# Long paths and cycles in hypercubes with faulty vertices

Jiří Fink *
Department of Applied Mathematics
Faculty of Mathematics and Physics
Charles University
Malostranské náměstí 25
118 00 Prague 1
E-mail: fink@kam.mff.cuni.cz

Petr Gregor †
Department of Theoretical Computer Science and Mathematical Logic
Charles University
Malostranské náměstí 25
118 00 Prague 1
E-mail: gregor@ktiml.mff.cuni.cz

**Abstract**

A fault-free path in the $n$-dimensional hypercube $Q_n$ with $f$ faulty vertices is said to be *long* if it has length at least $2^n - 2f - 2$. Similarly, a fault-free cycle in $Q_n$ is long if it has length at least $2^n - 2f$. If all faulty vertices are from the same bipartite class of $Q_n$, such length is the best possible. We show that for every set of at most $2n - 4$ faulty vertices in $Q_n$ and every two fault-free vertices $u$ and $v$ satisfying a simple necessary condition on neighbors of $u$ and $v$, there exists a long fault-free path between $u$ and $v$. This number of faulty vertices is tight and improves the previously known results. Furthermore, we show for every set of at most $n^2/10 + n/2 + 1$ faulty vertices in $Q_n$ where $n \geq 15$ that $Q_n$ has a long fault-free cycle. This is a first quadratic bound, which is known to be asymptotically optimal.

## 1 Introduction

The $n$-dimensional hypercube $Q_n$ is a (bipartite) graph with all binary vectors of length $n$ as vertices and with edges joining every two vertices that differ in exactly one coordinate. The application of hypercubes as interconnection networks inspired many questions related to their fault-tolerance. In particular, in this paper we consider a problem of long fault-free cycles and long fault-free paths between two given vertices in hypercubes in which some vertices are faulty.

This problem is sometimes considered in a more general setting also with faulty edges, not only vertices. Assume that we have $f_v$ faulty vertices and $f_e$ faulty edges in $Q_n$. A path or a cycle in $Q_n$ is said to be *fault-free* if it contains no faulty vertex and no faulty edge. Furthermore, a cycle in $Q_n$ is *long* if it has length at least $2^n - 2f_v$. Similarly, a path in $Q_n$ is *long* if it has length at least $2^n - 2f_v - 2$. Note that every long path between vertices $u$ and $v$ has length at least $2^n - 2f_v - 1$ if $d(u, v)$ is odd, where $d(u, v)$ is the distance between $u$ and $v$. Furthermore, if all faulty vertices belong to the same bipartite class of $Q_n$, then every long fault-free cycle and long fault-free path is the longest possible. In this view, the problem of long fault-free cycles and paths is a relaxation of a substantially more difficult problem of Hamiltonian cycles and paths in hypercubes with balanced faulty vertices in the sense that in the former problem we are allowed to choose another up to $f_v$ vertices that will be avoided (see e.g. [4] for some references on the latter problem).

As far as we know, the problem of long fault-free cycles in hypercubes was first studied by Tseng [14] who showed that such cycle in $Q_n$ exists if $f_v + f_e \leq n - 1$, $f_v \leq n - 1$, and $f_e \leq n - 4$. This bound was slightly improved by Sengupta [12] to $f_v + f_e \leq n - 1$, and $f_v > 0$ or $f_e \leq n - 2$. Then it was substantially strengthened by Fu [5] to $f_v \leq 2n - 4$ (and $f_e = 0$), and further naturally

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generalized by Hsieh [7] to $f_v + f_e \leq 2n - 4$ and $f_e \leq n - 2$. The latest improvement due to Castañeda and Gotchev [3] is for $f_v \leq 3n - 7$ ($f_e = 0$) and $n \geq 5$. Note that all these bounds are linear in the dimension $n$. We provide a first quadratic bound on $f_v$ (and $f_e = 0$), which is known to be asymptotically optimal.

The similar problem for paths was first studied by Fu [6] who showed that there is a long fault-free path in $Q_n$ between every two fault-free vertices if $f_v \leq n - 2$ (and $f_e = 0$). Hung, Chang, and Sun [8] showed that even a little longer path exists under similar conditions. More precisely, there is a fault-free path in $Q_n$ of length at least $2^n - 2f_v$ between every two fault-free vertices if $f_v \leq n - 2$ ($f_e = 0$) and at least one vertex from each bipartite class is faulty.

Recently, the bound of Fu was improved by Kueng, Liang, Hsu, and Tan [10] to $f_v \leq 2n - 5$ (and $f_e = 0$), but with an additional (strong) condition that every vertex has at least two fault-free neighbors. We show for $f_v \leq 2n - 4$ (and $f_e = 0$) that a much weaker condition is both necessary and sufficient (up to one exception in $Q_4$). We also show that our bound is tight.

Let us also mention related results on bipancyclicity and bipanconnectivity. Tsai [13] showed that every fault-free edge and every fault-free vertex of $Q_n$ lies on a fault-free cycle of every even length from 4 to $2^n - 2f_v$ if $f_v \leq n - 2$ (and $f_e = 0$). Ma, Liu, and Pan [11] showed that if $f_v + f_e \leq n - 2$, then $Q_n$ contains a fault-free path of length $l$ between every two fault-free vertices $u$ and $v$ for every $l$ from $d(u, v) + 2$ to $2^n - 2f_v - 1$ such that $l - d(u, v)$ is even. There are also many results on long fault-free cycles and paths in various modifications of hypercubes, which we do not list here.

## 2 Main results

A long fault-free path between $u$ and $v$ in $Q_n$ with a set $F$ of faulty vertices is shortly called an *$(F, u, v)$-path*. An edge $uv \in E(Q_n)$ is *fault-free* if both vertices $u$ and $v$ are fault-free. Note that for $n \geq 2$, every long path has length at least 2 if $|F| \leq 2n - 4$. A vertex $u$ is *surrounded* by $F$ if $F$ contains all neighbors of $u$. Furthermore, a triple $(F, u, v)$ is *blocked* in $Q_n$ if

$$u \text{ is surrounded by } F \cup \{v\} \text{ in } Q_n \text{ or } v \text{ is surrounded by } F \cup \{u\} \text{ in } Q_n; \tag{1}$$

otherwise $(F, u, v)$ is *free* in $Q_n$. The reference to the underlying graph $Q_n$ may be omitted if it is clear from the context. Clearly, if $(F, u, v)$ is *blocked*, there is no fault-free path between $u$ and $v$ of length more than 1. Thus, the triple $(F, u, v)$ must be free for the existence of an $(F, u, v)$-path if $|F| \leq 2n - 4$. The following theorem shows that this necessary condition is also sufficient, up to one exception in $Q_4$.

**Theorem 2.1.** *Let $F$ be a set of at most $2n - 4$ faulty vertices of $Q_n$ where $n \geq 2$. For every two fault-free vertices $u$ and $v$, there exists a long fault-free path between $u$ and $v$ in $Q_n$ if and only if both (1) and (2) does not hold.*

On Figure 1 we have the following configuration for $n = 4$ and $|F| = 2n - 4$:

$$\begin{aligned} &\text{there are two vertices } a \text{ and } b \text{ with } d(a, b) = 4 \text{ in } Q_4 \text{ such that} \\ &F \cup \{u, v, a, b\} \text{ are the all 8 vertices of one bipartite class of } Q_4. \end{aligned} \tag{2}$$

Observe in this configuration that every fault-free path between $u$ and $v$ has length at most 4 because the graph $Q_4 \setminus (F \cup \{u, v\})$ has two components and no fault-free path between $u$ and $v$ can visit both components. Hence, there is no $(F, u, v)$-path although $|F| \leq 2n - 4$ and $(F, u, v)$ is free. Note that there are two non-isomorphic exceptional configurations since $d(u, v)$ can be 2 or 4.

Moreover, observe that the inequality $|F| \leq 2n - 4$ in Theorem 2.1 is tight for every $n \geq 4$. On Figure 2 we can see three configurations of $2n - 3$ faulty vertices and two fault-free vertices $u$ and $v$ in $Q_n$ such that $(F, u, v)$ is free. Clearly, in all these configurations there is only one fault-free path between $u$ and $v$ of length 1 or 2, which is not long.

We prove Theorem 2.1 by induction on the dimension $n$. In Section 4 we prove the base of induction by a tedious case analysis for $n \leq 4$. In Section 5 we prove the induction step. In

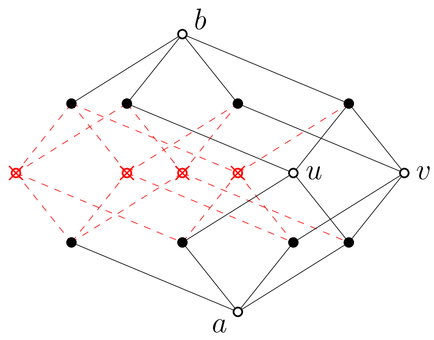



Figure 1: The exceptional configuration (2) in $Q_4$. The crossed points represent the faulty vertices and $u$, $v$ are the prescribed endvertices for a requested long fault-free path.

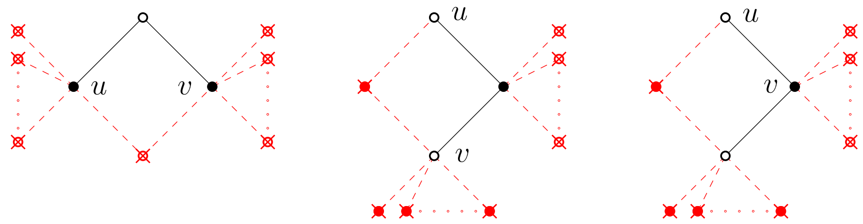



Figure 2: $|F| = 2n-3$, $n \geq 4$, and $(F, u, v)$ is free, but there is no $(F, u, v)$-path.

Section 6 we prove that for a sufficiently large $n$, we can find a long fault-free cycle in $Q_n$ with a quadratic number of faulty vertices.

**Theorem 2.2.** *Let $F$ be a set of at most $\frac{n^2}{10} + \frac{n}{2} + 1$ faulty vertices of $Q_n$ where $n \geq 15$. Then $Q_n$ contains a long fault-free cycle.*

On the other hand, Koubek [9] and independently Castañeda and Gotchev [2] noticed that for $n \geq 4$ there is a set $F$ of $\binom{n}{2} - 1$ faulty vertices such that $Q_n$ has no long fault-free cycle, so Theorem 2.2 is asymptotically optimal. Such a set $F$ can be, for example, a set consisting of all but one vertex at distance 2 from the vertex $(0, \ldots, 0)$.

It remains an open question whether the bound given by Theorem 2.2 can be improved to meet the upper bound of $\binom{n}{2} - 2$ vertices, as Castañeda and Gotchev [2] conjectured.

## 3 Preliminaries

The main obstacle in the proof of Theorem 2.1 are vertices surrounded by faulty vertices. In the following auxiliary propositions we mainly show that there are only few such obstacles.

**Proposition 3.1.** *Let $F$ be a set of at most $2n-3$ faulty vertices in $Q_n$ where $n \geq 2$. Then, at most one vertex of $Q_n$ is surrounded by $F$.*

*Proof.* Suppose on the contrary that two vertices $u$ and $v$ of $Q_n$ are surrounded by $F$. Since each of them has $n$ faulty neighbors, and they have at most 2 faulty neighbors in common, it follows that $|F| \geq 2n-2$, a contradiction. □ □

In the following proposition we show that at most one triple $(F, u, v)$ is blocked when $|F| \leq 2n-4$ and the vertex $u$ is fixed and not surrounded by $F$ itself.

**Proposition 3.2.** *Let $F$ be a set of at most $2n-4$ faulty vertices in $Q_n$ where $n \geq 2$, and let $u \in V(Q_n)$ be not surrounded by $F$. Then, $(F, u, v)$ is blocked for at most one vertex $v \in V(Q_n)$.*

*Proof.* First, assume that $u$ has exactly one fault-free neighbor $v$. Thus, $u$ is surrounded by $F \cup \{v\}$ and not surrounded by $F \cup \{w\}$ for any other vertex $w$. By Proposition 3.1, no other vertex than $u$ is surrounded by $F \cup \{v\}$. It follows that no vertex is surrounded by $F \cup \{u\}$, so $v$ is the only vertex such that $(F, u, v)$ is blocked.

Now assume that $u$ has at least 2 fault-free neighbors. Thus, $u$ is not surrounded by $F \cup \{w\}$ for any vertex $w$. By Proposition 3.1, at most one vertex $v$ is surrounded by $F \cup \{u\}$. Therefore, $(F, u, v)$ is blocked for at most one vertex $v$. □ □

Next, we show that at most one triple $(F, u, v)$ is blocked when $|F| \leq 2n - 5$ and $uv$ is required to be a fault-free edge.

**Proposition 3.3.** *Let $F$ be a set of at most $2n - 5$ faulty vertices in $Q_n$ where $n \geq 3$. Then, $(F, u, v)$ is blocked for at most one fault-free edge $uv \in E(Q_n)$.*

*Proof.* Suppose on the contrary that triples $(F, u, v)$ and $(F, u', v')$ are blocked for two fault-free edges $uv, u'v' \in E(Q_n)$. Assume that $u$ is surrounded by $F \cup \{v\}$, and $u'$ is surrounded by $F \cup \{v'\}$. Observe that $u \neq u'$ since $v$ and $v'$ are fault-free. But then, both $u$ and $u'$ are surrounded by $F \cup \{v, v'\}$, which contradicts Proposition 3.1. □ □

The following proposition is useful in situations when we have a long fault-free path $P$ in $Q_L$ and we need to find an edge $a_L b_L$ on $P$ such that there is a long fault-free path between $a$ and $b$ in $Q_R$.

**Proposition 3.4.** *Let $F$ be a set of at most $2n - 4$ faulty vertices in $Q_n$ where $n \geq 2$. For every path $P$ in $Q_n$, if $P$ contains at least three fault-free edges $uv$ such that $(F, u, v)$ is blocked, then it contains a fault-free edge $ab$ such that $(F, a, b)$ is free.*

*Proof.* Let $uv$ be a fault-free edge of $P$ such that $(F, u, v)$ is blocked, and both $u$ and $v$ are inner vertices of $P$. Such edge exists since only two edges of $P$ can contain an endvertex. Assume that $u$ is surrounded by $F \cup \{v\}$, and let $w$ be the other neighbor of $v$ on $P$. Furthermore, assume that $u'$ is surrounded by $F \cup \{v'\}$ for some other fault-free edge $u'v'$ of $P$. We show that the edge $vw$ of $P$ is fault-free and $(F, v, w)$ is free.

Since both $u$ and $u'$ have exactly $n - 1$ faulty neighbors and $|F| \leq 2n - 4$, they must have two faulty neighbors in common. Thus $d(u, u') = 2$ and all faulty vertices together with $v$ (and $v'$) belong to the same bipartite class of $Q_n$. Hence $w$ is fault-free and moreover, $v$ is not surrounded by $F \cup \{w\}$. Since $u$ is surrounded by $F \cup \{v\}$, it follows from Proposition 3.1 that $w$ is not surrounded by $F \cup \{v\}$. Therefore, $(F, v, w)$ is free for a fault-free edge $vw$ of $P$. □ □

In order to apply induction, we need to split the hypercube $Q_n$ with up to $2n-4$ faulty vertices into two $(n - 1)$-dimensional subcubes $Q_L$ and $Q_R$ so that both $Q_L$ and $Q_R$ contain at most $2n - 6$ faulty vertices. This is obtained by fixing some coordinate $i \in [n]$ where $[n] = \{1, \ldots, n\}$. Formally, we define the subcube $Q_L^i$ as the subgraph of $Q_n$ induced by vertices that have 0 on the $i$-th coordinate. Similarly, the subcube $Q_R^i$ is the subgraph of $Q_n$ induced by vertices that have 1 on the $i$-th coordinate. The index $i$ in $Q_L^i$ and $Q_R^i$ is omitted when it is clear or irrelevant. For $x \in V(Q_L)$, let $x_R$ be the (only) neighbor of $x$ in $Q_R$. Similarly for $x \in V(Q_R)$, let $x_L$ be the (only) neighbor of $x$ in $Q_L$.

**Proposition 3.5.** *Let $F$ be a set of at most $2n - 4$ vertices in $Q_n$ where $n \geq 5$. Then $Q_n$ can be split into $Q_L$ and $Q_R$ such that both subcubes contain at most $2n - 6$ faulty vertices, unless $n = 5$, $|F| = 6$, and $F$ consists of some vertex $w \in V(Q_n)$ and all his neighbors.*

*Proof.* If $|F| \leq 1$, we may split $Q_n$ arbitrarily. If $2 \leq |F| \leq 2n - 5$, we choose two faulty vertices and split $Q_n$ so that they are in different subcubes. Clearly, in both these cases both $Q_L$ and $Q_R$ contain at most $2n - 6$ faulty vertices. Now we assume that $|F| = 2n - 4$.

Let $A$ be the binary $|F| \times n$ matrix with faulty vertices in its rows. Assume that $Q_n$ cannot be split into $Q_L$ and $Q_R$ such that both subcubes contain at most $2n - 6$ faulty vertices. That is, each column of $A$ contains at most one 1, or at most one 0. Without loss of generality we may assume that each column contains at most one 1. Thus $A$ contains at most $n$ 1's. Hence $A$ has at most $n+1$ rows as all rows are different. Since $n+1 < 2n-4$ for $n \geq 6$, it follows that $n=5$ and $F$ consists of the vertex $(0,0,\dots,0)$ and all his neighbors. □ □

Let us recall that a path between $u$ and $v$ is long if it has length at least $2^n - 2|F| - 2$. We represent paths by sequences of vertices, i.e. $(u_1, u_2, \dots, u_k)$ is a path $P$ between $u_1$ and $u_k$ of length $|E(P)| = k-1$ if all vertices $u_1, \dots, u_k$ are distinct and $u_i u_{i+1}$ is an edge for every $i \in [k-1]$. This allows us to define concatenation of paths as concatenation of their sequences. For example, if $P_1$ is a path between $u_1$ and $v_1$ and $P_2$ is a path between $u_2$ and $v_2$ such that $P_1$ and $P_2$ are vertex-disjoint and $v_1 u_2$ is an edge, then $(P_1, P_2)$ is a path between $u_1$ and $v_2$ of length $|E(P_1)| + |E(P_2)| + 1$.

# 4 Long fault-free paths - small dimension

In this section we present the base of induction for Theorem 2.1. The case $n=2$ is obvious since $|F| \leq 2n-4 = 0$. For $n=3$ we even prove a stronger statement with one additional faulty vertex than in Theorem 2.1. Namely, for $|F| \leq 2n-3 = 3$ and every two fault-vertices $u$ and $v$ there exists an $(F,u,v)$-path if $(F,u,v)$ is free. Note that the opposite implication does not hold since the edge $uv$ itself (if it exists) is an $(F,u,v)$-path when $|F| = 3$.

**Lemma 4.1.** *Let $F$ be a set of at most 3 vertices of $Q_3$, and let $u$ and $v$ be two fault-free vertices. If $(F,u,v)$ is free, then there exists an $(F,u,v)$-path.*

*Proof. Case 1:* $|F| = 3$.

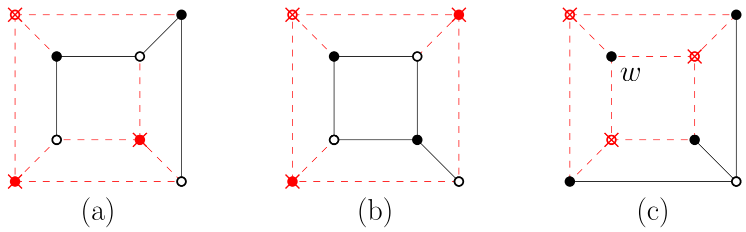



Figure 3: All configurations (up to isomorphism) of 3 faulty vertices in $Q_3$.

We want to find a path of length at least $2^3 - 3 \cdot 2 - 2 = 0$, so it suffices to show that $u$ and $v$ belong to the same component of $Q_3 \setminus F$ if $(F,u,v)$ is free. There are tree configurations (up to isomorphism) of $F$ with $|F| = 3$; see Figure 3. Observe that $Q_3 \setminus F$ on Figure 3(a,b) is connected. Also $Q_3 \setminus (F \cup \{w\})$ on Figure 3(c) is connected and $w$ is surrounded by $F$. Hence the statement holds.

*Case 2:* $|F| = 2$.

The graph $Q_3 \setminus F$ is connected because $Q_3$ is 3-connected, so there exists a path $P$ between $u$ and $v$ in $Q_3 \setminus F$. We want to find a fault-free path between $u$ and $v$ of length at least $2^3 - 2 \cdot 2 - 2 = 2$. If $d(u,v) \geq 2$, then $P$ has this length.

Now assume that $d(u,v) = 1$. There exist two disjoint edges $x_i y_i$ such that $ux_i$ and $y_i v$ are edges of $Q_3$ for $i \in \{1,2\}$. If $x_i, y_i \notin F$ for some $i \in \{1,2\}$, then $(u, x_i, y_i, v)$ is a requested path. If $x_1, x_2 \in F$ or $y_1, y_2 \in F$, then $(F,u,v)$ is blocked. It remains to find an $(F,u,v)$-path for the case where $F = \{x_1, y_2\}$ (or isomorphically $F = \{x_2, y_1\}$). See Figure 4 for such path.

*Case 3:* $|F| \leq 1$.

This case follows from the previous result by Fu [6] for at most $n-2$ faulty vertices. □ □

Assume that $Q_n$ is split into $Q_L$ and $Q_R$. The sets of faulty vertices in $Q_L$ and $Q_R$ are denoted by $F_L$ and $F_R$, respectively.

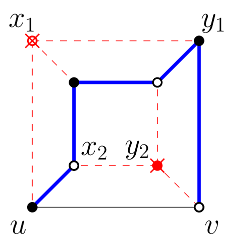



Figure 4: The $(F, u, v)$-path in Case 2 of Lemma 4.1.

In $Q_4$ we are often in a situation when $Q_4$ is split into $Q_L$ and $Q_R$ so that $u \in V(Q_L)$ and $v \in v(Q_R)$. We would like to find a vertex $x$ in $Q_L$ such that there exist an $(F_L, u, x)$-path $P_L$ and an $(F_R, x_R, v)$-path $P_R$ and their concatenation $P = (P_L, P_R)$ is an $(F, u, v)$-path. Now, we present sufficient conditions on the vertex $x$ to apply such construction.

**Lemma 4.2.** *Let $Q_4$ be split into $Q_L$ and $Q_R$ so that $u \in V(Q_L)$, $v \in V(Q_R)$, $|F_L| \leq 3$, $|F_R| \leq 3$ and there exists a fault-free vertex $x$ in $Q_L$ such that $x_R \notin F_R$, $(F_R, v, x_R)$ is free in $Q_R$ and at least one of the following conditions holds.*

*(a) $(F_L, u, x)$ is free in $Q_L$, and $d(u, x)$ or $d(v, x_R)$ is odd.*

*(b) There exists a fault-free path $P_L$ between $u$ and $x$ in $Q_L$ of length at least $2^3 - 2|F_L| - 1$.*

*(c) $d(u, v)$ is even, $|F_L| = 3$, and $x = u$.*

*Then there exists an $(F, u, v)$-path in $Q_4$.*

*Proof.* There exists an $(F_R, x_R, v)$-path $P_R$ in $Q_R$ by Lemma 4.1. In the first case, there exists an $(F_L, u, x)$-path $P_L$ by Lemma 4.1. In the third case, let $P_L$ be the trivial path between $u$ and $x$. We show that the path $P = (P_L, P_R)$ has sufficient length in all three cases.

(a) Without lost of generality we assume that $d(u, x)$ is odd. Then the length of $P$ is $|E(P)| = |E(P_L)| + 1 + |E(P_R)| \geq 2^3 - 2|F_L| - 1 + 1 + 2^3 - 2|F_R| - 2 = 2^4 - 2|F| - 2$.

(b) $|E(P)| = |E(P_L)| + 1 + |E(P_R)| \geq 2^3 - 2|F_L| - 1 + 1 + 2^3 - 2|F_R| - 2 = 2^4 - 2|F| - 2$.

(c) Since $d(x_R, v)$ is odd we have $|E(P)| \geq 1 + 2^3 - 2|F_R| - 1 \geq 2^4 - 2|F| - 2$. □

□

Note that if $d(u, v)$ is even, then one of $d(u, x)$ and $d(v, x_R)$ is odd for every vertex $x$ in $Q_L$. Let $N(u)$, $N_L(u)$ and $N_R(u)$ be the sets of neighbors of $u$ in $Q_n$, $Q_L$ and $Q_R$, respectively. We conclude this section with the following lemma that serves as the basis for induction in the proof of Theorem 2.1 for $n = 4$.

**Lemma 4.3.** *Let $F$ be a set of at most 4 faulty vertices in $Q_4$. For every two fault-free vertices $u$ and $v$, there is an $(F, u, v)$-path if and only if $(F, u, v)$ is free and (2) does not hold.*

*Proof.* The necessity was discussed in Section 2.

*Case 1: We can split $Q_4$ so that $|F_L| = 4$ or $|F_R| = 4$.*

Assume that $|F_L| = 4$. Let $u' = u$ if $u \in V(Q_R)$, otherwise $u' = u_R$. Similarly, let $v' = v$ if $v \in V(Q_R)$, otherwise $v' = v_R$. Clearly, there is an $(F_R, u', v')$-path in $Q_R$ which is a long path in $Q_4$. We prolong this path by the edge $uu_R$ if $u \in V(Q_L)$ and $vv_R$ if $v \in V(Q_L)$ and we obtain an $(F, u, v)$-path in $Q_4$.

For the rest of the proof, we assume that $|F_L| \leq 3$ and $|F_R| \leq 3$ for every splitting of $Q_4$ into $Q_L$ and $Q_R$, which is one of the conditions of Lemma 4.2. Furthermore, we assume that $u \in V(Q_L)$ for every splitting of $Q_4$, otherwise we exchange the roles of $Q_L$ and $Q_R$. We distinguish the following cases.

*Case 2: We can split $Q_4$ so that $v \in V(Q_R)$, $|F_L| = 3$ or $|F_R| = 3$, and moreover, if $d(u,v)$ is odd, then $u$ is not surrounded by $F_L$ in $Q_L$ and $v$ is not surrounded by $F_R$ in $Q_R$.*

Without lost of generality we assume that $|F_L| = 3$. Since $|F_R| \leq 1$, $(F_R, z, v)$ is free in $Q_R$ for every vertex $z$ in $Q_R$. If $u$ is surrounded by $F_L$ in $Q_L$, then $d(u,v)$ is even and $u_R \notin F_R$. This configuration satisfies conditions of Lemma 4.2(c) for $x = u$. So we assume that $u$ is not surrounded by $F_L$ in $Q_L$.

Observe on Figure 3 that there are at least 3 vertices different from $u$ in the component of $Q_L \setminus F_L$ containing $u$. Since $|F_R \cup \{v\}| \leq 2$, there is a vertex $x \in V(Q_L)$ satisfying the requirements of Lemma 4.2(b).

*Case 3: We can split $Q_4$ so that $u, v \in V(Q_L)$, $|F_L| = 0$ and $|F_R| \leq 3$.*

Observe that for every edge $ab$ in $Q_L$ such that $\{a,b\} \neq \{u,v\}$ there exists an $(F_L, u, v)$-path containing $ab$. Assume that $|F_R| = 3$. There exists fault-free edge $ab$ in $Q_R$ such that $\{a,b\} \neq \{u_R, v_R\}$ because $Q_3$ has 12 edges and one faulty vertex makes only 3 edges faulty. Let $P_L$ be an $(F_L, u, v)$-path in $Q_L$ containing the edge $a_L b_L$. We obtain an $(F, u, v)$-path from $P_L$ by replacing the edge $a_L b_L$ with the path $(a_L, a, b, b_L)$.

Now assume that $|F_R| \leq 2$. There exist at least 5 fault-free edges in $Q_R$ different from $u_R v_R$ because $Q_3$ has 12 edges and one faulty vertex makes only 3 edges faulty. If $(F_R, x, y)$ is blocked in $Q_R$ for some fault-free edge $xy$ in $Q_R$, then there are 2 faulty vertices in $Q_R$ in distance 2 and there is only another one fault-free edge $x'y'$ such that $(F_R, x', y')$ is blocked in $Q_R$. Hence, there exists a fault-free edge $ab$ in $Q_R$ different from $u_R v_R$ such that $(F_R, a, b)$ is free in $Q_R$. Let $P_R$ be an $(F_R, a, b)$-path in $Q_R$ and $P_L$ be an $(F_L, u, v)$-path in $Q_L$ containing $a_L b_L$. Let $P$ be obtained from $P_L$ by replacing the edge $a_L b_L$ with the path $P_R$. Since the length of $P$ is $|E(P_L)| - 1 + 2 + |E(P_R)| \geq 2^4 - 2|F| - 1$, it follows that $P$ is an $(F, u, v)$-path.

*Case 4: $d(u,v)$ is even.*

We split $Q_4$ so that $u \in V(Q_L)$ and $v \in V(Q_R)$. If there exists splitting such that moreover $u_R \in F$ or $v_L \in F$, then we apply it. If $|F_R| = 3$ or $|F_L| = 3$, then this configuration satisfies the requirements of Case 2. So, we assume that $|F_R| \leq 2$ and $|F_L| \leq 2$.

By Proposition 3.2, there exists at most one vertex $l$ in $Q_L$ such that $(F_L, l, u)$ is blocked in $Q_L$ and at most one vertex $r$ of $Q_R$ such that $(F_R, r, v)$ is blocked in $Q_R$. If there exists a vertex $x \in V(Q_L)$ such that $x, x_R \notin F \cup \{u, v, l, r\}$, then there exists an $(F, u, v)$-path by Lemma 4.2(a). When there is no such vertex $x$?

Note that $|F \cup \{u, v, r, l\}| \leq 8$ and $Q_L$ has 8 vertices. There is no requested vertex $x$ if and only if

$$\text{for every vertex } y \text{ of } Q_L \text{ exactly one of } y \text{ and } y_R \text{ belongs to } F \cup \{u, v, l, r\}. \tag{3}$$

Our aim is to show that we have the exceptional configuration (2) if (3) holds. So we assume for the rest of this case that (3) holds. Hence $|F_L| = |F_R| = 2$ and vertices $l$ and $r$ exist.

We know that $u$ is surrounded by $F_L \cup \{l\}$ in $Q_L$ or $l$ is surrounded by $F_L \cup \{u\}$ in $Q_L$. Now, we show that $u$ is not surrounded by $F_L \cup \{l\}$ in $Q_L$. Suppose on the contrary that $u$ is surrounded by $F_L \cup \{l\}$ in $Q_L$. If $d(u,v) = 2$, then $v_L \in N_L(u) = F_L \cup \{l\}$ which contradicts (3). Now, $d(u,v) = 4$. Let $f$ be some faulty neighbor of $u$. It follows from (3) that $u_R \notin F$ and $v_L \notin F$ which contradicts our requirements on splitting because it is possible to split $Q_4$ by the dimension in which $f$ and $u$ differ. Similarly, $r$ is not surrounded by $F_R \cup \{v\}$.

Since $l$ is surrounded by $F_L \cup \{u\}$, vertices of $F_L \cup \{u\}$ belong to the same bipartite class $A$ of $Q_4$ and $l$ belongs to the other bipartite class $B$ of $Q_4$. Let $a$ be the only vertex of $Q_L$ in $A$ that does not belong to $F_L \cup \{u\}$. Similarly, the three vertices of $F_R \cup \{v\}$ belong to the same bipartite class and let $b$ be the fourth vertex of that bipartite class in $Q_R$. Since $u$ and $v$ are in the same bipartite class $A$, the vertices of $F \cup \{u, v, a, b\}$ form the bipartite class $A$. It follows from (3) that $a_R = r$ and $b_L = l$. See Figure 5 for an illustration.

We have $d(a,b) \geq 3$ because $a \in V(Q_L)$, $b \in V(Q_R)$, $a_R = r$, $N_R(r) = F_R \cup \{v\}$ and $b \notin F_R \cup \{v\}$. Since $a$ and $b$ belong to the same bipartite class, it follows that $d(a,b) = 4$. Hence, we conclude that if (3) holds, then we have the exceptional configuration (2).

*Case 5: $d(u,v)$ is odd.*

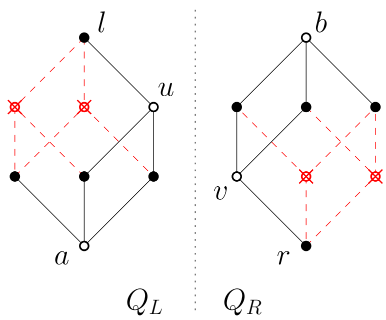



Figure 5: Case 4 in Lemma 4.3: the exceptional configuration (2).

First, we show that we can split $Q_4$ so that $u \in V(Q_L)$, $v \in V(Q_R)$, $u$ is not surrounded by $F_L$ in $Q_L$, $v$ is not surrounded by $F_R$ in $Q_R$ and $u_R \in F_R \cup \{v\}$.

If $d(u, v) = 1$ then we split $Q_4$ by the dimension in which $u$ and $v$ differs. Then, $u_R = v$ and the vertex $u$ is not surrounded by $F_L$ in $Q_L$ and $v$ is not surrounded by $F_R$ in $Q_R$, otherwise $(F, u, v)$ would be blocked.

Now, we assume that $d(u, v) = 3$. Let $Q_A$ be the smallest subcube of $Q_4$ containing $u$ and $v$. Since $d(u, v) = 3$, the dimension of $Q_A$ is 3 and let $Q_B$ be the complementary subcube. If there is no faulty vertex in $Q_A$, then we have the configuration of Case 3. If there exists a faulty vertex $f$ in $Q_A$, then $f$ is a neighbor of $u$ or $v$, say $u$, so we split $Q_4$ by the dimension in which $f$ and $u$ differs so $u \in V(Q_L)$ and $v \in V(Q_R)$. Furthermore, $u$ is not surrounded by $F_L$ in $Q_L$, because $(F, u, v)$ is free and $u_R = f$. If $v$ is surrounded by $F_R$ in $Q_R$, then $u_R = f$ is in $N_R(v) = F_R$ as $|F_R| \leq 3$ which contradicts the assumption that $d(u, v) = 3$.

Now, $Q_4$ is split so that $u \in V(Q_L)$, $v \in V(Q_R)$, $u$ is not surrounded by $F_L$ in $Q_L$, $v$ is not surrounded by $F_R$ in $Q_R$ and $u_R \in F_R \cup \{v\}$. If $|F_R| = 3$ or $|F_L| = 3$, then we have Case 2. So we assume that $|F_R| \leq 2$ and $|F_L| \leq 2$.

First, we assume that $u$ has only one fault-free neighbor $u'$ in $Q_L$. The triple $(F, u', v)$ is free and all neighbors of $u$ are in $F \cup \{u', v\}$. Observe on Figure 2 that in the exceptional configuration (2) there is no vertex surrounded by faulty vertices and end-vertices. Hence, the triple $(F, u', v)$ does not form the exceptional configuration (2). There exists an $(F, u', v)$-path by Case 4 which we prolong by the edge $uu'$ to obtain an $(F, u, v)$-path.

Next, we assume that $v$ has only one fault-free neighbor in $Q_R$. Observe that $d(u, v) = 1$, otherwise $u_R \notin F_R \cup \{v\}$. Thus, $v_L = u$ and by exchanging the roles of $Q_L$ and $Q_R$ and the roles of $u$ and $v$, we may proceed as in the previous paragraph. Now, both $u$ and $v$ have at least two fault-free neighbors in their subcubes.

Note that there is at most one faulty vertex in $N_L(u)$ and at most one faulty vertex in $N_R(u_R)$ because $u_R \in F \cup \{v\}$. By Proposition 3.2, there exists at most one vertex $l$ in $Q_L$ such that $(F_L, u, l)$ is blocked in $Q_L$. If a vertex $l$ exists, then there is no faulty vertex in $N_L(u)$. Hence, there is at most one vertex $x$ in $N_L(u)$ such that $x \in F$ or $(F_L, u, x)$ is blocked. Similarly, there is at most one vertex $x$ in $N_L(u)$ such that $x_R \in F$ or $(F_R, v, x_R)$ is blocked. Therefore, there exists a vertex $x$ in $N_L(u)$ satisfying the condition of Lemma 4.2(a). □ □

# 5 Long fault-free paths - general dimension

In this section we present the proof of our main result on long fault-free paths.

**Theorem 2.1.** *Let $F$ be a set of at most $2n-4$ faulty vertices of $Q_n$ where $n \geq 2$. For every two fault-free vertices $u$ and $v$, there exists an $(F, u, v)$-path in $Q_n$ if and only if $(F, u, v)$ is free and we do not have the exceptional configuration (2).*

*Proof.* The necessity was discussed in Section 2. We proceed by induction on $n$. The statement holds for $n \leq 4$ by the previous section. Now we assume that $n \geq 5$ and we have two fault-free vertices $u$ and $v$ in $Q_n$ such that $(F, u, v)$ is free.

First, we consider the case when $u$ or $v$ has exactly one neighbor uncovered by $F \cup \{u, v\}$. Assume that $u$ has the only neighbor $u'$ uncovered by $F \cup \{v\}$. Clearly, the vertex $v$ is not surrounded by $F \cup \{u'\}$. Let $v'$ be the vertex $v$ if $v$ has at least two neighbors uncovered by $F \cup \{u'\}$, otherwise let $v'$ be the only neighbor of $v$ uncovered by $F \cup \{u'\}$. Since $|F| \leq 2n - 4$, the vertex $u'$ has at least two neighbors uncovered by $F \cup \{v\}$. Moreover, if $u'$ has exactly two such neighbors, then all faulty vertices and the vertex $v$ are neighbors of $u$ or $u'$, so $v$ has at most 3 vertices covered by $F \cup \{u'\}$, and thus $v' = v$. Hence, $u'$ and $v'$ have at least two neighbors uncovered by $F \cup \{u', v'\}$. Furthermore, every $(F, u', v')$-path avoids $u$ (and $v$ if $v' \neq v$), so it can be prolonged to an $(F, u, v)$-path. Therefore, in the following we assume that both $u$ and $v$ have at least two neighbors uncovered by $F \cup \{u, v\}$.

Our aim is to split $Q_n$ into $Q_L$ and $Q_R$ such that $|F_L| \leq 2n - 6$ and $|F_R| \leq 2n - 6$ where $F_L = F \cap V(Q_L)$ and $F_R = F \cap V(Q_R)$. By Proposition 3.5, this can be done with the only exception when $n = 5$, $|F| = 6$, and $F$ consists of some vertex $w$ and all his neighbors. But when this exception happens, we may remove the vertex $w$ from $F$ since it cannot be visited by any path that is fault-free with respect to $F \setminus \{w\}$, so we may assume that the requested split exists.

In what follows, note that whenever we apply induction for a free triple $(F', a, b)$ in $Q_L$ or in $Q_R$, the configuration (2) cannot occur since $d(a, b)$ is odd or $|F'| < 2n - 6$. We assume that $u \in V(Q_L)$ and we distinguish the following cases.

*Case 1: $v \in V(Q_R)$.*

We may assume that $|F_L| \geq |F_R|$. Thus $|F_R| \leq n - 2$. See Figure 6 for an illustration.

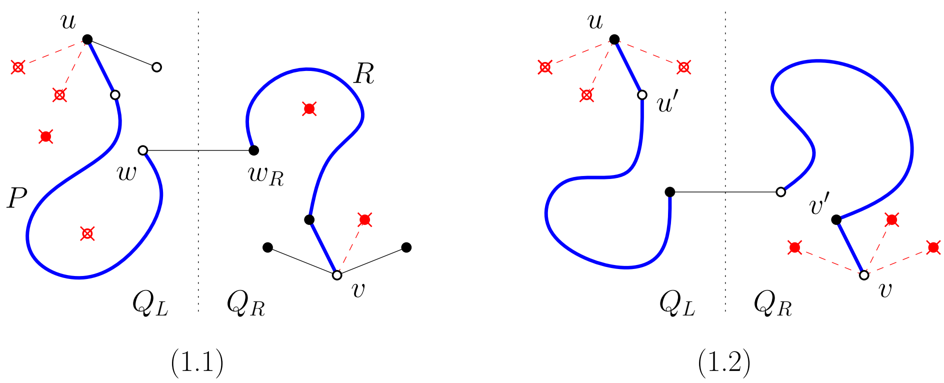



Figure 6: The construction of an $(F, u, v)$-path in Case 1 of Theorem 2.1.

*Subcase 1.1: Both vertices $u$ and $v$ have at least 2 fault-free neighbors in their subcubes.*

It follows for every $w \in V(Q_L)$ that if $(F_L, u, w)$ is blocked in $Q_L$, then $w$ is surrounded by $F_L \cup \{u\}$ in $Q_L$. Similarly for every $w_R \in V(Q_R)$, if $(F_R, v, w_R)$ is blocked in $Q_R$, then $w_R$ is surrounded by $F_R \cup \{v\}$ in $Q_R$.

We claim that there is a vertex $w \in V(Q_L)$ such that $d(u, w)$ is odd, $w_R \neq v$, both $w$ and $w_R$ are fault-free, $(F_L, u, w)$ is free in $Q_L$, and $(F_R, v, w_R)$ is free in $Q_R$. Let $A = \{w \in V(Q_L) \mid d(u, w) \text{ is odd}\}$. We say that a vertex $x \in V(Q_n)$ *eliminates* a vertex $w \in A$ if $w = x$, or $w_R = x$, or $w$ is surrounded by $F_L \cup \{u\}$ and $x$ is a neighbor of $w$, or $w_R$ is surrounded by $F_R \cup \{v\}$ and $x$ is a neighbor of $w_R$. Thus, every vertex $w \in A$ that is not eliminated by any vertex from $F \cup \{v\}$ satisfies the claim. By Proposition 3.1, at most one vertex in $A$ is surrounded by $F_L \cup \{u\}$ in $Q_L$, and at most one vertex $w \in A$ has the neighbor $w_R$ surrounded by $F_R \cup \{v\}$ in $Q_R$. Hence, every vertex from $F \cup \{v\}$ eliminates at most one vertex from $A$. Therefore the claim holds as

$$|A| - |F| - 1 \geq 2^{n-2} - 2n + 3 \geq 1 \text{ for } n \geq 5.$$

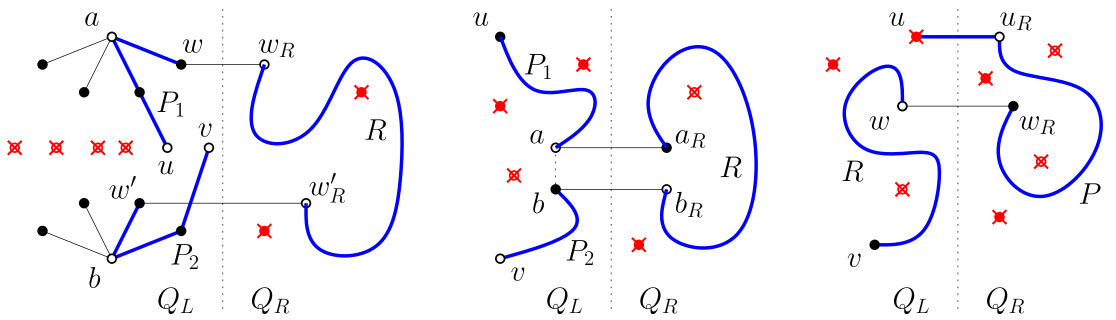
Let $w \in V(Q_L)$ be a vertex satisfying the claim above. By induction, there is an $(F_L, u, w)$-path $P$ in $Q_L$ of length at least $2^{n-1} - 2|F_L| - 1$, and an $(F_R, w_R, v)$-path $R$ in $Q_R$. Therefore, by adding the edge $ww_R$ we obtain an $(F, u, v)$-path $(P, R)$ of length at least $2^{n-1} - 2|F_L| - 1 + 2^{n-1} - 2|F_R| - 2 + 1 = 2^n - 2|F| - 2$.

*Subcase 1.2: Vertex $u$ or $v$ has only 1 fault-free neighbor in its subcube.*

Assume that $u$ has the only fault-free neighbor $u'$ in $Q_L$. Let $v'$ be the vertex $v$ if $v$ has at least two fault-free neighbors in $Q_R$, otherwise let $v'$ be the only fault-free neighbor of $v$ in $Q_R$. Clearly, both $u'$ and $v'$ have at least two fault-free neighbors in their subcubes. By the previous case, there is an $(F, u', v')$-path $P$. Then, $(u, P)$ if $v' = v$, or $(u, P, v)$ if $v' \neq v$, is an $(F, u, v)$-path.

*Case 2: $v \in V(Q_L)$.*

Since both $u$ and $v$ have at least two neighbors uncovered by $F \cup \{u, v\}$, it follows that $(F_L, u, v)$ is free in $Q_L$. See Figure 7 for an illustration.




(2.1) (2.2) (2.3)

Figure 7: The construction of an $(F, u, v)$-path in Case 2 of Theorem 2.1.

*Subcase 2.1: We have the exceptional configuration (2) in $Q_L$.*

Assume that $a, b \in V(Q_L)$ are the vertices in the exceptional configuration (2). Let $w$ and $w'$ be some neighbors of $a$ and $b$, respectively, such that $w_R$ and $w'_R$ are fault-free. Since $|F_R| \leq 2$, the triple $(F_R, w_R, w'_R)$ is free in $Q_R$. Thus, by induction, there is $(F_R, w_R, w'_R)$ path $R$ in $Q_R$. Furthermore, there are disjoint fault-free paths $P_1$ between $u$ and $w$, and $P_2$ between $w'$ and $v$, both of length 3. Therefore, by adding the edges $ww_R$ and $w'_R w'$ we obtain an $(F, u, v)$-path $(P_1, R, P_2)$ of length at least $2^{n-1} - 2|F_R| - 2 + 2 \cdot 3 + 2 = 2^n - 2|F| - 2$.

*Subcase 2.2: We do not have the exceptional configuration (2) in $Q_L$. Moreover, at least one of $u_R$ and $v_R$ is faulty, or $|F_R| \leq 2n - 7$, or $d(u, v)$ is odd.*

Applying induction we obtain an $(F_L, u, v)$-path $P$ in $Q_L$. We claim that there is an edge $ab$ on $P$ so that the edge $a_R b_R \in E(Q_R)$ is fault-free and also $(F_R, a_R, b_R)$ is free. At most $2|F_R|$ edges $a_R b_R \in E(Q_R)$ with $ab$ on $P$ are faulty. However, if at least one of $u_R$ and $v_R$ is faulty, it is less than $2|F_R|$ edges. Furthermore, by Proposition 3.4, we may assume that $(F_R, a_R, b_R)$ is blocked for at most 2 fault-free edges $a_R b_R \in E(Q_R)$ with $ab$ on $P$, otherwise we are done. However, if $|F_R| \leq 2n - 7$, then by Proposition 3.3, $(F_R, a_R, b_R)$ is blocked only for at most 1 fault-free edge $a_R b_R \in E(Q_R)$ with $ab \in E(P)$. Thus, some edge $ab$ on $P$ satisfying the claim exists as

$$\left.\begin{array}{l} E(P) - 2|F_R| - 1 \text{ for } d(u,v) \text{ even} \\ E(P) - 2|F_R| - 2 \text{ for } d(u,v) \text{ odd} \end{array}\right\} \geq 2^{n-1} - 2|F| - 3 \geq 2^{n-1} - 4n + 5 \geq 1 \text{ for } n \geq 5.$$

Hence by induction, there is an $(F_R, a_R, b_R)$-path $R$ in $Q_R$ of length at least $2^{n-1} - 2|F_R| - 1$. Therefore, by removing the edge $ab$ and adding the edges $aa_L$, and $b_L b$ we obtain an $(F, u, v)$-path $(P_1, R, P_2)$ of length at least $2^{n-1} - 2|F_L| - 2 + 2^{n-1} - 2|F_R| - 1 - 1 + 2 = 2^n - 2|F| - 2$ where $P_1$ and $P_2$ are the subpaths of $P \setminus \{ab\}$.

*Subcase 2.3: Both $u_R$ and $v_R$ are fault-free, $|F_R| = 2n - 6$, and $d(u, v)$ is even.*

By Proposition 3.1, at most one of $u_R$ and $v_R$ is surrounded by $F_R$ in $Q_R$. Assume that $u_R$ is not surrounded by $F_R$ in $Q_R$. We put $F'_L = F_L \cup \{u\}$, so $|F'_L| \leq 3 < 2n - 6$. Note that $v$ has at least two neighbors in $Q_L$ that are not in $F'_L$ since $d(u,v)$ is even. It follows for every $w \in V(Q_L)$ that if $(F'_L, v, w)$ is blocked, then $w$ is surrounded by $F'_L \cup \{v\}$.

We claim that there is a vertex $w \in V(Q_L)$ such that $d(v,w)$ is odd, both $w$ and $w_R$ are fault-free, $(F'_L, v, w)$ is free in $Q_L$, and $(F_R, u_R, w_R)$ is free in $Q_R$. Let $A = \{w \in V(Q_L) \mid d(u,w) \text{ is odd}\}$. By Proposition 3.2, $(F_R, u_R, w'_R)$ is blocked for at most one vertex $w' \in A$. If that happens for some $w' \in A$, let $A' = A \setminus \{w'\}$, otherwise let $A' = A$.

We say that a vertex $x \in V(Q_n)$ *eliminates* a vertex $w \in A'$ if $w = x$, or $w_R = x$, or $w$ is surrounded by $F'_L \cup \{v\}$ and $x$ is a neighbor of $w$. Thus, every vertex $w \in A'$ that is not eliminated by any vertex from $F$ satisfies the claim. By Proposition 3.1, at most one vertex in $A$ is surrounded by $F'_L \cup \{v\}$. Hence every vertex from $F$ eliminates at most one vertex from $A$. Therefore the claim holds as

$$|A'| - |F| \geq 2^{n-2} - 2n - 3 \geq 1 \text{ for } n \geq 5.$$

Hence by induction, there is an $(F_R, u_R, w_R)$-path $P$ in $Q_R$ of length at least $2^{n-1} - 2|F_R| - 1$. Furthermore, there is an $(F'_L, w, v)$-path $R$ in $Q_L$ that avoids $u$ and has length at least $2^{n-1} - 2(|F_L|+1) - 1$. Therefore, by adding the edges $uu_R$ and $w_R w$, we obtain an $(F, u, v)$-path $(u, P, R)$ of length at least $2^{n-1} - 2|F_R| - 1 + 2^{n-1} - 2|F_L| - 1 - 2 + 2 = 2^n - 2|F| - 2$. □ □

# 6 Long fault-free cycles

Let $D \subseteq [n]$ be a set of $d = |D|$ coordinates of $Q_n$. We can consider every vertex $x$ of $Q_n$ as a pair $x = (u, v)_D$ where $u \in \{0,1\}^{n-d}$ and $v \in \{0,1\}^d$ are projections of $x$ on the coordinates of $[n] \setminus D$ and $D$, respectively. For $u \in \{0,1\}^{n-d}$ we denote by $Q_D(u)$ the $d$-dimensional *subcube* of $Q_n$ induced by vertices $V_D(u) = \{(u,v)_D \mid v \in \{0,1\}^d\}$. In other words, $Q_D(u)$ is the subcube of $Q_n$ with coordinates $[n] \setminus D$ fixed by $u$. The index $D$ in $(u,v)_D$ is omitted whenever clear from the context.

Let $F$ be a set of faulty vertices of $Q_n$. Recall that a cycle in $Q_n$ is *long* if it has length at least $2^n - 2|F|$. For a set $D \subseteq [n]$ and $u \in \{0,1\}^{n-d}$ we define $F_D(u) = F \cap V_D(u)$. Assume that we want to find a long fault-free cycle in $Q_n$.

Our approach is based on subcube partitioning similar as in the work of Bruck et al. [1] where the hypercube is partitioned into subcubes so that each subcube contains a large fault-free component. However, instead of using the same partitioning as in [1], we apply recent results by Wiener [15] on edge multiplicity of traces in set systems which gives better bounds. We proceed as follows.

First, we find a set $D \subseteq [n]$ such that $|F_D(u)| \leq 2d - 4$ for every $u \in \{0,1\}^{n-d}$ where $d = |D|$. Then, for some Hamiltonian cycle $(u^1, u^2, \ldots, u^{2^{n-d}}, u^{2^{n-d}+1} = u^1)$ of $Q_{n-d}$ we choose in each subcube $Q_D(u^i)$ two appropriate vertices $a^i$ and $b^i$ such that $a^i b^{i+1} \in E(Q_n)$ for every $i \in [2^{n-d}]$. Next, applying Theorem 2.1 we find long fault-free paths between $a^i$ and $b^i$ in each subcube $Q_D(u^i)$. Finally, we glue these paths together and obtain a desired long fault-free cycle in $Q_n$. See Figure 8 for an illustration.

The crucial step is the determination of the set $D$. Although the following theorem by Wiener [15] was originally formulated for set systems, here we take the liberty to formulate it for vertices of the hypercube.

**Theorem 6.1** (Wiener [15]). *Let $F$ be a set of at least $2n$ vertices of $Q_n$, and let $d = \left\lceil \frac{n^2}{2|F|-n-2} \right\rceil$. Then, there exists a set $D \subseteq [n]$, $|D| = d$ such that $|F_D(u)| \leq d+1$ for every $u \in \{0,1\}^{n-d}$.*

For the choice of vertices $a^i$ and $b^i$ we employ the following separate lemma. Recall that a triple $(F, u, v)$ is blocked for $F \subseteq V(Q_n)$ and $u, v \in V(Q_n)$ if $u$ is surrounded by $F \cup \{v\}$ or $v$ is surrounded by $F \cup \{u\}$, otherwise $(F, u, v)$ is free.

**Lemma 6.2.** *Let $F$ be a set of faulty vertices of $Q_n$ where $n \geq 5$, and let $D \subseteq [n]$ be such that $d = |D| = 5$ and $|F_D(u)| \leq 6$ for every $u \in \{0,1\}^{n-d}$. Let $(u^1, u^2, \ldots, u^{2^{n-d}}, u^{2^{n-d}+1} = u^1)$ be a Hamiltonian cycle of $Q_{n-d}$. Then, there are fault-free vertices $a^i$ and $b^i$ in each $Q_D(u^i)$ such that*

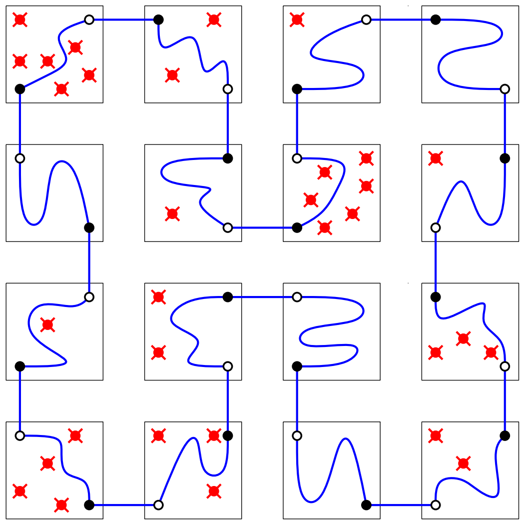


Figure 8: The construction of a long fault-free cycle in Theorem 2.2.

- *$d(a^i, b^i)$ is odd,*
- *$(F_D(u^i), a^i, b^i)$ is free in $Q_D(u^i)$,*
- *$a^i b^{i+1} \in E(Q_n)$ where $b^{2^{n-d}+1} = b^1$,*

*for every $i \in [2^{n-d}]$.*

*Proof.* We determine vertices $a^i$ and $b^i$ in this order: $a^1$, $b^2$, $a^2, \ldots, b^{2^{n-d}}$, $a^{2^{n-d}}$, $b^{2^{n-d}+1} = b^1$. Since $u^i$ and $u^{i+1}$ are neighbors in $Q_{n-d}$, every vertex in $Q_D(u^i)$ has one neighbor in $Q_D(u^{i+1})$. Let $A$ and $B$ be the bipartite classes of $Q_n$. We will choose $a^i = (u^i, v^i)$ from $A \cap V_D(u^i)$ and obtain $b^{i+1} = (u^{i+1}, v^i)$ from $B \cap V_D(u^{i+1})$. Thus $d(a^i, b^i)$ is odd and $a^i b^{i+1} \in E(Q_n)$.

There are 16 vertices in $A_i = A \cap V_D(u^i)$ since $Q_D(u^i)$ is isomorphic to $Q_5$. At most 6 of them are faulty since $|F_D(u^i)| \leq 6$. Furthermore, at most 6 of them have faulty neighbor in $Q_D(u^{i+1})$ since $|F_D(u^{i+1})| \leq 6$.

In each of the cases $i = 1$, $1 < i < 2^{n-d}$, and $i = 2^{n-d}$, we show that amongst the 4 remaining vertices of $A_i$, there are at most two vertices, denoted by $x^i$ and $y^i$, that are not eligible for the choice of $a^i$.

*Case $i = 1$.* By Proposition 3.1, at most one vertex $x^1 \in A_1$ is surrounded by $F_D(u^1)$ in $Q_D(u^1)$. Furthermore, at most one vertex $y^1 \in A_1$ has the neighbor in $Q_D(u^2)$ surrounded by $F_D(u^2)$ in $Q_D(u^2)$.

*Case $1 < i < 2^{n-d}$.* By Proposition 3.2, $(F_D(u^i), x^i, b^i)$ is blocked in $Q_D(u^i)$ for at most one vertex $x^i \in A_i$. By Proposition 3.1, at most one vertex $y^i \in A_i$ has the neighbor in $Q_D(u^{i+1})$ surrounded by $F_D(u^{i+1})$ in $Q_D(u^{i+1})$.

*Case $i = 2^{n-d}$.* By Proposition 3.2, $(F_D(u^i), x^i, b^i)$ is blocked in $Q_D(u^i)$ for at most one vertex $x^i \in A_i$. Furthermore, at most one vertex $y^i \in A_i$ has the neighbor $z$ in $Q_D(u^1)$ such that $(F_D(u^1), a^1, z)$ is blocked in $Q_D(u^1)$.

Hence, by choosing vertices $a^i$ and $b^i$ for every $i \in [2^{n-d}]$ such that

$$a^i = (u^i, v^i) \in A_i \setminus (\{x^i, y^i\} \cup F_D(u^i) \cup F_D^*(u^{i+1})) \text{ for some } v^i \in \{0,1\}^d,$$
$$b^{i+1} = (u^{i+1}, v^i) \text{ and } b^1 = b^{2^{n-d}+1},$$

where $F_D^*(u^{i+1})$ is the set of vertices of $Q_D(u^i)$ that have a faulty neighbor in $Q_D(u^{i+1})$, we obtain that both $a^i$ and $b^i$ are fault-free, and $(F_D(u^i), a^i, b^i)$ is free in $Q_D(u^i)$ for every $i \in [2^{n-d}]$. □ □

Now we are ready to prove Theorem 2.2.

**Theorem 2.2.** *Let $F$ be a set of at most $\frac{n^2}{10} + \frac{n}{2} + 1$ faulty vertices of $Q_n$ where $n \geq 15$. Then $Q_n$ contains a long fault-free cycle.*

*Proof.* Let $F' \supseteq F$ be some set of exactly $\left\lfloor \frac{n^2}{10} + \frac{n}{2} + 1 \right\rfloor$ vertices of $Q_n$. Thus $|F'| \geq 2n$ as $n \geq 15$ and by Theorem 6.1, there is a set $D \subseteq [n]$ such that $d = |D| = 5$ and $|F_D(u)| \leq |F'_D(u)| \leq 6$ for every $u \in \{0,1\}^{n-d}$. Let $(u^1, u^2, \ldots, u^{2^{n-d}}, u^{2^{n-d}+1} = u^1)$ be some Hamiltonian cycle of $Q_{n-d}$.

By Lemma 6.2, there are fault-free vertices $a^i$ and $b^i$ in each $Q_D(u^i)$ such that $d(a^i, b^i)$ is odd, $(F_D(u^i), a^i, b^i)$ is free in $Q_D(u^i)$, and $a^i b^{i+1} \in E(Q_n)$ for every $i \in [2^{n-d}]$ where $b^{2^{n-d}+1} = b^1$.

Hence by Theorem 2.1, in each $Q_D(u^i)$ there is a fault-free path $P_i$ between $b^i$ and $a^i$ of length at least $2^d - 2|F_D(u^i)| - 1$. Concatenating these paths with edges $a^i b^{i+1} \in E(Q_n)$ we obtain a fault-free cycle $(P_1, P_2, \ldots, P_{2^{n-d}}, b^1)$ of length at least

$$2^{n-d} \cdot 2^d - \sum_{i \in [2^{n-d}]} 2|F_D(u^i)| - 2^{n-d} + 2^{n-d} = 2^n - 2|F|.$$

□ □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