Problem 1 (Independent set)**.** A set of vertices S of a graph (V, E) is called *independent* if no two vertices of S are joined by an edge. Formulate the problem of finding the maximal independent set using Integer Linear Programming.

Solution. First, we need to define a set of variables of a Integer Linear Programming problem. Since the goal of the Independent set problem is to find a subset S of vertices V, we it is natural to use variables $\boldsymbol{x} \in \{0, 1\}^V$ which correspond to the characteristic vector of S in V.

Second, we specify a set of constrains on variables $\boldsymbol{x}$ to ensure that they encode an independent set. As it stated above, variables $\boldsymbol{x}$ must be binary which is ensured by constrains $\mathbf{0} \le \boldsymbol{x} \le \mathbf{1}$ and $\boldsymbol{x} \in \mathbb{Z}^V$. Furthermore, we have to guarantee that variables $\boldsymbol{x}$ encode an independent set S which means that for every edge $uv \in E$ an independent set contains at most one vertex of $\{u, v\}$. Since $\boldsymbol{x}_u + \boldsymbol{x}_v$ is exactly the number of vertices of $\{u, v\}$ in an independent set which is provided by a condition $\boldsymbol{x}_u + \boldsymbol{x}_v \le 1$. The last inequality must hold for every edge $uv \in E$, so we use the incidence matrix A to the last inequality in a matrix form $A^{\mathrm{T}}\boldsymbol{x} \le \mathbf{1}$.

Third, we specify the objective function. Since $\mathbf{1}^{\mathrm{T}}\boldsymbol{x}$ corresponds to the number of vertices in an independent set, our objective function maximizes $\mathbf{1}^{\mathrm{T}}\boldsymbol{x}$.

In summary, the formulation we are asked to provide is the following.

$$
\begin{array}{ll}
\text{Maximize} & \mathbf{1}^{\mathrm{T}}\boldsymbol{x} \\
\text{subject to} & A^{\mathrm{T}}\boldsymbol{x} \le \mathbf{1} \\
& \mathbf{0} \le \boldsymbol{x} \le \mathbf{1} \\
& \boldsymbol{x} \in \mathbb{Z}^V
\end{array}
\tag{1}
$$

Now, we need to prove that a solution of (1) already gives an optimal solution of a given instance of the Independent set problem. This means that

1. every optimal solution of $\boldsymbol{x}$ of (1) is an characteristic vector of a maximal independent set,
2. a characteristic vector of every independent is a feasible solution to (1), and
3. if (1) has no solution, then there is no solution to the independent set problem.

Since the trivial solution $\boldsymbol{x} = \mathbf{0}$ is a feasible to (1) and the objective function $\mathbf{1}^{\mathrm{T}}\boldsymbol{x}$ is bounded above by the number of vertices, (1) always has an optimal solution and the third requirement is satisfied. For arbitrary independent set S, the characteristic vector $\boldsymbol{x}$ of S satisfies all conditions in (1), so the second requirement is satisfied.

In order to prove the first requirement, let $\boldsymbol{x}^\star$ be an optimal solution of (1). Clearly, $\boldsymbol{x}^\star$ is a binary vector, so we can define a set $S = \{v \in V;\ \boldsymbol{x}_v = 1\}$ of vertices. The inequality $A^{\mathrm{T}}\boldsymbol{x} \le \mathbf{1}$ prevents any pair of adjacent vertices being both in S, so the set S corresponding to $\boldsymbol{x}^\star$ is an independent set. It remains to prove the optimality of S. Indeed, suppose for the sake of contradiction that there exists an independent set S' of larger size than S. From the second requirement it follows that the characteristic vector $\boldsymbol{x}'$ of S' is feasible to (1). From $\mathbf{1}^{\mathrm{T}}\boldsymbol{x}^\star = |S^\star| < |S'| = \mathbf{1}^{\mathrm{T}}\boldsymbol{x}'$ it follows that $\boldsymbol{x}^\star$ is not an optimal solution of (1) which is a contradiction.

Problem 2 (Sum-zero problem)**.** Formulate the following problem using Integer Linear Programming: For a given integers $a_1, \dots, a_n \in \mathbb{Z}$ find a non-empty set $S \subseteq \{1, \dots, n\}$ such that the sum $\sum_{i \in S} a_i$ is as close to zero as possible, i.e. minimize $|\sum_{i \in S} a_i|$.

Solution. In order to simplify the notation, let $\boldsymbol{a}$ be a vector which composes all given integers $a_1, \dots, a_n \in \mathbb{Z}$. A feasible solution of the Sum-zero problem is a non-empty subset $S \subseteq \{1, \dots, n\}$, so we introduce variables $\boldsymbol{x} \in \{0, 1\}^n$ which represent a characteristic vector of S in $\{1, \dots, n\}$ and we set a condition $\mathbf{1}^{\mathrm{T}}\boldsymbol{x} \ge 1$ to ensure that $\boldsymbol{x}$ encodes a non-empty set. Although an objective function $|\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}|$ is a proper mathematical function, its is not a linear function, so we cannot use it a Linear programming problem. Hence, we use a standard approach to handle the absolute value: Introduce a new variable z, add constrains $z \ge \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}$ and $z \ge -\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}$, and replace the objective function by a function minimizing z.

In summary, the formulation we are asked to provide is the following.

$$\begin{array}{ll} \text{Minimize} & z \\ \text{subject to} & z - \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \geq 0 \\ & z + \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \geq 0 \\ & \mathbf{1}^{\mathrm{T}}\boldsymbol{x} \geq 1 \\ & \mathbf{0} \leq \boldsymbol{x} \leq \mathbf{1} \\ & \boldsymbol{x} \in \mathbb{Z}^{V} \end{array} \tag{2}$$

Since the problem (2) has a feasible solution $\boldsymbol{x} = \mathbf{1}$ and $z = |\mathbf{1}^{\mathrm{T}}\boldsymbol{x}|$ and the objective function z is bounded below by 0, the problem (2) always has an optimal solution. So, let $\boldsymbol{x}^{\star}, z^{\star}$ be an optimal solution of (2) and we prove that $S^{\star} = \{i;\ \boldsymbol{x}_i^{\star} = 1\}$ is an optimal solution of the Sum-zero problem. Since $\mathbf{1}^{\mathrm{T}}\boldsymbol{x} \geq 1$, the subset $S^{\star}$ is a feasible solution. Suppose for the sake of contradiction that there exists a non-empty subset $S' \subseteq \{1, \dots, n\}$ such that $|\sum_{i \in S'} a_i| < |\sum_{i \in S^{\star}} a_i|$. Let $\boldsymbol{x}'$ be a characteristic vector of S' in $\{1, \dots, n\}$ and z' be $|\mathbf{1}^{\mathrm{T}}\boldsymbol{x}'|$. Clearly, $\boldsymbol{x}', z'$ is a feasible solution of (2). From $z' = |\mathbf{1}^{\mathrm{T}}\boldsymbol{x}'| = |\sum_{i \in S'} a_i| < |\sum_{i \in S^{\star}} a_i| = |\mathbf{1}^{\mathrm{T}}\boldsymbol{x}^{\star}| \leq z^{\star}$ it follows that $\boldsymbol{x}^{\star}, z^{\star}$ is not an optimal solution of (2) which is a contradiction.