Problem 1 is for homework. Solutions must be submitted before the next lecture (not tutorial!) to be evaluated. Students are not allowed to keep submitted solutions after evaluation.

Problem 1. First, prove that the following two definitions of the n-dimensional crosspolytope are equivalent.

- $\{\boldsymbol{x} \in \mathbb{R}^n;\ \sum_{i=1}^{n} |\boldsymbol{x}_i| \leq 1\}$
- $\{\boldsymbol{x} \in \mathbb{R}^n;\ \boldsymbol{dx} \leq 1$ for all $\boldsymbol{d} \in \{-1,1\}^n\}$

Second, prove that the number of k-dimensional faces of the crosspolytope is $2^{k+1}\binom{n}{k+1}$.

Problem 2. Let n-dimensional hypercube be the set $\{\boldsymbol{x} \in \mathbb{R}^n;\ \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{1}\}$. Let n-dimensional simplex be the convex hull of $n+1$ affinely independent points such that no point belongs into the convex hull of the other points. Determine the number of k-dimensional faces of the n-dimensional hypercube and the n-dimensional simplex.

Problem 3. The intersection of two faces of a polyhedron P is a face of P.

Problem 4. Write a linear programming program which decides whether a given oriented graph contains an oriented cycle.

Problem 5. The travelling salesperson problem (TSP) asks the following question: Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?

Formulate the Travelling salesman problem using (integer) linear programming.

Problem 6. A square at the center in the point (r_1, r_2) with sides of length s is the square with vertices $\{(r_1 + s_1, r_2 + s_2);\ s_1, s_2, \in \{-s/2, s/2\}\}$.

An input of the problem is a set of points $b_1, \dots, b_n$. For every point b_i, we have to assign a square at the center b_i with sides of length $a_i >= 0$. The area of the intersection of every pair of squares has to be zero, i.e. the intersection has to be at most an side. For instance, you can assign to points $(0,0)$ and $(5,4)$ squares with sides of length 2 and 1, or 6 and 4, or 10 and 0; however, sides of length 4 and 7 are forbidden. If a square has a side of length 0, then it is allowed to lie on a side (vertex) of a square but it must not lie inside another square.

Model this problem using linear programming.

Problem 7. Suppose we have a system of linear inequalities that also contains strict inequalities. One that may look like this:

$$\begin{array}{rcrcrcrcr}
5x & + & 3y & & & & & \leq & 8 \\
2x & & & - & 5z & & & < & -3 \\
6x & + & 5y & & & + & 2w & = & 5 \\
 & & & & 3z & + & 2w & > & 5 \\
 & & & & & x,y,z,w & & \geq & 0
\end{array}$$

Is there a way to check if this system has a feasible solution using a linear program?

Does this mean that linear programming allows strict inequalities? Not really. As a strange example, construct a "linear program with a strict ineqality" that satisfies the following:

- There is a simple finite upper bound on its optimum value;
- there is a feasible solution; and
- there is no optimal solution.

This may not happen for a linear program – for a bounded LP, once there exists a feasible solution, there exists also an optimal solution.