Problem 2 is for homework. Solutions must be submitted before the next lecture (not tutorial!) to be evaluated. Students are not allowed to keep submitted solutions after evaluation.

Problem 1. The intersection of two faces of a polyhedron P is a face of P.

Problem 2. The convex hull of points $(0,1,0,1,0), (0,1,0,\frac{10}{11},\frac{10}{11})$, $(0,0,1,1,0)$, $(0,0,1,\frac{10}{11},\frac{10}{11})$ is a face F of a polyherdon P given by conditions

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & \leq & 1 \\ & & x_4 & + & 10x_5 & \leq & 10 \\ & & 10x_4 & + & x_5 & \leq & 10 \\ & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4, x_5} & \geq & 0. \end{array}$$

Find an objective function $c^T x$ such that the set of all optimal solution of the linear problem $\max\left\{c^T x;\ x \in P\right\}$ is exactly F. Prove that your objective function already gives the face F.

Problem 3. Prove that a non-empty set $F \subseteq \mathbb{R}^n$ is a face of a polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n;\ A\boldsymbol{x} \leq \boldsymbol{b}\}$ if and only if F is the set of all optimal solutions of a linear programming problem $\min\left\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x};\ A\boldsymbol{x} \leq \boldsymbol{b}\right\}$ for some vector $\boldsymbol{c} \in \mathbb{R}^n$.

Problem 4. Prove that the n-dimensional ball is not a polyhedron.

Problem 5. First, prove that the following two definitions of the n-dimensional crosspolytope are equivalent.

- $\{\boldsymbol{x} \in \mathbb{R}^n;\ \sum_{i=1}^n |\boldsymbol{x}_i| \leq 1\}$
- $\{\boldsymbol{x} \in \mathbb{R}^n;\ \boldsymbol{dx} \leq 1$ for all $\boldsymbol{d} \in \{-1,1\}^n\}$

Second, prove that the number of k-dimensional faces of the crosspolytope is $2^{k+1}\binom{n}{k+1}$.

Problem 6. Prove that the set of faces of a polyhedron and the inclusion form a partially ordered set.

Problem 7. Prove that every non-empty polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n;\ A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ has a vertex.

Problem 8. Prove that every polytope has a vertex.

Problem 9. Prove that every proper (inclusion) maximal face is a facet.

Problem 10. Prove that all proper (inclusion) minimal faces have the same dimension.

Problem 11. Every proper face is an intersection of facets.