Problem 1 is for homework. Solutions must be submitted before the next lecture (not tutorial!) to be evaluated. Students are not allowed to keep submitted solutions after evaluation.

Problem 1. Prove that a non-empty set $A \subseteq \mathbb{R}^n$ is affine if and only if for every pair of points of A the line defined by those two points is contained in A.

Problem 2. Prove that for vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ the following statements are equivalent.

- Vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k$ are affinely independent.
- Vectors $\boldsymbol{v}_1 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_k - \boldsymbol{v}_0$ are linearly independent.
- The origin $\mathbf{0}$ is not a non-trivial combination $\sum \alpha_i \boldsymbol{v}_i$ such that $\sum \alpha_i = 0$ and $\boldsymbol{\alpha} \neq \mathbf{0}$.

Problem 3. Prove that the affine hull of a set $S \subseteq \mathbb{R}^n$ is the set of all affine combinations of S.

Problem 4. Let $S \subseteq \mathbb{R}^n$. Prove that $\operatorname{span}(S) = \operatorname{aff}(S \cup \{\mathbf{0}\})$.

Problem 5. Suppose we have a system of linear inequalities that also contains strict inequalities. One that may look like this:

$$\begin{array}{ccccccccc} 5x & + & 3y & & & & & \leq & 8 \\ 2x & & & - & 5z & & & < & -3 \\ 6x & + & 5y & & & + & 2w & = & 5 \\ & & & & 3z & + & 2w & > & 5 \\ & & & & & & x,y,z,w & \geq & 0 \end{array}$$

Is there a way to check if this system has a feasible solution using a linear program?

Does this mean that linear programming allows strict inequalities? Not really. As a strange example, construct a "linear program with a strict ineqality" that satisfies the following:

- There is a simple finite upper bound on its optimum value;
- there is a feasible solution; and
- there is no optimal solution.

This may not happen for a linear program – for a bounded LP, once there exists a feasible solution, there exists also an optimal solution.

Problem 6. Using the graphical methods find the optimal solutions of two objective functions

- $\min \boldsymbol{x}_1 + \boldsymbol{x}_2$
- $\max \boldsymbol{x}_1 + \boldsymbol{x}_2$

subject to the following conditions.

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 3 & -1 \\ -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} \geq \begin{pmatrix} 14 \\ 0 \\ 0 \\ -7 \\ 8 \end{pmatrix}$$

Problem 7. A steel factory produces wire ropes of length L which is bought by an electricity distributor to reinforcement various parts of the grid. The distributor estimates that cables of length $a_1, \ldots, a_n$ are needed for the reinforcement (assume that $a_1, \ldots, a_n \leq L$). So, the question is how to cut wire ropes of length L into cables of length $a_1, \ldots, a_n$ so that the sum of length of unused parts is minimal. Model this problem using a single Integer linear programming problem.

Problem 8. Write a linear programming program which decides whether a given oriented graph contains an oriented cycle.

Problem 9. The travelling salesperson problem (TSP) asks the following question: Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?

Formulate the Travelling salesman problem using (integer) linear programming.

Problem 10. The k-colouring problem of a graph (V, E) is assigning one of k colours to every vertex such that no two adjacent vertices share the same colour. Using Integer linear programming determine whether a given graph is k-colourable for a given integer k.

Problem 11. A square at the center in the point (r_1, r_2) with sides of length s is the square with vertices $\{(r_1 + s_1, r_2 + s_2);\ s_1, s_2, \in \{-s/2, s/2\}\}$.

An input of the problem is a set of points $b_1, \ldots, b_n$. For every point b_i, we have to assign a square at the center b_i with sides of length $a_i >= 0$. The area of the intersection of every pair of squares has to be zero, i.e. the intersection has to be at most an side. For instance, you can assign to points $(0, 0)$ and $(5, 4)$ squares with sides of length 2 and 1, or 6 and 4, or 10 and 0; however, sides of length 4 and 7 are forbidden. If a square has a side of length 0, then it is allowed to lie on a side (vertex) of a square but it must not lie inside another square.

Model this problem using linear programming.

Problem 12 (The Classic Transportation Problem I)**.** An unknown country has n bakeries and m shops. Every day, the i-th bakery bakes b_i breads and the j-the shop sells s_j breads. The transportation of one bread from the i-th bakery to the j-th shop cost $c_{i,j}$. Describe the following problem using linear programming: Find the cheapest transportation of all breads so that all breads are transported according to the number of baked and sold breads.

Furthermore, find necessary and sufficient conditions for the existence of a feasible and an optimal solutions.

Problem 13 (The Classic Transportation Problem II)**.** It turned out that whenever i-th bakery supplies j-th shop (with positive number of breads), logistic is needed to organize the transportation which cost an extra $l_{i,j}$. Using linear programming formulate the problem of minimizing the total transportation and logistic expenses.