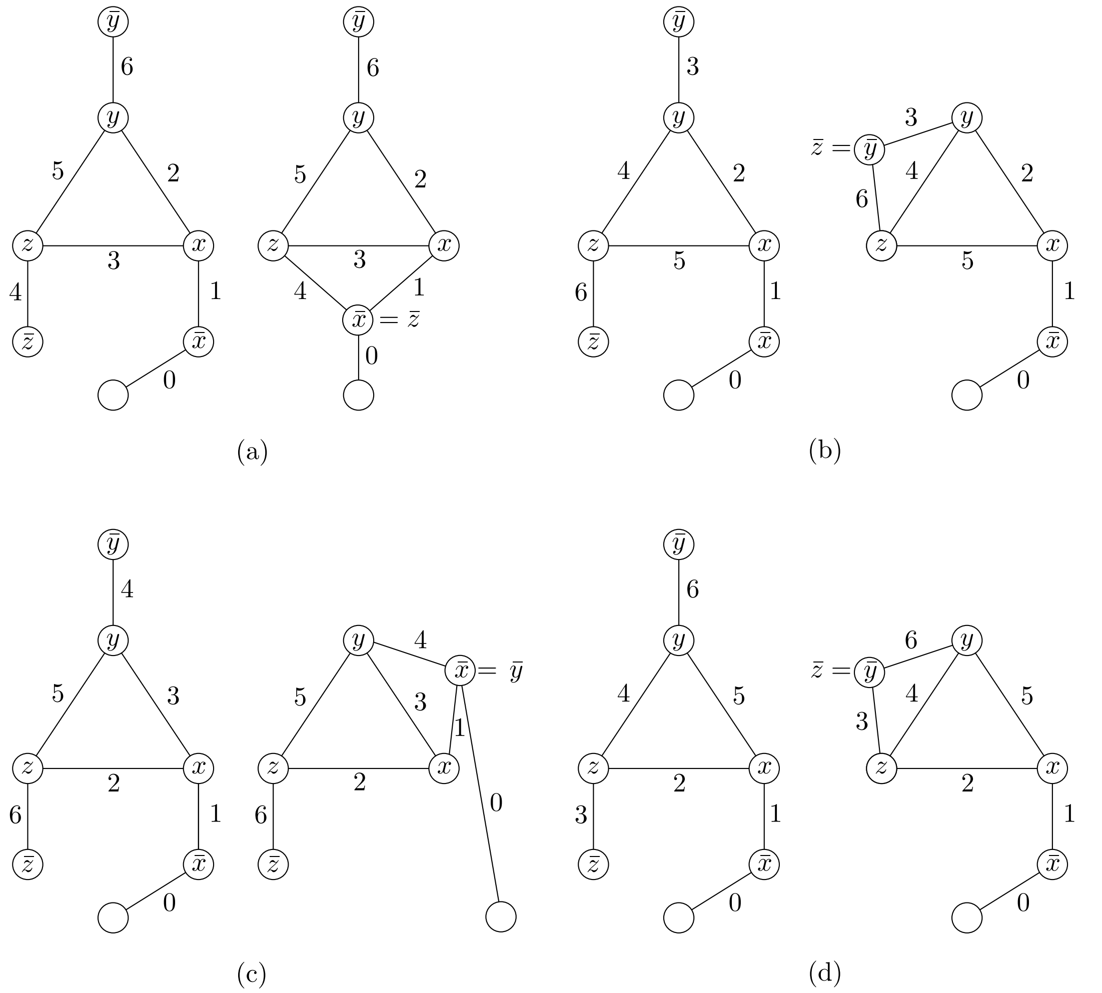


Figure 8: Cases (a)-(d) of Lemma 7.3

To see why the inequalities in all four cases hold, note that in case (a), $\bar{y} = \bar{z}$ would contradict $(*)$, while $\bar{x} = \bar{y}$ implies $e_{i+7} = e_i$, which means that $m = 7$, contrary to our assumption that $m \geq 8$. Inequalities in the other cases hold for analogical reasons.

It only remains to verify that the four cases are mutually exclusive. To that end, observe that if (a) or (b) holds, then $e_{i+1}, e_{i+2} = \bar{x}x, xy$, and therefore by condition (1) of Lemma 7.1, $\bar{x}xy \in E(H)$. Moreover, as $\bar{x} \notin \{\bar{y}, z\}$, we have $\bar{x}y \notin E(G)$ and therefore the only way how to satisfy condition (2) for $\bar{x}xy$ is to make edges $\bar{x}x$ and xy consecutive in the sequential ordering of $E(G)$. Since in cases (c) and (d) we have $e_{i+1} = \bar{x}x$, but both e_i and e_{i+2} different from xy, neither of these two cases may occur simultaneously with (a) or (b).

Similarly, in case (a) we have $e_{i+4}, e_{i+5} = z\bar{z}, yz$, which means that $yz\bar{z} \in E(H)$. Since $y\bar{z} \notin E(G)$ in this case, edges $z\bar{z}$ and yz must be consecutive in the sequential ordering of $E(G)$. That, however, does not happen in case (b), and therefore this case cannot occur together with (a). The mutual exclusiveness of cases (c) and (d) follows from an analogical argument applied to edges xz and $z\bar{z}$, which must be consecutive in case (d), but not in case (c). Hence we can conclude that the cases (a)-(d) are mutually exclusive. $\square$

The next lemma resolves the case when we run across a vertex of degree three which is not incident with a triangle.

Lemma 7.4. *Let G be a graph with $\Delta(G) \leq 3$, H be a 3-uniform hypergraph with $V(H) = V(G)$, and $e_1, e_2, \ldots, e_m$ be a sequential ordering of $E(G)$ satisfying conditions (1) and (2) of Lemma 7.1. Let v be a vertex of G not incident with a triangle and vx, vy, vz be pairwise distinct edges of G.*

(i) *If $e_i = vx$ for some $i \in [m]$, then*

$$e_{i+1} = vy \text{ iff } xvy \in E(H) \quad \text{and} \quad e_{i+1} = vz \text{ iff } xvz \in E(H).$$

(ii) *If $e_j \in \{vy, vz\}$ for some $j \in [m]$, then*

$$e_{j+1} \in \{vy, vz\} \setminus \{e_j\} \text{ iff } vyz \in E(H).$$

Proof. First note that the necessity ("$\Rightarrow$") part of all three equivalences follows directly from condition (1). To verify the sufficiecy ("$\Leftarrow$"), recall that v is not incident with a triangle, and therefore $\{xy, yz, xz\} \cap E(G) = \emptyset$. Consequently, to satisfy condition (2) for xvy (xvz, vyz), edges vy and vy (vx and vz, or vy and vz, respectively) must be consecutive in the sequential ordering of $E(G)$. □

Since vertices of degree two may be traversed only in one way in our construction, we have covered all the possibilities and are therefore ready to provide the algorithm.

Theorem 7.5. *There exists a polynomial-time algorithm which for every $P \in \mathcal{L}^3_{2,3}$ returns a Hamiltonian cycle of P if it exists, and "No" otherwise.*

Proof. For an input $P \in \mathcal{L}^3_{2,3}$, first construct the graph G_P and hypergraph H_P. Then check whether $|E(G_P)| = |E(H_P)|$. If it is not satisfied, then return "No". Otherwise if $|E(H_P)| < 8$, then solve the problem by exhaustive search. Thus we can assume that $|E(G_P)| = |E(H_P)| \geq 8$. If G_P contains a triangle then, by Lemma 7.3, either there exists an edge $x\bar{x}$ such that x belongs to the triangle but the edge $x\bar{x}$ is not contained in the triangle and one of statements (a)-(d) of Lemma 7.3 is satisfied, or the algorithm can return "No". In the former case set $e_1 = x\bar{x}$ and, by the satisfied case (a)-(d) of Lemma 7.3, append the next five edges and corresponding hyperedges to the output sequence. Moreover, remember the last edge of the output sequence and the vertex $\bar{y}$ if (a) or (d) is satisfied, otherwise remember the vertex $\bar{z}$. If G_P does not contain a triangle, then select an arbitrary edge uv of G_P, set $e_1 = uv$ and remember the edge uv and the vertex v.

In the general step the algorithm has the last edge of the output sequence e_i and a vertex v incident with e_i. There are three cases to be distinguished:

- v is incident with a triangle and one of the cases (a)-(d) of Lemma 7.3 applies, then the next five edges and corresponding hyperedges are appended to the output sequence,
- v is a vertex of degree three not included in a triangle and Lemma 7.4 applies, then two edges incident with v and the corresponding hyperedges are sent to output,
- otherwise the only way to continue is to output the edge incident with v different from e.

If none of these cases applies, a sequence with the desired properties does not exist.

To guarantee that each edge and hyperedge may be used at most once, we mark each (hyper)edge as used once included in the output sequence. Once all m edges of G_P have been sent to the output, it suffices to check whether there exists an unused hyperedge which contains both e_m and e_1. If true, then return $e_1, h_1, \ldots, e_m, h_m$, where each h_i is the hyperedge such that $h_i = e_i \cup e_{i+1}$, and "No" otherwise. Note that no (hyper)edge may be omitted, as the final output sequence then contains $|E(G_P)|$ edges and $|E(G_P)|$ hyperedges.

Finally, recall that by Lemma 7.1, the output sequence $e_1, h_1, \ldots, e_m, h_m$ is indeed a Hamiltonian cycle of P as required, while the running time of the described algorithm is clearly polynomial. □

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