ARTICLE

Gray codes extending quadratic matchings

Tomáš Dvořák[1] | Jiří Fink[1]

[1]Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

Correspondence
Tomáš Dvořák, KSVI MFF UK, Malostranské nám. 25, 118 00 Prague, Czech Republic
Email: Tomas.Dvorak@mff.cuni.cz

Funding information
Research supported by the Czech Science Foundation under grant GA14-10799S. The second author was also supported by Charles University project UNCE/SCI/004.

Is it true that every matching in the n-dimensional hypercube Q_n can be extended to a Gray code? More than two decades have passed since Ruskey and Savage asked this question and the problem still remains open. A solution is known only in some special cases, including perfect matchings or matchings of linear size. This paper shows that the answer to the Ruskey-Savage problem is affirmative for every matching of size at most $\frac{n^2}{16} + \frac{n}{4}$. The proof is based on an inductive construction which extends balanced matchings in the completion of the hypercube $K(Q_n)$ by edges of Q_n into a Hamilton cycle of $K(Q_n)$. On the other hand, we show that for every $n \geq 9$ there is a balanced matching in $K(Q_n)$ of size $\Theta(2^n/\sqrt{n})$ which cannot be extended in this way.

KEYWORDS
Gray code, Hamilton cycle, hypercube, matching, Ruskey and Savage problem

1 | INTRODUCTION

An *n-bit cyclic Gray code* is a cyclic ordering of all n-bit strings such that consecutive strings differ in exactly one bit. The applications of Gray codes fall into such diverse areas as signal encoding, image processing, information retrieval or data compression, and constructions of Gray codes satisfying various additional properties have been widely studied [14].

It should be noted that in the literature, cyclic Gray codes are sometimes disguised as Hamilton cycles in a special class of graphs called hypercubes. The *n-dimensional hypercube Q_n* is the graph whose vertex set consists of all n-bit strings, an edge joining two vertices whenever they differ in exactly one bit. An n-bit cyclic Gray code then corresponds to a Hamilton cycle of Q_n. It is therefore natural to describe properties of Gray codes in graph-theoretic terms, referring to the class of hypercubes.

There are several appealing problems related to Gray codes. Probably the most prominent of them was the notorious Middle Levels Conjecture: despite the attention it has attracted, it took over three decades until Mütze answered it affirmatively [15]. Another long-standing problem was formulated in 1993 by Ruskey and Savage [16]: does every matching in a hypercube extend to a cyclic Gray code? The question still remains open, even though partial results have been obtained in several special cases. There are also solutions to related problems, including necessary and sufficient conditions for the existence of Gray codes avoiding given matchings [3] or a recent verification of the Vandenbussche-West conjecture [17] saying that every matching in a hypercube extends to a 2-factor [10].

An affirmative answer to the Ruskey-Savage problem in the case of perfect matchings was provided by the second author of this paper [8]. Note that this implies a positive solution e.g. for every induced matching, as it may always be extended to a perfect one [17]. However, it does not settle the problem in general, as hypercubes contain matchings that are maximal with respect to inclusion but still not perfect [11].

The simplicity of the method of [8] inspired several generalizations [1, 12], but none of them addresses the problem of imperfect matchings. As far as arbitrary matchings are concerned, a positive solution has been verified for $n \le 5$ by a computer search [22] and besides that, there are only partial results dealing with matchings of linear size [2, 4, 18]: the most recent result for matchings of size at most $3n - 10$ was obtained by Wang and Zhang [19]. Matchings of quadratic size were studied by the first author of this paper in [5] where they were extended to a long cycle which not necessarily visits all the vertices. The main result of this paper provides an affirmative answer to the Ruskey-Savage question for matchings of quadratic size.

Theorem 1.1 *Every matching in Q_n, $n \ge 2$, of size at most $\frac{n^2}{16} + \frac{n}{4}$ can be extended to a cyclic Gray code.*

The proof of Theorem 1.1 generalizes the method pioneered in [8] which actually shows that every perfect matching of the complete graph $K(Q_n)$ built on the vertices of Q_n can be extended to a Hamilton cycle of $K(Q_n)$ using only edges of Q_n. One may ask whether such a statement can be generalized to an arbitrary matching of $K(Q_n)$. If true, this would imply a positive solution to the Ruskey-Savage problem. However, it is easy to see that such a generalization does not hold since a single edge joining two vertices of the same bipartite class of Q_n cannot be extended to a Hamilton cycle of $K(Q_n)$ by edges of Q_n. More generally, every matching of $K(Q_n)$ extendable in this way is balanced in the sense that it is incident with the same number of vertices from both bipartite classes of Q_n (Observation 4.1). Unfortunately, this natural necessary condition is insufficient to guarantee the desired extendability, even for a proper subclass of balanced matchings in $K(Q_n)$ formed by the matchings in $B(Q_n)$ which denotes the complete bipartite graph on the vertices of Q_n containing all edges of Q_n.

Theorem 1.2 *For every $n \ge 9$ there is a matching of size $2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} + 1 = \Theta(2^n/\sqrt{n})$ in $B(Q_n)$ which cannot be extended to a Hamilton cycle of $B(Q_n)$ using edges of Q_n.*

The paper is organized as follows. The next section introduces concepts and notations, supplemented by an informal outline of the proof of the main theorem. Section 3 describes basic tools that we use as building blocks for our construction. Section 4 studies the problem in small dimensions while Section 5 describes the inductive construction that settles the general case. The text is concluded with Section 6 which summarizes the main results of the paper, both positive and negative.

2 | PRELIMINARIES

In this section we introduce terminology and notation used throughout the paper. In the rest of the text, n always denotes a positive integer while $[n]$ stands for the set $\{1, 2, \ldots, n\}$. Vertex and edge sets of a graph G are denoted by

$V(G)$ and $E(G)$, respectively. Given a set $V \subseteq V(G)$, $G[V]$ denotes the subgraph of G induced by V while $G - V$ stands for the subgraph $G[V(G) \setminus V]$. If $V = \{v\}$, we simplify the notation by using $G - v$ instead of $G - \{v\}$.

2.1 | Paths and cycles

Given a nonnegative integer m, a sequence $(x_1, x_2, \ldots, x_{m+1})$ of pairwise distinct vertices such that x_i and x_{i+1} are adjacent for all $i \in [m]$ is a *path between* x_1 and x_{m+1} of *length* m. We denote the vertex set $\{x_1, x_2, \ldots, x_{m+1}\}$ of such a path P by $V(P)$. Note that the case that $m = 0$ is allowed, as P is then a path of length 0 consisting of a single vertex. In the rest of this text we use the following convention: instead of "let P_{ab} be a path between a and b" we only say "let P_{ab} be a path", as the endvertices of P_{ab} are specified by the subscripts a and b. Let P_{ab} and P_{bc} be paths such that $V(P_{ab}) \cap V(P_{bc}) = \{b\}$. Then $P_{ab} + P_{bc}$ denotes the path between a and c, obtained as a concatenation of P_{ab} with P_{bc} (where b is taken only once). Observe that the operation $+$ is associative. A collection $\{P_i\}_{i=1}^k$ of paths in a graph G forms a *disjoint path cover* of G if $\{V(P_i)\}_{i=1}^k$ partitions $V(G)$.

A *cycle* of length n is a path $(x_1, x_2, \ldots, x_n)$ where x_1 is adjacent to x_n. The sets of vertices $\{x_1, x_2, \ldots, x_n\}$ and edges $\{x_1x_2, x_2x_3, \ldots, x_nx_1\}$ of a cycle C are denoted by $V(C)$ and $E(C)$, respectively. A *Hamilton cycle* (*Hamilton path*) of a graph G is a cycle (path) that visits each vertex of G exactly once.

2.2 | Hypercubes

The n-dimensional hypercube Q_n is a graph with all n-bit strings as vertices, an edge joining two vertices whenever they differ in a single bit. The *parity* $\chi(v)$ of a vertex $v = v_1v_2 \cdots v_n$ and the *balance* $\beta(S)$ of a set $S \subseteq V(Q_n)$ are defined by $\chi(v) := \prod_{i=1}^n (-1)^{v_i}$ and $\beta(S) := \sum_{v \in S} \chi(v)$, respectively. Set S is called *balanced* if $\beta(S) = 0$. The *weight* $|v|$ of $v \in V(Q_n)$ is defined by $|v| := |\{i \mid v_i = 1\}|$. We use $d(u, v)$ to denote the Hamming distance of $u, v \in V(Q_n)$, i. e. $d(u, v) := |\{i \mid u_i \neq v_i\}|$. Vertices u, v of Q_n are *diametrical* if $d(u, v) = n$.

Let $K(Q_n)$ denote the complete graph on the set of vertices of Q_n. The set $\dim(uv)$ of *dimensions* of an edge $uv \in E(K(Q_n))$, $u = u_1 \cdots u_n$, $v = v_1 \cdots v_n$, is defined by $\dim(uv) := \{i \in [n] \mid u_i \neq v_i\}$. For a vertex $v \in V(Q_n)$ let v^d denote the vertex of Q_n such that $\dim(vv^d) = \{d\}$. The edge vv^d is called a *short edge of dimension d*. Given a subgraph G of $K(Q_n)$, we say that a set of edges $E \subseteq E(G)$ is *G-extendable* if there is a Hamilton cycle C of G such that $E(C) = E \cup E'$ where $E' \subseteq E(Q_n)$.

We use $B(Q_n)$ to denote the spanning subgraph of $K(Q_n)$ containing only edges $uv \in E(K(Q_n))$ such that $d(u, v)$ is odd. While $K(Q_n)$ is a completion of Q_n, $B(Q_n)$ may be viewed as a bipartite completion of Q_n.

Given a string $v = v_1v_2 \cdots v_n$ and a set $D \subseteq [n]$, we use v_D to denote the string $v_{j_1}v_{j_2} \cdots v_{j_d}$ where $j_1, j_2, \ldots, j_d$ is an increasing sequence of all elements of D. Given a nonempty set $D \subseteq [n]$ of size $d \geq 1$ and a vertex $u \in V(Q_{n-d})$, the subgraph of Q_n induced by the set $\{v \in V(Q_n) \mid v_{\bar{D}} = u\}$ where $\bar{D} := [n] \setminus D$ is denoted by $Q_D(u)$ and called a *subcube of dimension d*. Subcubes $Q_D(v)$ and $Q_D(v^i)$ are called *adjacent over dimension $i \in \bar{D}$*. Given $V \subseteq V(Q_n)$ and a set $\mathcal{S}$ of subcubes of Q_n, we use

- $V_D(u)$ to denote $V \cap V(Q_D(u))$,
- $V(\mathcal{S})$ to denote $\bigcup_{Q \in \mathcal{S}} V(Q)$,
- $K(\mathcal{S})$ to denote $K(Q_n)[V(\mathcal{S})]$.

Note that in further text we consider only sets of subcubes that are pairwise vertex-disjoint.

2.3 | Matchings

A *matching* is a set of edges such that no two of them have a vertex in common. A matching M in a graph G is called a *perfect* matching of G if every vertex of G is incident with an edge of M. Given a matching M, we use $V(M)$ to denote the set of all vertices incident with an edge of M.

Let M be a matching in $K(Q_n)$. M is called *balanced* if $V(M)$ is a balanced subset of $V(Q_n)$. Given subcubes Q, Q' and a set $\mathcal{S}$ of subcubes of Q_n,

- $M(Q)$ denotes the matching $\{uv \in M \mid u, v \in V(Q)\}$,
- $M(\mathcal{S})$ denotes the matching $\{uv \in M \mid u, v \in V(\mathcal{S})\}$,
- $M(Q, Q')$ denotes the matching $\{uv \in M \mid u \in V(Q), v \in V(Q')\}$,
- $M(Q, \mathcal{S})$ denotes the matching $\{uv \in M \mid u \in V(Q), v \in V(\mathcal{S})\}$.

Observe that removing all edges of some fixed dimension d splits Q_n into two $(n-1)$-dimensional subcubes $Q^0 := Q_{[n]\setminus\{d\}}(0)$ and $Q^1 := Q_{[n]\setminus\{d\}}(1)$. We use M_d^i to denote $M \cap E(K(Q^i))$ for both $i \in \{0, 1\}$ and $M_d^2 := M \setminus (M_d^0 \cup M_d^1)$.

Recall that an n-bit cyclic Gray code may be viewed as a Hamilton cycle of Q_n. In the rest of this paper we often resort to this equivalent formulation and speak about Hamilton cycles rather than Gray codes as it allows to use standard graph-theoretic terminology.

2.4 | Sketch of the proof of Theorem 1.1

Let M be an arbitrary matching in Q_n of size at most $\frac{n^2}{16} + \frac{n}{4}$. If the dimension is sufficiently large, apply Theorem 3.2 due to Wiener [20, 21] to partition Q_n into a set of 5-dimensional subcubes such that each of them contains at most six vertices of $V(M)$. Then apply Lemma 5.2 to form a matching $\bar{M} \supseteq M$ such that the set of vertices of $V(\bar{M})$ in an arbitrary subcube is balanced while its size remains reasonably small and, moreover, a connectivity condition is satisfied. Finally, apply Theorem 5.3 to extend $\bar{M}$ to a Hamilton cycle. The proof of Theorem 5.3 is based on Lemma 3.3 which is a refinement of a method originally devised to extend perfect matchings to Gray codes [9, 12].

In the case of small dimensions, the extendability of M follows from Corollary 4.6 which is built on our recent results on disjoint path covers of hypercubes.

3 | TOOLS

Now we are ready to describe the tools that are necessary for our constructions of Hamilton cycles in Sections 4 and 5. First recall several well-known properties of hypercubes.

Proposition 3.1 *Let $n \geq 2$.*

(1) [6, Proposition 3.1] *If $\{P_{u_i v_i}\}_{i=1}^m$ forms a disjoint path cover of $Q_n[V]$ for some $V \subseteq V(Q_n)$, then $\frac{1}{2}\sum_{i=1}^m (\chi(u_i) + \chi(v_i)) = \beta(V)$.*

(2) [13, Proposition 2.3] *There is a Hamilton path P_{uv} in Q_n for every $u, v \in V(Q_n)$ with $\chi(u) \neq \chi(v)$.*

(3) [4, Lemma 3.3] *There is a disjoint path cover $\{P_{uv}, P_{xy}\}$ of Q_n for every distinct $u, v, x, y \in V(Q_n)$ with $\chi(u) \neq \chi(v)$ and $\chi(x) \neq \chi(y)$.*

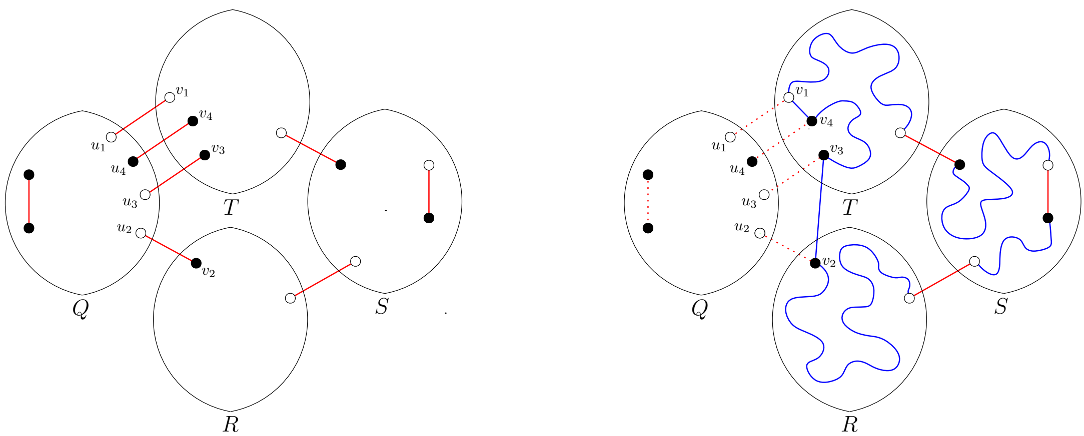
(4) [7, Corollary 4.9] *There is a disjoint path cover $\{P_{u_i v_i}\}_{i=1}^k$ of Q_n, $n \geq 5$, for every $k \in [n-1]$ and distinct $\{u_i, v_i\}_{i=1}^k \subseteq V(Q_n)$ with $\chi(u_i) \neq \chi(v_i)$ for all $i \in [k]$.*

(5) [8, Theorem] *Every perfect matching of $K(Q_n)$ is $K(Q_n)$-extendable.*

The following result of Wiener [20, 21] is employed in Section 6 to partition a hypercube into 5-dimensional subcubes.

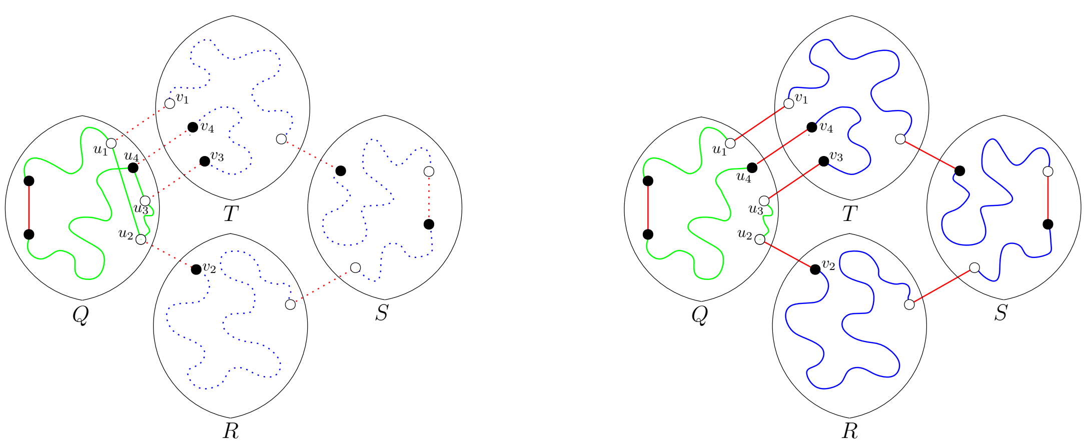
Theorem 3.2 ([20, Theorem 2.5]) *Let S be a set of vertices of Q_n of size $s \geq 2n$ and $d = \lceil \frac{n^2}{2s-n-2} \rceil$. Then there is a set $D \subseteq [n]$ such that $|D| = d$ and $|S_D(u)| \leq d+1$ for every $u \in \{0,1\}^{n-d}$.*

The following lemma serves as a cornerstone for our inductive constructions in the next two sections. While in Section 4 it is only applied in the special case when $|\mathcal{S}| = 2$, the general formulation is needed in Section 5 (Theorem 5.3). Note that it refines a method which was originally devised to extend perfect matchings to Gray codes [9, 12].



(a) Matching M (red lines) in $K(\mathcal{S})$, $\mathcal{S} = \{Q, R, S, T\}$, $\mathcal{S}' = \{R, S, T\}$, $N = \{u_i v_i\}_{i=1}^4$.

(b) Matchings $P' = \{v_1 v_4, v_2 v_3\}$ (blue lines) and $M(\mathcal{S}')$ (red lines) extended (blue paths) to a Hamilton cycle C' of $K(\mathcal{S}')$.



(c) Matchings $P'' = \{u_1 u_2, u_3 u_4\}$ (green lines) and $M(Q)$ (red line) extended (green paths) to a Hamilton cycle C'' of $K(Q)$.

(d) Replacing $u_1 u_2$ with $(u_1, v_1) + P_{v_1 v_2} + (v_2, u_2)$ and $u_3 u_4$ with $(u_3, v_3) + P_{v_3 v_4} + (v_4, u_4)$ extends C'' to a Hamilton cycle C of $K(\mathcal{S})$.

FIGURE 1 Illustration of the proof of Lemma 3.3. Black and white circles represent vertices of different parities.

Lemma 3.3 *Let $\mathcal{S}$ be a set of pairwise vertex-disjoint subcubes of Q_n, $n \geq 2$, $Q \in \mathcal{S}$ and M be a matching in $K(\mathcal{S})$. Put $\mathcal{S}' := \mathcal{S} \setminus \{Q\}$, $N := M(Q, \mathcal{S}')$ and let P' be a perfect matching of $K(\mathcal{S}')[V(N) \cap V(\mathcal{S}')]$. If*

(i) $N \neq \emptyset$,
(ii) *$V(M) \cap V(Q)$ is balanced,*
(iii) *$M(\mathcal{S}') \cup P'$ is $K(\mathcal{S}')$-extendable,*
(iv) *every balanced matching in $K(Q)$ of size $|M(Q)| + \frac{1}{2}|N|$ is $K(Q)$-extendable,*

then M is $K(\mathcal{S})$-extendable.

Proof Let $u_1v_1, u_2v_2, \ldots, u_kv_k$ be all edges of N where $u_i \in V(Q)$ and $v_i \in V(\mathcal{S}')$ for all $i \in [k]$. Note that (i)-(ii) imply that k is positive and even and therefore we can select a perfect matching P' of $K(\mathcal{S}')[\{v_1, v_2, \ldots, v_k\}]$. By (iii), there is a Hamilton cycle C' of $K(\mathcal{S}')$ such that $E(C') = M(\mathcal{S}') \cup P' \cup E'$ and all edges of E' are short, see Figure 1(b).

The removal of edges of P' splits C' into a disjoint path cover of $K(\mathcal{S}')$ and we can without loss of generality assume that it consists of paths $P_{v_1v_2}, P_{v_3v_4}, \ldots, P_{v_{k-1}v_k}$. Put $P'' := \{u_1u_2, u_3u_4, \ldots, u_{k-1}u_k\}$. By (iv) there is a Hamilton cycle C'' of $K(Q)$ such that $C'' = M(Q) \cup P'' \cup E''$ where $E'' \subseteq E(Q)$, see Figure 1(c).

It remains to replace each edge $u_iu_{i+1} \in P''$ in C'' with the path $(u_i, v_i) + P_{v_iv_{i+1}} + (v_{i+1}, u_{i+1})$. This extends C'' to a Hamilton cycle C of $K(\mathcal{S})$ such that $E(C) = M \cup E' \cup E''$, see Figure 1(d). Since $E' \cup E'' \subseteq E(Q_n)$, it follows that M is $K(\mathcal{S})$-extendable. ∎

4 | SMALL DIMENSIONS

In this section we show that Fink's result on extendability of perfect matchings in $K(Q_n)$ — stated above as part (5) of Proposition 3.1 — may be generalized to arbitrary matchings at least for small values of n. First observe that balancedness is necessary for such an extension to exist.

Observation 4.1 *If a matching M in $K(Q_n)$ is $K(Q_n)$-extendable, then M must be balanced.*

Proof If C is a Hamilton cycle of $K(Q_n)$ such that $E(C) \setminus M \subseteq E(Q_n)$, then $E(C) \setminus M$ forms a disjoint path cover of Q_n by paths with endvertices $V(M)$. Part (1) of Proposition 3.1 says that then $\frac{1}{2}\beta(V(M)) = \beta(V(Q_n)) = 0$ which means that the matching M is balanced. ∎

We show that for small dimensions, this natural necessary condition is also sufficient. The next two lemmas settle the cases $n \in \{3, 4\}$.

Lemma 4.2 *Every balanced matching M in $K(Q_3)$ is $K(Q_3)$-extendable. Moreover, if $V(Q_3) \setminus V(M)$ consists of diametrical vertices u, v of Q_3 and $uu' \in E(Q_3) \setminus M$ is a given short edge, then $M \cup \{uu'\}$ is $K(Q_3)$-extendable too.*

Proof Let M be a balanced matching in $K(Q_3)$. For $|M| \in \{1, 2, 4\}$ the statement follows from parts (2),(3) and (5) of Proposition 3.1, respectively. So assume that $|M| = 3$ and let u, v be the two vertices of Q_3 not incident with any edge of M. Note that as both $V(Q_3)$ and $V(M)$ are balanced, $\{u, v\}$ must be balanced as well. If uv is a short edge, then $M \cup \{uv\}$ is a balanced matching of size four which can be extended by the previous part of the proof. Hence it remains to settle the case that u, v are diametrical vertices of Q_3.

Since $M = \{u_i, v_i\}_{i=1}^{3}$ is balanced, there are only two options to be considered (Figure 2):

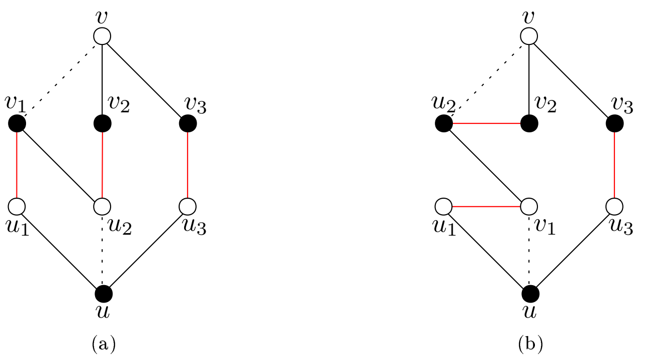


FIGURE 2 A special case in the proof of Lemma 4.2. Red and black lines depict edges of M and the remaining (short) edges of the Hamilton cycle, respectively.

(a) either $\chi(u_i) \neq \chi(v_i)$ for all $i \in [3]$, then we can assume that uu_i and vv_i are short edges for all $i \in [3]$ and $uu' = uu_1$,

(b) or $\chi(u_1) = \chi(v_1) = \chi(u_3) \neq \chi(v_3) = \chi(u_2) = \chi(v_2)$, then we can assume that uu_1, uv_1, uu_3 as well as vu_2, vv_2 and vv_3 are short edges and $uu' \in \{uu_1, uu_3\}$.

Note that there must be a short edge from v_1 to u_2 or u_3 (in case (a)) and from v_1 to u_2 or v_2 (in case (b)). In both cases assuming without loss of generality the former,

$$u, u_1, v_1, u_2, v_2, v, v_3, u_3$$

is the desired Hamilton cycle passing through uu'. ∎

Building on the previous lemma supplemented with a necessary case analysis, we can derive a similar result for the dimension $n = 4$.

Lemma 4.3 *Every balanced matching M in $K(Q_4)$ is $K(Q_4)$-extendable.*

Proof (A) Case $|M| = 8$ follows from part (5) of Proposition 3.1.

(B) If $|M| = 7$ then $V(Q_4) \setminus V(M) = \{u, v\}$ and as both M and $V(Q_n)$ are balanced, we have $\chi(u) \neq \chi(v)$. If uv is a short edge, then $M \cup \{uv\}$ is a perfect matching and the statement follows from part (A). Otherwise it must be the case that $d(u, v) = 3$ and hence Q_4 may be split by some dimension d into two 3-dimensional subcubes Q^0, Q^1 such that u and v are diametrical vertices of Q^0. Note that then $V(M) \cap V(Q^0)$ is balanced. Consequently, if $M_d^2 \neq \emptyset$, we may apply Lemmas 3.3 and 4.2 and the statement follows. Otherwise M_d^1 is a balanced matching in Q^1 and hence by Lemma 4.2 it may be extended by short edges to a Hamilton cycle C^1 of Q^1. Note that $E(C^1) \setminus M_d^1$ must contain a short edge $u^d w^d$ for some $w \in V(Q^0)$. Apply Lemma 4.2 again to obtain a Hamilton cycle C^0 of Q^0 extending $M_d^0 \cup \{uw\}$ by short edges. Removing uw and $u^d w^d$ from C^0 and C^1, respectively, and adding uu^d and ww^d combines both cycles into the desired Hamilton cycle of $K(Q_4)$ that extends M by short edges.

(C) If $|M| = 6$ then $V(Q_4) \setminus V(M)$ is a balanced set $\{u, v, w, x\}$. If two of these four vertices, say u and v, are incident with a short edge, then $M \cup \{uv\}$ is a balanced matching of size 7 and therefore extendable by part (B). If not, then Q_4 may be split into two subcubes such that u, v and w, x are diametrical vertices of Q^0 and Q^1, respectively. In this case we may use the same construction as in part (B).

(D) If $|M| \leq 5$, then $V(Q_4) \setminus V(M)$ is a balanced set of at least six vertices. It is easy to see that at least two of them must be adjacent and hence the extendability follows from part (C). ∎

Before proceeding to higher dimensions, we need the following "balancing" lemma.

Lemma 4.4 *For every balanced matching M in $K(Q_n)$ with $|M| \leq 2^{n-2}$ and $d \in [n]$ there is a balanced matching $\bar{M} \supseteq M$ such that*

(i) *$V(\bar{M}) \cap V(Q^i)$ is a balanced set of at most $2|M|$ vertices for both $i \in \{0, 1\}$,*
(ii) *$\bar{M} \setminus M$ consists only of short edges of dimension d*

where $Q^i := Q_{[n]\setminus\{d\}}(i)$.

Proof For both $i \in \{0, 1\}$ and $p \in \{-1, 1\}$ put $V^i_p := \{v \in V(M) \cap V(Q^i) \mid \chi(v) = p\}$ and note that if $|V^i_{-1}| = |V^i_1|$ for $i = 0$, then it holds also for $i = 1$ and it suffices to set $\bar{M} := M$. Hence we can without loss of generality assume that $|V^0_{-1}| < |V^0_1|$. Put $a := |V^0_{-1}|$ and $b := |V^1_1|$, then $|V^0_1| = a + c$ for some $c > 0$ and since M is balanced, we have $|V^1_{-1}| = b + c$. Note that $|V(M)| = 2(a + b + c)$ and therefore $|M| = a + b + c$. Then

$$|\{v \in V(Q^0) \mid v \in V^0_{-1} \text{ or } v^d \in V^1_1\}| \leq a + b \leq 2^{n-2} - c.$$

Since $|\{v \in V(Q^0) \mid \chi(v) = -1\}| = 2^{n-2}$, it follows that we can select c vertices $v_1, v_2, \ldots, v_c$ from this set such that $\{v_i, v^d_i\}^c_{i=1} \cap V(M) = \emptyset$. Then $\bar{M} := M \cup \{v_i v^d_i\}^c_{i=1}$ is the balanced matching satisfying the statement of the lemma. ∎

Armed with the previous lemma, we can show that small matchings are extendable also in larger dimensions. The following proposition will serve later as the induction basis for the proof of Theorem 5.3.

Proposition 4.5 *Every balanced matching M in $K(Q_n)$ such that $n \geq 2$ and $|M| \leq 8$ is $K(Q_n)$-extendable.*

Proof Case $n = 2$ may be verified by inspection, while cases $n = 3, 4$ are settled by Lemmas 4.2 and 4.3. If $n \geq 5$, select $d \in [n]$ such that $M^2_d \neq \emptyset$. Then apply Lemma 4.4 to add a sufficient number of short edges of dimension d and form a balanced matching $\bar{M} \supseteq M$ such that

$$V(\bar{M}) \cap V(Q^i) \text{ is a balanced set of at most } 2|M| \leq 16 \text{ vertices for both } i \in \{0, 1\} \tag{$*$}$$

where $Q^i := Q_{[n]\setminus\{d\}}(i)$.

Next we apply Lemma 3.3 to show that $\bar{M}$ is $K(Q_n)$-extendable. To that end, put $\mathcal{S} := \{Q^0, Q^1\}$, $N := \bar{M}(Q^0, Q^1)$ and observe that $(*)$ implies that $|V(N) \cap V(Q^1)|$ is even and therefore there is a perfect matching P' of $K(Q^1)[V(N) \cap V(Q^1)]$. The assumptions (i)-(ii) of Lemma 3.3 hold by the choice of d and by $(*)$, respectively. To prove that $\bar{M}$ is $K(Q_n)$-extendable, we argue by induction on $n \geq 5$. If $n = 5$, the remaining assumptions (iii)-(iv) of Lemma 3.3 follow from Lemma 4.3. If $n > 5$, $(*)$ implies that neither $|\bar{M}(Q^1) \cup P'|$ nor $|\bar{M}(Q^0)| + \frac{1}{2}|N|$ exceeds 8 and therefore the validity of (iii)-(iv) follows from the induction hypothesis. Hence in any case, $\bar{M}$ is $K(Q_n)$-extendable by Lemma 3.3. As $\bar{M} \setminus M \subseteq E(Q_n)$ by Lemma 4.4 (ii), M is $K(Q_n)$-extendable as well and the proof is complete. ∎

Employing our findings on disjoint path covers of hypercubes, we can make the above proposition even a little stronger. This will prove useful in the last section where the following corollary settles the case of small dimensions in the proof of the main result (Theorem 1.1).

Corollary 4.6 *Every balanced matching M in $K(Q_n)$ such that $n \geq 2$ and $|M| \leq \max(8, n-1)$ is $K(Q_n)$-extendable.*

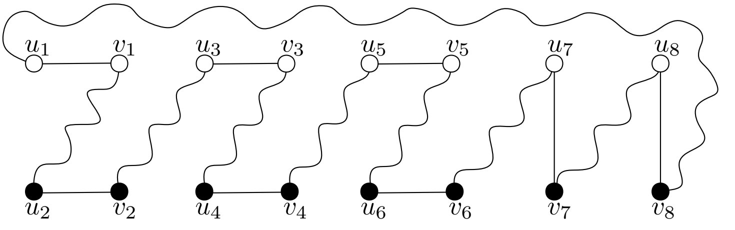


FIGURE 3 Illustration of the proof of Corollary 4.6 for $m = 6$ and $k = 8$.

Proof For $n \leq 9$ the statement follows from Proposition 4.5. To settle the case that $n \geq 10$, let $M = \{u_i v_i\}_{i=1}^{k}$ where $k \in [n-1]$. Since M is balanced, we can without loss of generality assume that there is even $m \geq 0$ such that $\chi(u_1) = \chi(u_{2i-1}) = \chi(v_{2i-1}) \neq \chi(u_{2i}) = \chi(v_{2i})$ for all $i \in [m/2]$ while $\chi(u_1) = \chi(u_i) \neq \chi(v_i)$ for all $i \in [k] \setminus [m]$. Put

$$\mathcal{P} := \{v_1 u_2, v_2 u_3, v_3 u_4, v_4 u_5, \dots, v_{m-1} u_m, v_m u_{m+1}, v_{m+1} u_{m+2}, \dots, v_{k-1} u_k, v_k u_1\}$$

and observe that by part (4) of Proposition 3.1, there is a disjoint path cover $\{P_{uv}\}_{uv \in \mathcal{P}}$ of Q_n, see Figure 3. Then $\bigcup_{uv \in P} E(P_{uv}) \cup M$ is edge set of a Hamilton cycle C of $K(Q_n)$ such that $E(C) = M \cup E$ where $E \subseteq E(Q_n)$. Hence M is $K(Q_n)$-extendable in this case, too. ∎

Considering the results of this section, it is natural to ask: can we generalize Proposition 3.1 (5) to an arbitrary balanced matching in $K(Q_n)$? In Section 6 we show that the answer to this question is negative.

5 | CONSTRUCTION

We start with a refinement of a "balancing" lemma whose goal is similar to that of Lemma 4.4, but the assumptions are more specialized and formulated in much greater detail.

Lemma 5.1 *Let M be a matching in $B(Q_n)$, $a, b, c, e, f \in \mathbb{N}$, $p \in \{-1, 1\}$ and Q, R be adjacent subcubes of dimension $m \in [n-1]$ in Q_n such that*

- *vertices of Q and R are incident with at most e and at most f edges of M, respectively,*
- *$V(M(Q,R)) \cap V(Q)$ contains a and b vertices of parities p and $-p$, respectively,*
- *all edges of $M(Q,R)$ are short.*

If $e + f - a - 2b + c \leq 2^{m-1}$, then there is a matching M' in Q_n such that

(i) *M' consists of c edges between Q and R,*

(ii) $V(M') \cap V(M) = \emptyset$,

(iii) $V(M') \cap V(Q)$ *contains only vertices of parity* p.

Proof Suppose that subcubes Q and R are adjacent over dimension d. Denote the set of all vertices of Q of parity p by V_p, then $|V_p| = 2^{m-1}$ and

$$|\{v \in V_p \mid v \in V(M)\}| \leq e - b$$
$$|\{v \in V_p \mid v \notin V(M) \text{ and } v^d \in V(M)\}| \leq f - (a + b).$$

Since $e - b + f - (a + b) \leq 2^{m-1} - c$, there are vertices $v_1, v_2, \ldots, v_c \in V_p$ such that $\{v_i, v_i^d\} \cap V(M) = \emptyset$ for all $i \in [c]$. It remains to set $M' = \{v_1 v_1^d, v_2 v_2^d, \ldots, v_c v_c^d\}$. ∎

Employing the previous result, we derive even more refined version of a "balancing" lemma which will be used in the proof of the main theorem. This time we consider a more general situation when the hypercube is split into a set of 5-dimensional subcubes. However, to be able to formulate it, we need to introduce a new concept.

Let $\mathcal{S}$ be a set of pairwise vertex-disjoint subcubes of Q_n and $E \subseteq E(K(Q_n))$. We define the *interconnection graph* $\mathcal{I}(\mathcal{S}, E)$ *of* $\mathcal{S}$ *and* E as the graph with vertices $\mathcal{S}$ and edges between two subcubes whenever E contains an edge between a vertex of one subcube and a vertex of the other.

Lemma 5.2 *Let* M *be a matching in* Q_n, $n \geq 5$, *and* $D \subseteq [n]$ *such that* $|D| = 5$ *and* $\mathcal{S} := \{Q_D(u) \mid u \in \{0,1\}^{n-5}\}$ *satisfies* $|V(M) \cap V(Q)| \leq 6$ *for every* $Q \in \mathcal{S}$. *Then there is a matching* $\bar{M} \supseteq M$ *in* Q_n *such that*

(i) $\mathcal{I}(\mathcal{S}, \bar{M})$ *is connected,*

(ii) $V(\bar{M}) \cap V(Q)$ *is balanced,*

(iii) $|V(\bar{M}) \cap V(Q)| \leq 14$

for every subcube $Q \in \mathcal{S}$.

Proof Let $m := 2^{n-5}$ and $u_1, u_2, \ldots, u_m$ be a Hamilton cycle in Q_{n-5} (which exists e.g. by Proposition 3.1 (2)). For $i \in [m]$ put $Q^i := Q_D(u_i)$, then $Q^1, Q^2, \ldots, Q^m$ is an ordering of all subcubes of $\mathcal{S}$ such that any two consecutive subcubes are adjacent. We construct the desired matching $\bar{M}$ in two passes through this sequence: the 1st pass adds edges to make (ii) true while the 2nd one adds some more to guarantee (i).

(1st pass) Our goal is to construct matchings $M^{(i)} \supseteq M$ and $N^{(i)}$ for every $i \in [m-1]$ such that

(a) $V(M^{(i)}(Q^j, Q^k)) \cap V(Q^j)$ is balanced for all $j \in [i]$ and $k \in [m]$ such that Q^j and Q^k are adjacent,

(b) $V(N^{(i)}) = V(M^{(i)})$,

(c) $|\{e \in N^{(i)} \mid e \cap V(Q^j) \neq \emptyset\}| = |\{e \in M \mid e \cap V(Q^j) \neq \emptyset\}|$ for all $j \in [m]$.

Note that while $M^{(i)}$ is used to set up the desired matching $\bar{M}$, $N^{(i)}$ serves only as a tool to prove condition (iii): $N^{(i)}$ is useful for this purpose because it satisfies the invariant (c) while preserving the same set of endvertices as $M^{(i)}$ (property (b)).

To start the construction, put $M^{(0)} := N^{(0)} := M$ and assume that we have already constructed matchings $M^{(i-1)} \supseteq M$ and $N^{(i-1)}$ satisfying (a)–(c). Put $M^{(i,0)} := M^{(i-1)}$, $N^{(i,0)} := N^{(i-1)}$ and let Q^k be a subcube such that Q^i and Q^k are adjacent over dimension d while $V(M^{(i,0)}(Q^i, Q^k)) \cap V(Q^i)$ consists of a and b vertices of parities p and $-p$ for some

$p \in \{-1, 1\}$, respectively. We can without loss of generality assume that $a < b$. Note that condition (a) says that $k > i$ while (c) implies that both Q^i and Q^k are incident with at most 6 edges of $N^{(i,0)}$. As $b \geq 1$, we have

$$2 \cdot 6 - a - 2b + (b - a) \leq 11 < 2^{5-1}$$

and hence Lemma 5.1 guarantees the existence of $b - a$ edges $v_1 v_1^d, v_2 v_2^d, \ldots, v_{b-a} v_{b-a}^d$ joining vertices of Q^i of parity p with vertices of Q^k of parity $-p$, having no vertex in common with $V(N^{(i,0)}) = V(M^{(i,0)})$. Put $M^{(i,1)} := M^{(i,0)} \cup \{v_1 v_1^d, v_2 v_2^d, \ldots, v_{b-a} v_{b-a}^d\}$ and observe that at this point, (a) holds for $M^{(i,1)}$, Q^i and Q^k. To construct $N^{(i,1)}$, select $b - a$ edges $u_1 u_1^d, u_2 u_2^d, \ldots, u_{b-a} u_{b-a}^d$ of $M^{(i,0)}(Q^i, Q^k)$ such that u_i is a vertex of Q^i of parity p for all $i \in [b - a]$ and set

$$N^{(i,1)} := (N^{(i,0)} \setminus \{u_1 u_1^d, u_2 u_2^d, \ldots, u_{b-a} u_{b-a}^d\}) \cup \{u_1 v_1, u_1^d v_1^d, u_2 v_2, u_2^d v_2^d, \ldots, u_{b-a} v_{b-a}, u_{b-a}^d v_{b-a}^d\}.$$

Note that both (b) and (c) hold for $N^{(i,1)}$. Repeating this step for all, say r, subcubes adjacent to Q^i, we finally construct matchings $M^{(i,r)}$ and $N^{(i,r)}$. Set $M^{(i)} := M^{(i,r)}$ and note that now it satisfies (a) while (b)–(c) still hold for $N^{(i,r)}$.

Once we have constructed the final matchings $M^{(m-1)}$ and $N^{(m-1)}$, note that condition (a) implies that $V(M^{(m-1)}) \cap V(Q^j)$ is balanced for all $j \in [m - 1]$. Since $M^{(m-1)}$ itself is a balanced matching (trivially, as it consists only of short edges), it follows that $V(M^{(m-1)}) \cap V(Q^m)$ must be balanced as well. Hence we can conclude that condition (ii) holds for $M^{(m-1)}$. Moreover, conditions (b)–(c) imply that

(d) $|V(M^{(m-1)}) \cap V(Q^j)| \leq 12$ for all $j \in [m]$.

Hence condition (iii) holds for $M^{(m-1)}$ even with a certain reserve. This may prove to be useful in the 2nd pass.
(2nd pass) First select $p \in \{-1, 1\}$, put $\bar{M}^{(0)} := M^{(m-1)}$ and $\bar{N}^{(0)} := N^{(m-1)}$. Then go through all $i \in [m]$ and construct matchings $\bar{M}^{(i)}, \bar{N}^{(i)}$ such that

$$\bar{M}^{(i)} := \bar{M}^{(i-1)} \cup \{v v^d\}$$
$$\bar{N}^{(i)} := \bar{N}^{(i-1)} \cup \{v v^d\}$$

where v is a vertex of Q^i of parity p, d is the dimension over which Q^i and Q^{i+1} (for $i \in [m - 1]$) or Q^m and Q^1 (for $i = m$) are adjacent, and $\{v, v^d\} \cap V(\bar{M}^{(i-1)}) = \emptyset$. We claim that such a vertex v always exists: it suffices to apply Lemma 5.1 to the matching $\bar{N}^{(i-1)}$, subcubes Q^i, Q^{i+1} (for $i \in [m - 1]$) or Q^m, Q^1 (for $i = m$), $e = f = 7$, $c = 1$ and a, b unspecified. As $7 + 7 + 1 < 2^{5-1}$, there is a vertex $v \in V(Q^i)$ such that $\chi(v) = p$ and $\{v, v^d\}$ is disjoint with $V(\bar{N}^{(i-1)}) = V(\bar{M}^{(i-1)})$.

Once we have constructed the final matching $\bar{M}^{(m)}$, it remains to set $\bar{M} := \bar{M}^{(m-1)}$. Note that the 2nd pass guarantees the connectivity of $\mathcal{I}(\mathcal{S}, \bar{M})$ which fulfills (i). Since condition (ii) holds for $M^{(m-1)}$ and

(e) $(V(\bar{M}) \setminus V(M^{(m-1)})) \cap V(Q^i)$ for each $i \in [m]$ consists of two vertices of different parities,

it follows that (ii) holds for $\bar{M}$ as well. Finally, the validity of condition (iii) follows from (d) and (e). ∎

After the input matching was "balanced" by the previous lemma, the following theorem shows how to extend it to a Hamilton cycle. Both statements form the cornerstone of the proof of the main theorem in the last section.

Theorem 5.3 *Let $n \geq 5$, $\mathcal{S}$ be a set of pairwise vertex-disjoint 5-dimensional subcubes of Q_n, and M be a matching in $K(\mathcal{S})$ such that*

(i) *$\mathcal{I}(\mathcal{S}, M)$ is connected,*
(ii) *for every subcube $Q \in \mathcal{S}$, $V(M) \cap V(Q)$ is a balanced set of at most 16 vertices.*

Then M is $K(\mathcal{S})$-extendable.

Proof We argue by induction on the size of $\mathcal{S}$. If $|\mathcal{S}| = 1$, condition (ii) says that M is a balanced matching in $K(Q_5)$ of size at most 8 and therefore the statement follows from Proposition 4.5. So assume that $|\mathcal{S}| > 1$ and select a subcube $Q \in \mathcal{S}$ such that Q is a leaf of a spanning tree of $\mathcal{I}(\mathcal{S}, M)$. Note that then Q cannot be a cutvertex of $\mathcal{I}(\mathcal{S}, M)$ and therefore $\mathcal{I}(\mathcal{S}, M) - Q$ remains connected. Put $\mathcal{S}' := \mathcal{S} \setminus \{Q\}$ and let $v_1, v_2, \ldots, v_k$ be all vertices of $V(\mathcal{S}')$ joined by an edge of M to a vertex of Q. Note that condition (ii) applied to Q guarantees that k is even. Put $M' := M(\mathcal{S}') \cup \{v_1v_2, v_3v_4, \ldots, v_{k-1}v_k\}$ and note that $V(M') = V(M) \setminus V(Q)$ and therefore condition (ii) still holds for M'. Moreover, observe that $\mathcal{I}(\mathcal{S}, M) - Q$ is a spanning subgraph of $\mathcal{I}(\mathcal{S}', M')$ and thus the connectedness of the former implies the same for the latter, which means that condition (i) holds for $\mathcal{S}'$ and M' as well. Hence by the induction hypothesis, M' is $K(\mathcal{S}')$-extendable. This verifies condition (iii) of Lemma 3.3. Conditions (i)–(ii) of Lemma 3.3 follow from our assumptions (i)–(ii), and the latter together with Proposition 4.5 implies validity of the last condition (iv) of Lemma 3.3. Thus we can conclude that the $K(\mathcal{S})$-extendability of M follows from Lemma 3.3. ∎

6 | RESULTS

At this point we are ready to provide an affirmative answer to the Ruskey-Savage problem for every matching of size bounded by a quadratic function. Recall that both theorems of this section were already stated in the introduction.

Theorem 1.1 *Every matching in Q_n, $n \geq 2$, of size at most $\frac{n^2}{16} + \frac{n}{4}$ can be extended to a cyclic Gray code.*

Proof Let M be a arbitrary matching in Q_n such that $|M| \leq \frac{n^2}{16} + \frac{n}{4}$. For $n \leq 11$ we have $\lfloor \frac{n^2}{16} + \frac{n}{4} \rfloor \leq n - 1$ and therefore in this case the statement of the theorem follows from Corollary 4.6. For $n > 11$ select a set S such that $V(M) \subseteq S \subseteq V(Q_n)$ and $|S| = 2\lfloor \frac{n^2}{16} + \frac{n}{4} \rfloor$. Then $|S| \geq 2n$ while $\lceil \frac{n^2}{2|S|-n-2} \rceil = 5$ and therefore by Theorem 3.2, there is a set $D \subseteq [n]$ such that $|D| = 5$ and $|V(M_D(u))| \leq |S_D(u)| \leq 6$ for every $u \in \{0,1\}^{n-5}$. The statement of the theorem now follows from Lemma 5.2 and Theorem 5.3. ∎

Inspired by our results in the previous two sections, namely Corollary 4.6 and Theorem 5.3, we may be tempted to propose a generalization of the original problem as follows: is it true that every balanced matching in $K(Q_n)$ is $K(Q_n)$-extendable? The answer to this question is negative, not only for balanced matchings in $K(Q_n)$, but also for matchings in $B(Q_n)$, as demonstrated by the last result of this paper.

Theorem 1.2 *For every $n \geq 9$ there is a matching of size $2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} + 1 = \Theta(2^n/\sqrt{n})$ in $B(Q_n)$ which is not $B(Q_n)$-extendable.*

Proof Note that for $m \geq 8$ we have

$$\binom{m}{\lfloor \frac{m}{2} \rfloor} < \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m}{i} \leq \sum_{i=\lfloor \frac{m}{2} \rfloor + 1}^{m} \binom{m}{i}. \tag{$*$}$$

Put $d := 1$ and by removing all edges of dimension d, split Q_n into subcubes $Q^0 := Q_{[n]\setminus\{d\}}(0)$ and $Q^1 := Q_{[n]\setminus\{d\}}(1)$. Note that for every $u \in V(Q^0)$, $v \in V(Q^1)$ we have $0 \le |u| \le n-1$ while $1 \le |v| \le n$. Put $h := \lfloor \frac{n-1}{2} \rfloor$, $k := \binom{n-1}{h}$ and

$$\begin{aligned} \mathrm{Low}^0 &:= \{v \in V(Q^0) \mid |v| < h\}, & \mathrm{Low}^1 &:= \{v^d \mid v \in \mathrm{Low}^0\} \\ \mathrm{Mid}^0 &:= \{v \in V(Q^0) \mid |v| = h\}, & \mathrm{Mid}^1 &:= \{v^d \mid v \in \mathrm{Mid}^0\} \\ \mathrm{Up}^0 &:= \{v \in V(Q^0) \mid |v| > h\}, & \mathrm{Up}^1 &:= \{v^d \mid v \in \mathrm{Up}^0\}. \end{aligned}$$

Let $\mathrm{Mid}^0 = \{x_1, x_2, \ldots, x_k\}$, $\chi(x_1) = p$ and observe that Mid^0 and Mid^1 consist solely of vertices of parities p and $-p$, respectively. Put $m = n-1$ and note that $(*)$ guarantees that

$$k = \binom{m}{\lfloor \frac{m}{2} \rfloor} < |\mathrm{Low}^i| = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m}{i} \le |\mathrm{Up}^i| = \sum_{i=\lfloor \frac{m}{2} \rfloor + 1}^{m} \binom{m}{i}$$

for both $i \in \{0,1\}$. Since $\mathrm{Low}^0 \cup \mathrm{Low}^1$ and $\mathrm{Up}^0 \cup \mathrm{Up}^1$ are balanced, it follows that we can select distinct vertices $y_1, y_2, \ldots, y_{k+1} \in \mathrm{Low}^0 \cup \mathrm{Low}^1$ and $z_1, z_2, \ldots, z_{k+1} \in \mathrm{Up}^0 \cup \mathrm{Up}^1$ such that $\chi(z_i) = p = -\chi(y_i)$ for all $i \in [k+1]$. Then

$$M := \{x_i y_i\}_{i=1}^{k} \cup \{x_i^d z_i\}_{i=1}^{k} \cup \{y_{k+1} z_{k+1}\}$$

is a matching in $B(Q_n)$ and we claim that M is not $B(Q_n)$-extendable.

To verify the claim, assume, by way of contradiction, that there is a Hamilton cycle C of $B(Q_n)$ such that $E(C) \supseteq M$ and $E(C) \setminus M \subseteq E(Q_n)$. Put $\mathrm{LM}_n := Q_n[\mathrm{Low}^0 \cup \mathrm{Low}^1 \cup \mathrm{Mid}^0 \cup \mathrm{Mid}^1]$ and observe that $(E(C) \setminus M) \cap E(\mathrm{LM}_n)$ forms an edge set of a disjoint path cover $\mathcal{P}$ of LM_n such that the endvertices of all paths of $\mathcal{P}$ are exactly the vertices of $V(M) \cap V(\mathrm{LM}_n)$. Since the vertex set of LM_n is balanced and therefore $\beta(V(\mathrm{LM}_n)) = 0$, Proposition 3.1 (1) implies that

$$\sum_{P_{uv} \in \mathcal{P}} (\chi(u) + \chi(v)) = 0$$

as well. Note that each endvertex v contributes to this sum either by $2\chi(v)$ (if $P_{vv} \in \mathcal{P}$) or by $\chi(v)$ (if $P_{uv} \in \mathcal{P}$ or $P_{vu} \in \mathcal{P}$ for some $u \neq v$). Since the set $V(M) \cap V(\mathrm{LM}_n)$ consists of k vertices of parity p and $2k+1$ vertices of parity $-p$, it follows that $|\sum_{P_{uv} \in \mathcal{P}} (\chi(u) + \chi(v))| > 0$, which is a contradiction. This verifies the claim that the matching M is not $B(Q_n)$-extendable. ∎

Acknowledgments

The authors are much grateful to the anonymous referees for a number of constructive comments that substantially improved the exposition of this paper.

references

[1] A. Alahmadi, R.E.L. Aldred, A. Alkenani, R. Hijazi, P. Sole, and C. Thomassen, *Extending a perfect matching to a Hamiltonian cycle*, Discrete Math. Theor. Comput. Sci. 17 (2015), 241–254, http://dmtcs.episciences.org/2105.

[2] R. Caha, V. Koubek, *Hamiltonian cycles and paths with a prescribed set of edges in hypercubes and dense sets*, J. Graph Theory 51 (2006), 137–169, doi:10.1002/jgt.20128.

[3] D. Dimitrov, T. Dvořák, P. Gregor, and R. Škrekovski, *Gray codes avoiding matchings*, Discrete Math. Theor. Comput. Sci. 11 (2009), 123–148, http://dmtcs.episciences.org/457.

[4] T. Dvořák, *Hamiltonian cycles with prescribed edges in hypercubes*, SIAM J. Discrete Math. 19 (2005), 135–144, doi:10.1137/S0895480103432805.

[5] T. Dvořák, *Matchings of quadratic size extend to long cycles in hypercubes*, Discrete Math. Theor. Comput. Sci. 18:3 (2016), #12, http://dmtcs.episciences.org/2012.

[6] T. Dvořák, P. Gregor, *Partitions of faulty hypercubes into paths with prescribed endvertices*, SIAM J. Discrete Math. 22 (2008), 1448–1461, doi:10.1137/060678476.

[7] T. Dvořák, P. Gregor, V. Koubek, *Generalized Gray codes with prescribed ends*, Theor. Comput. Sci. 668 (2017), 70–94, doi:10.1016/j.tcs.2017.01.010.

[8] J. Fink, *Perfect matchings extend to Hamiltonian cycles in hypercubes*, J. Combin. Theory Ser. B 97 (2007), 1074–1076, doi:10.1016/j.jctb.2007.02.007.

[9] J. Fink, *Matching graphs of hypercubes and complete bipartite graphs*, Eur. J. Comb. 30 (2009), 1624–1629, doi:10.1016/j.ejc.2009.03.007.

[10] J. Fink, *Matchings extend into 2-factors in hypercubes*, Combinatorica, to appear.

[11] R. Forcade, *Smallest maximal matchings in the graph of the d-dimensional cube*, J. Combin. Theory Ser. B 14 (1973), 153–156, doi:10.1016/0095-8956(73)90059-2.

[12] P. Gregor, *Perfect matchings extending on subcubes to Hamiltonian cycles of hypercubes*, Discrete Math. 309 (2009), 1711–1713, doi:10.1016/j.disc.2008.02.013.

[13] I. Havel, *On hamiltonian circuits and spanning trees of hypercubes*, Čas. Pěst. Mat. 109 (1984), 135–152, http://dml.cz/dmlcz/108506.

[14] D.E. Knuth, The Art of Computer Programming, Volume 4A: Combinatorial Algorithms, Part 1, Addison-Wesley Professional, 2011.

[15] T. Mütze, *Proof of the middle levels conjecture*, Proc. Lond. Math. Soc. 112:4 (2016), 677–713, doi:10.1112/plms/pdw004.

[16] F. Ruskey, C. Savage, *Hamilton cycles which extend transposition matchings in Cayley graphs of S_n*, SIAM J. Discrete Math. 6 (1993), 152–166, doi:10.1137/0406012.

[17] J. Vandenbussche, D. B. West, *Matching extendability in hypercubes*, SIAM J. Discrete Math. 23 (2009), 1539–1547, doi:10.1137/080732687.

[18] F. Wang, H. Zhang, *Prescribed matchings extend to Hamiltonian cycles in hypercubes with faulty edges*, Discrete Math. 321 (2014), 35–44, doi:10.1016/j.disc.2013.12.014.

[19] F. Wang, H. Zhang, *Small matchings extend to Hamiltonian cycles in hypercubes*, Graphs Comb. 32:1 (2016), 363–376, doi:10.1007/s00373-015-1533-6.

[20] G. Wiener, *Edge multiplicity and other trace functions*, Electron. Notes Discrete Math. 29 (2007), 491–495, doi:10.1016/j.endm.2007.07.076.

[21] G. Wiener, *Rounds in combinatorial search*, Algorithmica 67 (2013), 315–323, doi:10.1007/s00453-013-9750-y.

[22] E. Zulkoski, C. Bright, A. Heinle, I. Kotsireas, K. Czarnecki, V. Ganesh, *Combining SAT solvers with computer algebra systems to verify combinatorial conjectures*, J. Autom. Reasoning 58:3 (2017), 313–339, doi:10.1007/s10817-016-9396-y.